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BOUND FOR THE SUM INVOLVING THE JACOBI SYMBOL IN $\mathbb{Z}[i]$

Kazuhiro Onodera

Abstract: We give a nontrivial estimate of a certain sum involving the Jacobi symbol in $\mathbb{Z}[i]$ which is a generalization of Heath-Brown's character sum estimate.

Keywords: character sum, Jacobi symbol

1. Introduction

In 1995 Heath-Brown [1] derived a powerful estimate

$$\sum_{m\leqslant M}^{*}\left|\sum_{n\leqslant N}^{*} a_n\left(\frac{n}{m}\right)\right|^2 \ll_\varepsilon (MN)^\varepsilon(M+N)\sum_{n\leqslant N}^{*}|a_n|^2$$

for any complex sequence $\{a_n\}$, where $\sum^*$ means that the range of summation is restricted to odd squarefree integers. As applications, he gave new mean-value, and zero-density, estimates for Dirichlet L-functions. Thereafter many people applied the above result and its corollaries to arithmetic problems such as the mean-value estimate for various kinds of L-functions [2, 4], or the nonvanishing of the central value of quadratic Dirichlet L-functions [5].

In this paper we consider the generalization of the above estimate to the case of Gaussian integers. Let $[\frac{\cdot}{\cdot}]$ denote the Jacobi symbol in $\mathbb{Z}[i]$ which is defined in § 2, and $N(n)$ the norm of n in $\mathbb{Z}[i]$. Then our main result is stated as follows.

Theorem 1. *Let M, N be positive integers and $\{a_n\}$ be arbitrary complex sequence. Then we have*

$$\sum_{N(m)\leqslant M}^{*}\left|\sum_{N(n)\leqslant N}^{*} a_n\left[\frac{n}{m}\right]\right|^2 \ll_\varepsilon (MN)^\varepsilon(M+N)\sum_{N(n)\leqslant N}^{*}|a_n|^2 \qquad (1.1)$$

for any $\varepsilon > 0$.

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The strategy of the proof is the same as in [1]. But we have to resolve some difficulties which occur in our case.

In §2, we recall some properties of the Jacobi symbol and the Gaussian sums in $\mathbb{Z}[i]$. The exact value of the Gaussian sum is determined in Proposition 2.2. Moreover in Lemma 2.3 we give a rough bound for the left hand side of (1.1). In §3, we prove various kinds of lemmas which will be used in the later section. In §4, we prove Lemma 4.6 which improve the bound given in Lemma 2.3. Applying the lemma repeatedly, we obtain Theorem 1.

2. Jacobi symbol and Gaussian sums in $\mathbb{Z}[i]$

In this section we recall definitions and some properties for the Jacobi symbol and the Gaussian sums in $\mathbb{Z}[i]$. Moreover we will give a rough bound, Lemma 2.3, for the left hand side of (1.1).

Let p be a Gaussian prime and q be a Gaussian integer with $(p, q) = 1$. Then we define the quadratic residue symbol as

$$\left[\frac{q}{p}\right] = \begin{cases} 1 & \text{if there exist some } x \in \mathbb{Z}[i] \text{ such that } x^2 \equiv q \bmod p, \\ -1 & \text{otherwise.} \end{cases}$$

Let n be an odd integer in $\mathbb{Z}[i]$ with a prime factorization $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ and m be a Gaussian integer. The Jacobi symbol is defined by

$$\left[\frac{m}{n}\right] = \begin{cases} \left[\frac{m}{p_1}\right]^{\alpha_1} \left[\frac{m}{p_2}\right]^{\alpha_2} \cdots \left[\frac{m}{p_r}\right]^{\alpha_r} & \text{if } (m, n) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Using the reciprocity law of the residue symbol [3, Proposition 5.1], we easily see that the same law holds for the Jacobi symbol as follows.

Proposition 2.1. *Let m, n be coprime Gaussian integers $\equiv 1 \pmod 2$. Then we have*

$$\left[\frac{m}{n}\right] = \left[\frac{n}{m}\right].$$

Moreover for $a + bi \equiv 1 \bmod 2$ with $a, b \in \mathbb{Z}$ we have

$$\left[\frac{i}{a+bi}\right] = (-1)^{\frac{b}{2}}, \qquad \left[\frac{1+i}{a+bi}\right] = \left(\frac{2}{a+b}\right),$$

where $(\frac{\cdot}{\cdot})$ is the Jacobi symbol in $\mathbb{Z}$.

We define the Gaussian sum for the Jacobi symbol by

$$\tau(n) = \sum_{m \bmod n} \left[\frac{m}{n}\right] e\left(\left[1, \frac{m}{n}\right]\right)$$

for an odd integer $n \in \mathbb{Z}[i]$, where $e(x) = \exp(2\pi i x)$ and $[z, w] = \mathrm{Re}(z\bar{w})$. This sum is not multiplicative, but we have the relation

$$\tau(mn) = \left[\frac{m}{n}\right] \left[\frac{n}{m}\right] \tau(m)\tau(n) \tag{2.1}$$

for odd integers m and n with $(m,n)=1$. We determine the exact value of $\tau(n)$ for a prime power $n=p^r$. We immediately see that $\tau(p^r)$ vanishes for an odd prime p and $r\geqslant 2$. The result for $\tau(p)$ is as follows.

Proposition 2.2. *Let p be an odd prime and write $p=p_1+p_2i$ with $p_1,p_2\in\mathbb{Z}$. Then we have*

$$\tau(p)=\left(\cos\frac{\pi p_1}{2}+\cos^2\frac{\pi p_2}{2}\right)N(p)^{\frac{1}{2}}. \tag{2.2}$$

Proof. To prove this proposition we employ the method used by Dirichlet to determine the exact value of the usual quadratic Gaussian sum. Let n be a nonzero Gaussian integer. Put

$$S(n)=\sum_{z \bmod n} e\left(\left[1,\frac{z^2}{n}\right]\right).$$

Then we can check that $\tau(p)=S(p)$, so we investigate the sum $S(n)$. We treat the case that $\mathrm{Re}(n)>0$ and $\mathrm{Im}(n)\geqslant 0$, the other cases being similar. Let $D(n)$ be the fundamental parallelogram for the lattice generated by n and in, namely

$$D(n)=\{sn+tin\in\mathbb{C}:0\leqslant s,t<1\}.$$

Moreover we set

$$D'(n)=\bigcup_{w\in\mathbb{Z}[i]\cap D(n)}\{z\in\mathbb{C}:0\leqslant \mathrm{Re}(z)-\mathrm{Re}(w),\mathrm{Im}(z)-\mathrm{Im}(w)<1\}.$$

By the double Poisson's summation formula we have

$$S(n)=\sum\sum_{|m_1|,|m_2|\leqslant M}\int_{D'(n)} e\left(\left[1,\frac{z^2}{n}+mz\right]\right)dz+O_n\left(\frac{\log^2 2M}{M}\right)$$

for any $M>1$, where we write $m=m_1+im_2$. Set

$$S_j=\sum\sum_{\substack{|m_1|,|m_2|\leqslant M\\ m\equiv j \bmod 2}} e\left(-\left[1,\frac{nm^2}{4}\right]\right)\int_{D'(n)+\frac{mn}{2}} e\left(\left[1,\frac{z^2}{n}\right]\right)dz$$

for $j\in\{0,1,i,1+i\}$. Then the main term is $S_0+S_1+S_i+S_{1+i}$. We first treat S_0. Then we have

$$\begin{aligned}S_0&=\sum_{|l_1|\leqslant M/2}\sum_{|l_2|\leqslant M/2}\int_{D'(n)+nl} e\left(\left[1,\frac{z^2}{n}\right]\right)dz\\&=\sum_{|l_1|\leqslant M/2}\sum_{|l_2|\leqslant M/2}\int_{D(n)+nl} e\left(\left[1,\frac{z^2}{n}\right]\right)dz+O_n(M^{-\frac{1}{2}})\end{aligned}$$

by partial integration. Hence we obtain

$$S_0=N(n)\int_{[-M',M'+1]\times[-M',M'+1]} e([1,nz^2])dz+O_n(M^{-\frac{1}{2}})$$

where $M' = [M/2]$. Thus S_0 converges to $\frac{1}{2}N(n)^{\frac{1}{2}}$ when M goes to $+\infty$. Similarly, we have

$$S_1 \to \frac{e(-[1,\frac{n}{4}])}{2}N(n)^{\frac{1}{2}}, \quad S_i \to \frac{e([1,\frac{n}{4}])}{2}N(n)^{\frac{1}{2}}, \quad S_{1+i} \to \frac{e(-[1,\frac{in}{2}])}{2}N(n)^{\frac{1}{2}}$$

as $M \to +\infty$. We combine these results to obtain

$$\begin{aligned} S(n) &= \frac{1 + e([1,n/4]) + e(-[1,n/4]) + e(-[1,in/2])}{2}N(n)^{\frac{1}{2}} \\ &= \left(\cos\frac{\pi n_1}{2} + \cos^2\frac{\pi n_2}{2}\right)N(n)^{\frac{1}{2}}, \end{aligned}$$

which gives Proposition 2.2. ■

Our next result gives a rough bound for the left of (1.1) which is important to get the bound in Theorem 1.

Lemma 2.3. *Let M, N be positive integers. Then*

$$\sideset{}{^*}\sum_{N(m)\leqslant M}\left|\sideset{}{^*}\sum_{N(n)\leqslant N} a_n\left[\frac{n}{m}\right]\right|^2 \ll (M+N^8)\sideset{}{^*}\sum_{N(n)\leqslant N}|a_n|^2.$$

To prove this lemma, we first obtain a bound for the following double sum of the Jacobi symbol.

Lemma 2.4. *Let M be a positive real number and let n be an odd squarefree integer. Then we have*

$$\sum_{N(m)\leqslant M}\left[\frac{m}{n}\right] \ll M^{\frac{1}{2}}N(n)^{\frac{3}{2}}. \tag{2.3}$$

Proof. We easily see that

$$\left[\frac{m}{n}\right]\tau(n) = \sum_{a \bmod n}\left[\frac{a}{n}\right]e\left(\left[1,\frac{am}{n}\right]\right). \tag{2.4}$$

The equations (2.1) and (2.2) imply that $|\tau(n)| = N(n)^{\frac{1}{2}}$ for n odd squarefree. Hence we have

$$\left|\sum_{N(m)\leqslant M}\left[\frac{m}{n}\right]\right| \leqslant \frac{1}{N(n)^{\frac{1}{2}}}\sum_{\substack{a \bmod n\\(a,n)=1}}\left|\sum_{N(m)\leqslant M} e\left(\left[1,\frac{am}{n}\right]\right)\right|,$$

where the condition $(a,n)=1$ implies that $\mathrm{Re}(\frac{a}{n})$ or $\mathrm{Im}(\frac{a}{n})$ is not a rational integer. For the terms in which both $\mathrm{Re}(\frac{a}{n})$ and $\mathrm{Im}(\frac{a}{n})$ are not integers, we have a bound

$$\ll \frac{1}{N(n)^{\frac{1}{2}}}\sum_{\substack{a \bmod n\\ \mathrm{Re}(\frac{a}{n}),\mathrm{Im}(\frac{a}{n})\notin\mathbb{Z}}}\frac{M^{\frac{1}{2}}}{|\sin(\pi\,\mathrm{Im}(\frac{a}{n}))|} \ll M^{\frac{1}{2}}N(n)^{\frac{3}{2}},$$

since $\|\operatorname{Im}(\frac{a}{n})\| \geqslant N(n)^{-1}$ where $\|x\| = \min_{n\in\mathbb{Z}} |x-n|$. Next we obtain a bound

$$\ll \frac{1}{N(n)^{\frac{1}{2}}} \sum_{\substack{a \bmod n \\ \operatorname{Re}(\frac{a}{n})\notin\mathbb{Z}, \operatorname{Im}(\frac{a}{n})\in\mathbb{Z}}} \frac{M^{\frac{1}{2}}}{|\sin(\pi \operatorname{Re}(\frac{a}{n}))|} \ll M^{\frac{1}{2}} N(n)^{\frac{3}{2}}.$$

Similarly, the terms in which $\operatorname{Re}(\frac{a}{n}) \in \mathbb{Z}$ and $\operatorname{Im}(\frac{a}{n}) \notin \mathbb{Z}$ are bounded by $O(M^{\frac{1}{2}} N(n)^{\frac{3}{2}})$. Combining the above estimates, we obtain (2.3). ■

Proof of Lemma 2.3. By Lemma 3.1 given below there exist some sequence $\{a'_n\}$ with $|a'_n| = |a_n|$ for all n such that

$$\sideset{}{^*}\sum_{N(m)\leqslant M} \Big| \sideset{}{^*}\sum_{N(n)\leqslant N} a_n \Big[\frac{n}{m}\Big] \Big|^2 \leqslant 2 \sum_{N(m)\leqslant M} \sideset{}{^*}\sum\sideset{}{^*}\sum_{N(n_1),N(n_2)\leqslant N} a'_{n_1} \overline{a'_{n_2}} \Big[\frac{m}{n_1 n_2}\Big].$$

We split the sum over n_1, n_2 into two parts depending on whether $(n_1) = (n_2)$ or not. Then we have

$$\begin{aligned}
&\sideset{}{^*}\sum_{N(m)\leqslant M} \Big| \sideset{}{^*}\sum_{N(n)\leqslant N} a_n \Big[\frac{n}{m}\Big] \Big|^2 \\
&\ll M \sideset{}{^*}\sum_{N(n)\leqslant N} |a_n|^2 + \Big| \sideset{}{^*}\sum\sideset{}{^*}\sum_{(n_1)\neq(n_2)} a'_{n_1} \overline{a'_{n_2}} \sum_{N(m)\leqslant M} \Big[\frac{m}{n_1 n_2}\Big] \Big| \\
&\ll M \sideset{}{^*}\sum_{N(n)\leqslant N} |a_n|^2 + \Big| \sum_{N(d)\leqslant 2N} \sideset{}{^*}\sum\sideset{}{^*}\sum_{\substack{(n_1,n_2)=(d) \\ (n_1)\neq(n_2)}} a'_{n_1} \overline{a'_{n_2}} \sum_{\substack{N(m)\leqslant M \\ (m,d)=1}} \Big[\frac{m}{n_1 n_2/d^2}\Big] \Big| \\
&\ll M \sideset{}{^*}\sum_{N(n)\leqslant N} |a_n|^2 \\
&\quad + \Big| \sum_{N(d)\leqslant 2N} \sideset{}{^*}\sum\sideset{}{^*}\sum_{\substack{(n_1,n_2)=(d) \\ (n_1)\neq(n_2)}} a'_{n_1} \overline{a'_{n_2}} \sum_{k|d} \mu(k) \Big[\frac{k}{n_1 n_2/d^2}\Big] \sum_{N(l)\leqslant M/N(k)} \Big[\frac{l}{n_1 n_2/d^2}\Big] \Big|.
\end{aligned}$$

Lemma 2.4 shows that the innermost sum is bounded by

$$O\left(\sqrt{\frac{MN(n_1 n_2)^3}{N(k)N(d)^6}}\right).$$

Hence the contribution of the second term is

$$\ll M^{\frac{1}{2}} N^3 \Big(\sideset{}{^*}\sum_{N(n)\leqslant N} |a_n| \Big)^2.$$

By Cauchy's inequality we have

$$\sideset{}{^*}\sum_{N(m)\leqslant M} \Big| \sideset{}{^*}\sum_{N(n)\leqslant N} a_n \Big[\frac{n}{m}\Big] \Big|^2 \ll (M + M^{\frac{1}{2}} N^4) \sideset{}{^*}\sum_{N(n)\leqslant N} |a_n|^2.$$

Since

$$M + M^{\frac{1}{2}} N^4 \leqslant (M^{\frac{1}{2}} + N^4)^2 \leqslant 2M + 2N^8,$$

Lemma 2.3 follows. ■

3. Preliminary lemmas

In this section we shall prove various lemmas which will be necessary to prove Theorem 1.

We put

$$\Sigma_1 = \sum_{N(m)\sim M}^{*} \Big| \sum_{N(n)\sim N} a_n \left[\frac{n}{m}\right] \Big|^2$$

where $N(m) \sim M$ means $M < N(m) \leqslant 2M$ and the sequence $\{a_n\}$ is supported on the odd squarefree Gaussian integers satisfying $N(n) \sim N$. We define

$$\mathcal{B}(M,N) = \sup \Sigma_1 / \sum_{N(n)\sim N} |a_n|^2$$

where the supremum is over all $\{a_n\}$ for which the denominator is non-vanishing. Then Theorem 1 is equivalent to

$$\mathcal{B}(M,N) \ll_\varepsilon (MN)^\varepsilon (M+N)$$

for any $\varepsilon > 0$.

First, we pick up two basic properties in Lemma 3.1 and 3.2. Our first result concerns the symmetry for $\mathcal{B}(M,N)$.

Lemma 3.1. *We have*

$$\mathcal{B}(M,N) \leqslant 2\mathcal{B}(N,M).$$

Moreover there exist some sequence $\{a'_n\}$ with $|a'_n| = |a_n|$ for all n such that

$$\sum_{N(m)\sim M}^{*} \Big| \sum_{N(n)\sim N}^{*} a_n \left[\frac{n}{m}\right] \Big|^2 \leqslant 2 \sum_{N(m)\sim M}^{*} \Big| \sum_{N(n)\sim N}^{*} a'_n \left[\frac{m}{n}\right] \Big|^2.$$

Proof. The duality principle shows that

$$\mathcal{B}(M,N) = \sup_{\{a_m\}} \left\{ \sum_{N(n)\sim N}^{*} \Big| \sum_{N(m)\sim M}^{*} a_m \left[\frac{n}{m}\right] \Big|^2 \Big/ \sum_{N(m)\sim M}^{*} |a_m|^2 \right\}.$$

We investigate the sum in the numerator of the right. We split the sum over n into two parts depending on the value of n (mod 2) and use the reciprocity law of the Jacobi symbol, so that we have

$$\begin{aligned}
&\sum_{N(n)\sim N}^{*}\left|\sum_{N(m)\sim M}^{*} a_m\left[\frac{n}{m}\right]\right|^2 \\
&\quad\leqslant \sum_{\substack{N(n)\sim N\\ n\equiv 1 \bmod 2}}^{*}\left|\sum_{N(m)\sim M}^{*} a_m\left[\frac{m}{n}\right]\right|^2+\sum_{\substack{N(n)\sim N\\ n\equiv i \bmod 2}}^{*}\left|\sum_{N(m)\sim M}^{*} a_m\left[\frac{i}{m}\right]\left[\frac{m}{n}\right]\right|^2 \\
&\quad\leqslant 2\mathcal{B}(N,M)\sum_{N(m)\sim M}^{*}|a_m|^2.
\end{aligned}$$

Hence the first part of Lemma 3.1 follows. To obtain the second part we should take $\{a_n'\}=\{a_n\}$ or $\{a_n\left[\frac{i}{n}\right]\}$, since

$$\begin{aligned}
&\sum_{N(m)\sim M}^{*}\left|\sum_{N(n)\sim N}^{*} a_n\left[\frac{n}{m}\right]\right|^2 \\
&\quad\leqslant \sum_{\substack{N(m)\sim M\\ m\equiv 1 \bmod 2}}^{*}\left|\sum_{N(n)\sim N}^{*} a_n\left[\frac{m}{n}\right]\right|^2+\sum_{\substack{N(m)\sim M\\ m\equiv i \bmod 2}}^{*}\left|\sum_{N(n)\sim N}^{*} a_n\left[\frac{i}{n}\right]\left[\frac{m}{n}\right]\right|^2 \\
&\quad\leqslant 2\max_{\{a_n'\}=\{a_n\},\{a_n[\frac{i}{n}]\}}\sum_{N(m)\sim M}^{*}\left|\sum_{N(n)\sim N}^{*} a_n'\left[\frac{n}{m}\right]\right|^2. \qquad \blacksquare
\end{aligned}$$

The next result implies that we may regard that $\mathcal{B}(M,N)$ is essentially increasing with respect to M and N.

Lemma 3.2. *There exists an absolute constant $C\geqslant 1$ as follows. If $M_1,N\geqslant 1$ and $M_2\geqslant CM_1\log(2M_1N)$, then*

$$\mathcal{B}(M_1,N)\ll\mathcal{B}(M_2,N).$$

Moreover, if $M,N_1\geqslant 1$ and $N_2\geqslant CN_1\log(2MN_1)$, then

$$\mathcal{B}(M,N_1)\ll\mathcal{B}(M,N_2).$$

Proof. Put $K=M_2/M_1$. First we suppose that m is an odd squarefree integer in $\mathbb{Z}[i]$ and satisfies $M_1<N(m)\leqslant\frac{3}{2}M_1$. We take an odd prime p with $K<N(p)<\frac{4}{3}K$. Then, since

$$\left|\sum_{N(n)\sim N}^{*} a_n\left[\frac{n}{m}\right]\right|^2\leqslant 2\left|\sum_{p\nmid n}^{*} a_n\left[\frac{n}{m}\right]\right|^2+2\left|\sum_{p\mid n}^{*} a_n\left[\frac{n}{m}\right]\right|^2,$$

we have

$$\sideset{}{^*}\sum_{M_1<N(m)\leqslant\frac{3}{2}M_1}\Big|\sideset{}{^*}\sum_{n}\Big|^2=\sideset{}{^*}\sum_{p|m}\Big|\sideset{}{^*}\sum_{n}\Big|^2+\sideset{}{^*}\sum_{p\nmid m}\Big|\sideset{}{^*}\sum_{n}\Big|^2$$
$$\leqslant\sideset{}{^*}\sum_{p|m}\Big|\sideset{}{^*}\sum_{n}\Big|^2+2\sideset{}{^*}\sum_{p\nmid m}\Big|\sideset{}{^*}\sum_{p\nmid n}\Big|^2+2\sideset{}{^*}\sum_{p\nmid m}\Big|\sideset{}{^*}\sum_{p|n}\Big|^2.$$

In the second sum on the right, pm is odd squarefree and $N(pm)$ lies in $(M_2, 2M_2]$. Hence we have

$$\sideset{}{^*}\sum_{\substack{M_1<N(m)\leqslant\frac{3}{2}M_1\\ p\nmid m}}\Big|\sideset{}{^*}\sum_{\substack{N(n)\sim N\\ p\nmid n}}a_n\left[\frac{n}{m}\right]\Big|^2=\sideset{}{^*}\sum_{\substack{M_1<N(m)\leqslant\frac{3}{2}M_1\\ p\nmid m}}\Big|\sideset{}{^*}\sum_{\substack{N(n)\sim N\\ p\nmid n}}b_n\left[\frac{n}{pm}\right]\Big|^2$$
$$\leqslant\sideset{}{^*}\sum_{N(r)\sim M_2}\Big|\sideset{}{^*}\sum_{N(n)\sim N}b_n\left[\frac{n}{r}\right]\Big|^2,$$

where $b_n=[\frac{n}{p}]a_n$. Thus we obtain a bound for the second sum

$$\leqslant\mathcal{B}(M_2,N)\sideset{}{^*}\sum_{N(n)\sim N}|a_n|^2.$$

For the third sum on the right we have

$$\sideset{}{^*}\sum_{\substack{N(m)\sim M_1\\ p\nmid m}}\Big|\sideset{}{^*}\sum_{\substack{N(n)\sim N\\ p|n}}a_n\left[\frac{n}{m}\right]\Big|^2\leqslant\sideset{}{^*}\sum_{N(m)\sim M_1}\Big|\sideset{}{^*}\sum_{\substack{N(n)\sim N\\ p|n}}a_n\left[\frac{n}{m}\right]\Big|^2$$
$$\leqslant\mathcal{B}(M_1,N)\sideset{}{^*}\sum_{\substack{N(n)\sim N\\ p|n}}|a_n|^2.$$

Hence we obtain

$$\sideset{}{^*}\sum_{M_1\leqslant N(m)\leqslant\frac{3}{2}M_1}\Big|\sideset{}{^*}\sum_{N(n)\sim N}a_n\left[\frac{n}{m}\right]\Big|^2$$
$$\leqslant\sideset{}{^*}\sum_{\substack{N(m)\sim M_1\\ p|m}}\Big|\sideset{}{^*}\sum_{N(n)\sim N}a_n\left[\frac{n}{m}\right]\Big|^2+2\mathcal{B}(M_2,N)\sideset{}{^*}\sum_{N(n)\sim N}|a_n|^2$$
$$+2\mathcal{B}(M_1,N)\sideset{}{^*}\sum_{\substack{N(n)\sim N\\ p|n}}|a_n|^2.$$

We now sum over all odd primes with $N(p) \in (K, \frac{3}{2}K)$. By the prime ideal theorem we have

$$\begin{aligned}
&\frac{K}{\log K} \sideset{}{^*}\sum_{M_1 < N(m) \leqslant \frac{3}{2}M_1} \Big| \sideset{}{^*}\sum_{N(n)\sim N} a_n \left[\frac{n}{m}\right] \Big|^2 \\
&\ll \frac{\log(2M_1)}{\log K} \sideset{}{^*}\sum_{N(m)\sim M_1} \Big| \sideset{}{^*}\sum_{N(n)\sim N} a_n \left[\frac{n}{m}\right] \Big|^2 \\
&\quad + \frac{K}{\log K}\mathcal{B}(M_2, N) \sideset{}{^*}\sum_{N(n)\sim N} |a_n|^2 + \frac{\log(2N)}{\log K}\mathcal{B}(M_1, N) \sideset{}{^*}\sum_{N(n)\sim N} |a_n|^2 \\
&\ll \left\{ \frac{\log(2M_1 N)}{\log K}\mathcal{B}(M_1, N) + \frac{K}{\log K}\mathcal{B}(M_2, N) \right\} \sideset{}{^*}\sum_{N(n)\sim N} |a_n|^2. \qquad (3.1)
\end{aligned}$$

For the other terms for which $N(m) \in (\frac{3}{2}M_1, 2M_1]$, we consider an odd prime p with $\frac{2}{3}K < N(p) < K$, then a similar inequality as (3.1) holds. Combining these inequalities we deduce that there exists an absolute constant $C \geqslant 1$ such that

$$\mathcal{B}(M_1, N) \leqslant \frac{CM_1 \log(2M_1 N)}{2M_2}\mathcal{B}(M_1, N) + C\mathcal{B}(M_2, N).$$

Therefore if $M_2 \geqslant CM_1 \log(2M_1 N)$, we obtain

$$\mathcal{B}(M_1, N) \ll \mathcal{B}(M_2, N).$$

Moreover, by Lemma 3.1 we have

$$\mathcal{B}(M, N_1) \leqslant 2\mathcal{B}(N_1, M) \ll \mathcal{B}(N_2, M) \leqslant 2\mathcal{B}(M, N_2)$$

for $N_2 \geqslant CN_1 \log(2MN_1)$. The proof of Lemma 3.2 is complete. ■

The following result shows that the sum involving two different sequences is bounded in terms of $\mathcal{B}$.

Lemma 3.3. *Let C be as in Lemma 3.2. Let $D \geqslant \frac{1}{2}$ and $\{a_n\}, \{b_n\}$ be complex sequences supported on the odd squarefree integers in $(N, 2N]$. Then there exist D_1, D_2 satisfying*

$$\frac{1}{2C\log(2MN)} \leqslant D_i \leqslant \frac{2D}{C\log(2MN)}$$

for $i = 1, 2$ and

$$\frac{D}{2C^2\log^2(2MN)} \leqslant D_1 D_2 \leqslant \frac{D}{C^2\log^2(2MN)},$$

such that
$$S := \sum_{N(d)\sim D} \sideset{}{^*}\sum_{N(m)\sim M} \left| \sum\sum_{(n_1,n_2)=1,\ d|n_1n_2} a_{n_1} b_{n_2} \left[\frac{m}{n_1n_2}\right] \right|$$
$$\ll_\eta (MN)^\eta (D_1D_2)^{\frac{1}{2}} \left\{ \mathcal{B}(M, N/D_1) \sum |a_n|^2 \right\}^{\frac{1}{2}}$$
$$\times \left\{ \mathcal{B}(M, N/D_2) \sum |b_n|^2 \right\}^{\frac{1}{2}}$$
for any $\eta > 0$.

Proof. If $D > 4N^2$, then $S = 0$ and Lemma 3.3 holds. We now suppose that $D \leqslant 4N^2$. If we write $d_1 = (n_1, d)$ and $d_2 = dd_1^{-1}$, then
$$S \leqslant \frac{1}{4} \sum\sum_{N(d_1d_2)\sim D} \sideset{}{^*}\sum_{N(m)\sim M} \left| \sum\sum_{\substack{(n_1,n_2)=1 \\ d_1|n_1, d_2|n_2}} a_{n_1} b_{n_2} \left[\frac{m}{n_1n_2}\right] \right|$$
$$= \frac{1}{16} \sum\sum_{N(d_1d_2)\sim D} \sideset{}{^*}\sum_{N(m)\sim M} \left| \sum_{N(l)\leqslant 2N} \mu(l) \sum_{d_1|n_1, l|n_1} \sum_{d_2|n_2, l|n_2} a_{n_1} b_{n_2} \left[\frac{m}{n_1n_2}\right] \right|.$$
We decompose the sum over d_1 and d_2 into dyadic ranges $E_i < d_i \leqslant 2E_i$ with $\frac{1}{2}D \leqslant E_1E_2 \leqslant D$. Since there are $O(\log N)$ possible pairs E_1, E_2, we have
$$S \ll (\log N) \sum_{N(l)\leqslant 2N} \sum_{N(d_1)\sim E_1} \sum_{N(d_2)\sim E_2} \sideset{}{^*}\sum_{N(m)\sim M} \left| \sum_{d_1|n_1, l|n_1} \sum_{d_2|n_2, l|n_2} a_{n_1} b_{n_2} \left[\frac{m}{n_1n_2}\right] \right|$$
for some choice of E_1 and E_2. By Cauchy's inequality we obtain
$$S \ll (\log N) \Sigma_a^{\frac{1}{2}} \Sigma_b^{\frac{1}{2}},$$
where
$$\Sigma_a = \sum_{N(l)\leqslant 2N} \sum_{N(d_1)\sim E_1} \sum_{N(d_2)\sim E_2} \sideset{}{^*}\sum_{N(m)\sim M} \left| \sum_{d_1|n, l|n} a_n \left[\frac{m}{n}\right] \right|^2$$
and
$$\Sigma_b = \sum_{N(l)\leqslant 2N} \sum_{N(d_1)\sim E_1} \sum_{N(d_2)\sim E_2} \sideset{}{^*}\sum_{N(m)\sim M} \left| \sum_{d_2|n, l|n} b_n \left[\frac{m}{n}\right] \right|^2.$$
We treat Σ_a. Since
$$\sum_{d_1|n, l|n} a_n \left[\frac{m}{n}\right] = \left[\frac{m}{d_1}\right] \sum_{l|d_1k} a_{d_1k} \left[\frac{m}{k}\right],$$
we have
$$\sideset{}{^*}\sum_{N(m)\sim M} \left| \sum_{d_1|n, l|n} a_n \left[\frac{m}{n}\right] \right|^2 \leqslant \mathcal{B}(M, N/N(d_1)) \sum_{l|d_1k} |a_{d_1k}|^2$$
$$= \mathcal{B}(M, N/N(d_1)) \sum_{d_1|n, l|n} |a_n|^2.$$

Lemma 3.2 shows that $\mathcal{B}(M, N/N(d_1)) \ll \mathcal{B}(M, \theta N/E_1)$ for $\theta = C\log(2MN)$. Hence we have

$$\begin{aligned}\Sigma_a &\ll \sum_{N(l)\leqslant 2N}\ \sum_{N(d_1)\sim E_1}\ \sum_{N(d_2)\sim E_2} \mathcal{B}(M, \theta N/E_1) \sum_{d_1|n, l|n} |a_n|^2 \\ &\ll_\eta N^{\eta/2} E_2 \mathcal{B}(M, \theta N/E_1) \sum_{N(n)\sim N} |a_n|^2\end{aligned}$$

for any $\eta > 0$. Similarly, it follows that

$$\Sigma_b \ll_\eta N^{\eta/2} E_1 \mathcal{B}(M, \theta N/E_2) \sum_{N(n)\sim N} |a_n|^2.$$

Thus we obtain

$$S \ll_\eta N^{3\eta/4}(E_1E_2)^{\frac{1}{2}} \left\{\mathcal{B}(M, \theta N/E_1) \sum |a_n|^2\right\}^{\frac{1}{2}} \left\{\mathcal{B}(M, \theta N/E_2) \sum |b_n|^2\right\}^{\frac{1}{2}}.$$

If we take $D_i = E_i/\theta$, then D_1 and D_2 satisfy the conditions in Lemma 3.3 and

$$S \ll_\eta (MN)^\eta (D_1D_2)^{\frac{1}{2}} \left\{\mathcal{B}(M, N/D_1) \sum |a_n|^2\right\}^{\frac{1}{2}} \left\{\mathcal{B}(M, N/D_2) \sum |b_n|^2\right\}^{\frac{1}{2}}.$$

This completes the proof of Lemma 3.3. ■

Our next result is an application of the double Poisson summation formula.

Lemma 3.4. *Let $W(x)$ be an infinitely differentiable function supported on a compact subset of $(0, \infty)$. Let q be an odd squarefree integer in $\mathbb{Z}[i]$. Then we have*

$$\sum_{m\in\mathbb{Z}[i]} W\left(\frac{N(m)}{M}\right)\left[\frac{m}{q}\right] = \frac{M\tau(q)}{N(q)} \sum_{h\in\mathbb{Z}[i]} \hat{W}\left(\sqrt{\frac{M}{N(q)}}h\right)\left[\frac{\bar{h}}{q}\right],$$

where $\tau(q)$ is the Gaussian sum defined in the previous section, and we put

$$\hat{W}(z) = \int_{\mathbb{C}} W(|t|^2) e(-[z, t]) dt$$

with $t = t_1 + it_2$ and $dt = dt_1dt_2$.

Proof. We split the sum on the left into parts depending on the value of $m \pmod q$, and apply the double Poisson summation formula to each, so that

$$\sum_{m\in\mathbb{Z}[i]} W\left(\frac{N(m)}{M}\right)\left[\frac{m}{q}\right]$$
$$= \sum_{a \bmod q}\left[\frac{a}{q}\right]\sum_{b\in\mathbb{Z}[i]} W\left(\frac{N(a+bq)}{M}\right)$$
$$= \sum_{a \bmod q}\left[\frac{a}{q}\right]\sum\sum_{h_1,h_2\in\mathbb{Z}}\int_{\mathbb{R}^2} W\left(\frac{N(a+q(x_1+ix_2))}{M}\right)e(-h_1x_1-h_2x_2)dx_1dx_2$$
$$= \frac{M}{N(q)}\sum_{a \bmod q}\left[\frac{a}{q}\right]\sum_{h\in\mathbb{Z}[i]} e\left(\left[h,\frac{a}{q}\right]\right)\hat{W}\left(\sqrt{\frac{M}{N(q)}}h\right).$$

Thus Lemma 3.4 follows from the relation (2.4). ■

The next result will be used to separate the variables in the proof of Lemma 4.2.

Lemma 3.5. *Suppose that $\rho : \mathbb{R}^2 \longrightarrow \mathbb{C}$ be an infinitely differentiable function satisfying*

$$\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta}\rho(x,y) \ll_{\alpha,\beta,A} (1+x^2+y^2)^{-A}$$

for $(x,y)\in\mathbb{R}^2$ and any $A>0$. Let

$$\rho_{j,k}(s_1,s_2) = \int_0^\infty\int_0^\infty \rho(jx,ky)x^{s_1-1}y^{s_2-1}dxdy,$$

for $j,k\in\{-1,1\}$, and let

$$\rho_{l,0}(s) = \int_0^\infty \rho(lx,0)x^{s-1}dx, \qquad \rho_{0,l}(s) = \int_0^\infty \rho(0,ly)y^{s-1}dy$$

for $l\in\{-1,1\}$. Then $\rho_{j,k}(s_1,s_2)$ is holomorphic in $\mathrm{Re}(s_1)=\sigma_1>0$ and $\mathrm{Re}(s_2)=\sigma_2>0$. Moreover $\rho_{l,0}(s)$ and $\rho_{0,l}(s)$ are holomorphic in $\mathrm{Re}(s)=\sigma>0$. We have

$$\rho_{j,k}(s_1,s_2) \ll_{A,\sigma_1,\sigma_2} |s_1s_2|^{-A} \qquad \text{and} \qquad \rho_{l,0}(s),\rho_{0,l}(s) \ll_{A,\sigma} |s|^{-A}$$

for any $A>0$. Moreover if $\sigma_1,\sigma_2>0$,

$$\rho(jx,ky) = \frac{1}{(2\pi i)^2}\int_{(\sigma_2)}\int_{(\sigma_1)} \rho_{j,k}(s_1,s_2)x^{-s_1}y^{-s_2}ds_1ds_2$$

for $x,y>0$ and $j,k\in\{-1,1\}$, and if $\sigma>0$,

$$\rho(lx,0) = \frac{1}{2\pi i}\int_{(\sigma)}\rho_{l,0}(s)x^{-s}ds \qquad \text{and} \qquad \rho(0,ly) = \frac{1}{2\pi i}\int_{(\sigma)}\rho_{0,l}(s)y^{-s}ds$$

for $x,y>0$ and $l\in\{-1,1\}$.

The next result is also an application of the double Poisson summation formula which will be use in the proofs of Lemma 4.2 and 4.3.

Lemma 3.6. *Suppose that $\varphi : \mathbb{R}^2 \to \mathbb{C}$ be an infinitely differential function satisfying*

$$\frac{\partial^{\alpha+\beta}}{\partial u^\alpha \partial v^\beta}\varphi(u,v) \ll_{\alpha,\beta,A} (1+u^2+v^2)^{-A}$$

for $(u,v) \in \mathbb{R}^2$ and any $A > 0$. Put $\varphi(z) = \varphi(z_1, z_2)$ for $z = z_1 + iz_2$ with $z_1, z_2 \in \mathbb{R}$. Let x be a nonzero complex number and put $X = |x|^2$. Let $0 < X_1 \leqslant X \leqslant X_2$ and $0 < J \leqslant \min\{X/X_1, X_2/X\}$. Then for any $k \in \mathbb{Z}[i]$ with $(k) \neq (0), (1)$, we have

$$\begin{aligned}\sum_{\substack{n\in\mathbb{Z}[i]\\(n,k)=1}} \varphi\left(\frac{n}{x}\right) = {} & \frac{\Phi(k)}{N(k)} X \int_{\mathbb{C}} \varphi(x)dx - \frac{1}{4}\varphi(0) \sum_{\substack{N(d)\leqslant X_2\\ d|k}} \mu(d) \\ & - \frac{1}{4} \sum_{\substack{N(d)>X_2\\ d|k}} \mu(d) \frac{X}{N(d)} \int_{\mathbb{C}} \varphi(x)dx \\ & + \frac{1}{4} \sum_{0\neq N(l)\leqslant L} \sum_{\substack{X_1<N(d)\leqslant X_2\\ d|k}} \mu(d)\frac{X}{N(d)} \hat{\varphi}\left(\frac{\bar{x}l}{\bar{d}}\right) \\ & + O_A(XJ^{-A}) \qquad (3.2)\end{aligned}$$

for any $A > 0$ and $L \geqslant (X_2/X)^2$, where $\Phi(k)$ is the Euler function and

$$\hat{\varphi}(a) = \int_{\mathbb{C}} \varphi(t) e(-[a,t])dt.$$

Proof. We remove the condition $(n, k) = 1$ by using the sum of the Möbius function to obtain

$$\sum_{(n,k)=1} \varphi\left(\frac{n}{x}\right) = \frac{1}{4} \sum_{d|k} \mu(d) \sum_{n\in\mathbb{Z}[i]} \varphi\left(\frac{nd}{x}\right).$$

First we consider the case in which $N(d) > X_2$. Since $(k) \neq (0), (1)$, we have

$$\frac{1}{4} \sum_{N(d)>X_2, d|k} \mu(d)\varphi(0) = -\frac{1}{4}\varphi(0) \sum_{N(d)\leqslant X_2, d|k} \mu(d).$$

The assumption for φ shows that

$$\begin{aligned}\frac{1}{4} \sum_{N(d)>X_2, d|k} \mu(d) \sum_{n\neq 0} \varphi\left(\frac{nd}{x}\right) & \ll_A \sum_{N(d)>X_2, d|k} \sum_{n\neq 0} \left(\frac{N(nd)}{N(x)}\right)^{-A} \\ & \ll_A X^A X_2^{1-A} = X(X_2/X)^{-A} \leqslant XJ^{-A}.\end{aligned}$$

Next we deal with the case in which $N(d) \leqslant X_2$. Using the double Poisson summation formula, we obtain

$$\frac{1}{4} \sum_{N(d)\leqslant X_2, d|k} \mu(d) \sum_{n\in\mathbb{Z}[i]} \varphi\left(\frac{nd}{x}\right) = \frac{1}{4} \sum_{N(d)\leqslant X_2, d|k} \mu(d) \frac{X}{N(d)} \sum_{l\in\mathbb{Z}[i]} \hat{\varphi}\left(\frac{\bar{x}l}{\bar{d}}\right).$$

We compute the terms with $l=0$ to be

$$\begin{aligned}
&\frac{1}{4}\sum_{N(d)\leqslant X_2, d|k}\mu(d)\frac{X}{N(d)}\hat{\varphi}(0)\\
&\qquad=\frac{1}{4}\sum_{d|k}\mu(d)\frac{X}{N(d)}\hat{\varphi}(0)-\frac{1}{4}\sum_{N(d)>X_2, d|k}\mu(d)\frac{X}{N(d)}\hat{\varphi}(0)\\
&\qquad=\frac{\Phi(k)}{N(k)}X\int_{\mathbb{C}}\varphi(x)dx-\frac{1}{4}\sum_{N(d)>X_2, d|k}\mu(d)\frac{X}{N(d)}\int_{\mathbb{C}}\varphi(x)dx.
\end{aligned}$$

For the remaining terms we have

$$\begin{aligned}
\sum_{N(d)\leqslant X_1, d|k}\sum_{l\neq 0}\mu(d)\frac{X}{N(d)}\hat{\varphi}\left(\frac{\bar{x}l}{\bar{d}}\right) &\ll_A \sum_{N(d)\leqslant X_1, d|k}\sum_{l\neq 0}\frac{X}{N(d)}\left(\frac{N(x)N(l)}{N(d)}\right)^{-A}\\
&\ll_A X^{1-A}\sum_{N(d)\leqslant X_1, d|k}N(d)^{A-1}\\
&\ll_A X^{1-A}X_1^A = X(X/X_1)^{-A}\leqslant XJ^{-A}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{X_1<N(d)\leqslant X_2, d|k}\sum_{N(l)>L}\mu(d)\frac{X}{N(d)}\hat{\varphi}\left(\frac{\bar{x}l}{\bar{d}}\right)\\
&\qquad\ll_A \sum_{X_1<N(d)\leqslant X_2, d|k}\sum_{N(l)>L}\frac{X}{N(d)}\left(\frac{XN(l)}{N(d)}\right)^{-A-2}\\
&\qquad\ll_A X^{-A-1}\sum_{X_1<N(d)\leqslant X_2, d|k}N(d)^{A+1}\sum_{N(l)>L}N(l)^{-A-2}\\
&\qquad\ll_A X^{-A-1}X_2^{A+2}L^{-A-1}\leqslant X^{A+1}X_2^{-A}=X(X_2/X)^{-A}\leqslant XJ^{-A}.
\end{aligned}$$

We combine the above results to obtain the formula (3.2). ■

Lemma 3.7. *Let W and $\hat{W}$ be as in Lemma 3.4. Then we have*

$$\int_{\mathbb{C}}\hat{W}(z^2)dz=\pi\int_0^\infty W(r^2)dr.$$

To prove this lemma, we first investigate a certain integral. Let α be a complex number and K be a positive real number. Set $D(K)=\{z\in\mathbb{C}:|z|\leqslant K\}$ and

$$I_\alpha(K)=\int_{D(K)}e([1,\alpha z^2])dz.$$

Lemma 3.8. *For a nonzero complex number α,*

$$I_\alpha(K)=\frac{1}{2|\alpha|}+O\left(\frac{1}{\sqrt{|\alpha|^3}K}\right) \tag{3.3}$$

where the implied constant is absolute.

Proof. We use the following properties of the J-Bessel function $J_0(z)$:

$$J_0(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iz\cos\theta} d\theta \qquad \text{(cf. [6, §2.2 (5)])}, \tag{3.4}$$

$$J_0(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4}\right) + O(|z|^{-\frac{3}{2}}) \tag{3.5}$$

when $|z|$ is large and $|\arg z| < \pi$ (cf. [6, §7.21 (1)]), and

$$\int_0^\infty J_0(t)dt = 1 \qquad \text{(cf. [6, §13.42 (7)])}. \tag{3.6}$$

By (3.4) and (3.6) we have

$$\begin{aligned} I_\alpha(K) &= \int_0^K \int_{-\pi}^{\pi} r e^{2\pi i|\alpha|r^2\cos\theta} d\theta dr \\ &= 2\pi \int_0^K rJ_0(2\pi|\alpha|r^2)dr \\ &= \frac{1}{2|\alpha|} + \frac{1}{2|\alpha|}\int_{2\pi|\alpha|K^2}^\infty J_0(t)dt. \end{aligned}$$

The asymptotic formula (3.5) shows that the second term on the right is $O(|\alpha|^{-\frac{3}{2}}K^{-1})$, which gives (3.3). ■

Proof of Lemma 3.7. Fix positive real numbers a and b such that the function $W(|t|^2)$ is supported in $D(a,b) := D(b) \setminus D(a)$. Then

$$\begin{aligned} \int_{\mathbb{C}} \hat{W}(z^2)dz &= \lim_{K\to\infty} \int_{D(K)} \left(\int_{D(a,b)} W(|t|^2) e(-[z^2, t])dt \right) dz \\ &= \lim_{K\to\infty} \int_{D(a,b)} W(|t|^2) I_t(K)dt \end{aligned}$$

by Fubini's Theorem. Hence the asymptotic formula (3.3) shows

$$\int_{\mathbb{C}} \hat{W}(z^2)dz = \frac{1}{2}\int_{D(a,b)} W(|t|^2)\frac{dt}{|t|} = \pi \int_0^\infty W(r^2)dr.$$ ■

4. Proof of Theorem 1

Let us consider the complex sequence $\{a_n\}$ where n runs over odd squarefree integers in $\mathbb{Z}[i]$ satisfying that $N(n) \sim N$. We assume $\sum |a_n|^2 \neq 0$. In this section we aim to prove that

$$\mathcal{B}(M,N) \ll_\varepsilon (MN)^\varepsilon (M+N) \tag{4.1}$$

for $\varepsilon > 0$, which is equivalent to Theorem 1. We introduce a weight function $W(x)$ given by

$$W(x) = \begin{cases} \exp\left(-\frac{1}{(2x-1)(2x-5)}\right) & \text{if } \frac{1}{2} < x < \frac{5}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Set

$$s(m) = \begin{cases} N(k) & \text{if } m \neq 0 \text{ and } m = kl^2 \text{ with } k \text{ squarefree}, \\ 0 & \text{if } m = 0, \end{cases}$$

and

$$\mathcal{B}(M,N,K) = \sup_{\{a_n\}} \left(\sum_{s(m)>K}^{\text{odd}} W\left(\frac{N(m)}{M}\right) \Big| \sum_{N(n)\sim N} a_n \left[\frac{m}{n}\right] \Big|^2 \right) \Big/ \sum_{N(n)\sim N} |a_n|^2,$$

where $\sum^{\text{odd}}$ means that the range of summation is restricted to odd integers. Then Lemma 3.1 tells us

$$\mathcal{B}(M,N) \ll \mathcal{B}(M,N,K)$$

for $K \leqslant M/2$. We give an estimate of $\mathcal{B}(M,N,K)$ as follows.

Lemma 4.1. *Let $\varepsilon > 0$ be given. Let $0 \leqslant K \leqslant M/2$ and $1 \leqslant D_0 \leqslant N$. For $\Delta \in \mathbb{Z}[i]$ we set*

$$\mathcal{C}(M,N,K,\Delta) = \sup_{\{a_n\}} \Sigma_2 \Big/ \sum_{N(n)\sim N} |a_n|^2$$

with

$$\Sigma_2 = \sum\sum_{(n_1,n_2)=\Delta} a_{n_1}\overline{a_{n_2}} \sum_{s(m)>K}^{\text{odd}} W\left(\frac{N(m)}{M}\right)\left[\frac{m}{n_1 n_2}\right].$$

Then we have

$$\mathcal{B}(M,N,K) \ll_\varepsilon N^\varepsilon \mathcal{B}(M,N_1,K) + \sum_{N(\Delta)\leqslant D_0} \mathcal{C}(M,N,K,\Delta) \tag{4.2}$$

for some positive real number $N_1 \leqslant N/D_0$.

Proof. We transform $\sum_{s(m)>K}^{\text{odd}} W\left(\frac{N(m)}{M}\right) |\sum_{N(n)\sim N} a_n \left[\frac{m}{n}\right]|^2$ into

$$\frac{1}{4} \sum_{N(\Delta)\neq 0} \sum_{s(m)>K}^{\text{odd}} W\left(\frac{N(m)}{M}\right) \sum\sum_{(n_1,n_2)=\Delta} a_{n_1}\overline{a_{n_2}} \left[\frac{m}{n_1 n_2}\right].$$

The part in which $N(\Delta) \leqslant D_0$ are bounded by

$$\sum_{N(\Delta)\leqslant D_0} \mathcal{C}(M,N,K,\Delta) \sum_{N(n)\sim N} |a_n|^2.$$

For each Δ with $N(\Delta) > D_0$ we have

$$\begin{aligned}
&\Bigg| \sum_{s(m)>K}^{\text{odd}} W\left(\frac{N(m)}{M}\right) \mathop{\sum\sum}_{(n_1,n_2)=\Delta} a_{n_1}\overline{a_{n_2}} \left[\frac{m}{n_1 n_2}\right] \Bigg| \\
&\qquad \leqslant \sum_{s(m)>K}^{\text{odd}} W\left(\frac{N(m)}{M}\right) \Bigg| \mathop{\sum\sum}_{(n_1,n_2)=\Delta} a_{n_1}\overline{a_{n_2}} \left[\frac{m}{n_1 n_2}\right] \Bigg| \\
&\qquad = \sum_{\substack{s(m)>K \\ (m,\Delta)=1}}^{\text{odd}} W\left(\frac{N(m)}{M}\right) \Bigg| \mathop{\sum\sum}_{(n_1,n_2)=1} a_{n_1\Delta}\overline{a_{n_2\Delta}} \left[\frac{m}{n_1 n_2}\right] \Bigg| \\
&\qquad = \frac{1}{4} \sum_{\substack{s(m)>K \\ (m,\Delta)=1}}^{\text{odd}} W\left(\frac{N(m)}{M}\right) \Bigg| \mathop{\sum\sum}_{n_1,n_2} a_{n_1\Delta}\overline{a_{n_2\Delta}} \left[\frac{m}{n_1 n_2}\right] \sum_{d|(n_1,n_2)} \mu(d) \Bigg| \\
&\qquad \leqslant \frac{1}{4} \sum_{N(d)\leqslant 2N} \sum_{s(m)>K}^{\text{odd}} W\left(\frac{N(m)}{M}\right) \Bigg| \sum_{d|n} a_{n\Delta} \left[\frac{m}{n}\right] \Bigg|^2 \\
&\qquad \leqslant \frac{1}{4} \sum_{N(d)\leqslant 2N} \mathcal{B}(M, N/N(\Delta), K) \sum_{d|n} |a_{n\Delta}|^2 \\
&\qquad \leqslant \frac{1}{4} \mathcal{B}(M, N/N(\Delta), K) \sum_{n} d(n) |a_{n\Delta}|^2 .
\end{aligned}$$

We note that there exists a positive real number $N_1 \leqslant N/D_0$ such that

$$\mathcal{B}(M, N/N(\Delta), K) \leqslant \mathcal{B}(M, N_1, K)$$

for any Δ with $N(\Delta) > D_0$. Hence we obtain that

$$\begin{aligned}
\Bigg| \sum_{N(\Delta)>D_0} \sum_{s(m)>K}^{\text{odd}} \cdots \Bigg| &\leqslant \frac{1}{4} \mathcal{B}(M, N_1, K) \sum_{N(\Delta)>D_0} \sum_{n} d(n) |a_{n\Delta}|^2 \\
&= \frac{1}{4} \mathcal{B}(M, N_1, K) \sum_{N(\Delta)>D_0} \sum_{\Delta|n} d\left(\frac{n}{\Delta}\right) |a_n|^2 \\
&\leqslant \frac{1}{4} \mathcal{B}(M, N_1, K) \sum_{n} d(n)^2 |a_n|^2 \\
&\ll_\varepsilon N^\varepsilon \mathcal{B}(M, N_1, K) \sum_{n} |a_n|^2 ,
\end{aligned}$$

and the lemma follows. ■

We decompose $\Sigma_2 = \Sigma_3 - \Sigma_4$ with

$$\Sigma_3 = \mathop{\sum\sum}_{(n_1,n_2)=\Delta} a_{n_1}\overline{a_{n_2}} \sum_{m\in\mathbb{Z}[i]}^{\text{odd}} W\left(\frac{N(m)}{M}\right) \left[\frac{m}{n_1 n_2}\right] \tag{4.3}$$

and

$$\Sigma_4 = \sum_{(n_1,n_2)=\Delta}\sum a_{n_1}\overline{a_{n_2}} \sum_{s(m)\leqslant K}^{\text{odd}} W\left(\frac{N(m)}{M}\right)\left[\frac{m}{n_1 n_2}\right]. \tag{4.4}$$

For Σ_3 we prove the following result.

Lemma 4.2. *Let $\varepsilon > 0$ be given. Let $N(\Delta) < N$ and*

$$N^2 M^{-1}(MN)^{\varepsilon} \leqslant K \leqslant M(MN)^{-\varepsilon}.$$

For any Gaussian integers n_1, n_2 we write

$$q = q(n_1, n_2) := \frac{n_1 n_2}{(n_1, n_2)^2}.$$

Moreover we put

$$\kappa(\Delta, q) = \frac{1}{4}\sum_{e|\Delta} \frac{\mu(e)}{\sqrt{N(e)}}\left[\frac{e}{q}\right]$$

and

$$m_0 = \min\left\{1, \frac{N/\sqrt{MB}}{D_1 D_2}\right\}.$$

Then we have

$$\Sigma_3 = M_3 + E_3 \sum_n |a_n|^2,$$

where

$$M_3 = \frac{\pi}{4}\sum_{N(b)\leqslant K}^{*} \sqrt{\frac{M}{N(b)}} \sum_{(n_1,n_2)=\Delta} a_{n_1}\overline{a_{n_2}} \frac{\Phi(q)}{2N(q)}\kappa(\Delta, q)\left[\frac{b}{q}\right]\int_0^{\infty} W(r^2)dr$$

and

$$E_3 \ll_{\varepsilon} 1 + (MN)^{\varepsilon} N(\Delta)\frac{M}{N} m_0 \sqrt{D_1 D_2}\,\mathcal{B}\Big(B, \frac{N}{D_1 N(\Delta)}\Big)^{\frac{1}{2}} \mathcal{B}\Big(B, \frac{N}{D_2 N(\Delta)}\Big)^{\frac{1}{2}}$$

for certain values of B, D_1 and D_2 with $D_1, D_2 \gg (MN)^{-\varepsilon}$ and $1 \ll B \ll K$.

Proof. Put $\mu = 1 + i$. The inner sum of Σ_3 becomes

$$\begin{aligned}\sum_{m\in\mathbb{Z}[i]}^{\text{odd}} W\left(\frac{N(m)}{M}\right)\left[\frac{m}{n_1 n_2}\right] &= \frac{1}{4}\sum_{e|\mu\Delta}\mu(e)\left[\frac{e}{q}\right]\sum_{m\in\mathbb{Z}[i]} W\left(\frac{N(m)}{M/N(e)}\right)\left[\frac{m}{q}\right] \\ &= \frac{1}{4}\sum_{e|\mu\Delta}\mu(e)\left[\frac{e}{q}\right]\frac{M\tau(q)}{N(eq)}\sum_{h\in\mathbb{Z}[i]}\hat{W}\left(\sqrt{\frac{M}{N(eq)}}h\right)\left[\frac{\bar{h}}{q}\right]\end{aligned}$$

by Lemma 3.4. Since $N(q) \geqslant N^2 N(\Delta)^{-2} > 1$, we observe that $[\frac{\bar{h}}{q}] = 0$ when $h = 0$. If $h \neq 0$, we can factorize $h = abc^2$, where $a = 1$ or $1 + i$, and b is an odd squarefree integer. Then we have

$$\Sigma_3 = \frac{M}{16} \sum_{e|\mu\Delta} \frac{\mu(e)}{N(e)} \sum_a \sideset{}{^*}\sum_b \sum_{c\neq 0} \times \sum_{(n_1,n_2)=1} a_{n_1\Delta}\overline{a_{n_2\Delta}} \frac{\tau(n_1n_2)}{N(n_1n_2)} \hat{W}\left(\sqrt{\frac{M}{N(en_1n_2)}}abc^2\right) \left[\frac{e\overline{abc^2}}{n_1n_2}\right].$$

For the terms with $N(b) > K$ the contribution is $O_\varepsilon(\sum |a_n|^2)$, since $\hat{W}(x) \ll_A |x|^{-A}$ and $K \geqslant N^2 M^{-1}(MN)^\varepsilon$. We next consider the terms with $N(b) \leqslant K$. We split the sum over b into subintervals $(B, 2B]$ with $B = 2^{-k}K$ for $k = 1, 2, \ldots, [\frac{\log K}{\log 2}] + 1$. Then the contribution for each range becomes

$$\Sigma_{3,B} = \frac{M}{16} \sum_{e|\mu\Delta} \frac{\mu(e)}{N(e)} \sum_a \sideset{}{^*}\sum_{N(b)\sim B} \sum_{(n_1,n_2)=1} a_{n_1\Delta}\overline{a_{n_2\Delta}} \frac{\tau(n_1n_2)}{N(n_1n_2)} \left[\frac{e\overline{ab}}{n_1n_2}\right] S(\overline{n_1n_2}),$$

where

$$S(\alpha) = \sum_{(c,\alpha)=1} \hat{W}\left(\sqrt{\frac{M}{N(en_1n_2)}}abc^2\right)$$

for $\alpha \in \mathbb{Z}[i]$ with $(\alpha) \neq (0), (1)$. We apply Lemma 3.6 to $S(\alpha)$ by taking

$$\varphi(z) = \hat{W}(z^2), \quad x = \frac{N(en_1n_2)^{\frac{1}{4}}}{M^{\frac{1}{4}}(ab)^{\frac{1}{2}}}, \quad X = \sqrt{\frac{N(en_1n_2)}{MN(ab)}},$$
$$X_1 = (MN)^{-\frac{\eta}{2}} \frac{\sqrt{N(e)}}{N(\Delta)} N(MB)^{-\frac{1}{2}}, \quad X_2 = (MN)^{\eta} \frac{\sqrt{N(e)}}{N(\Delta)} N(MB)^{-\frac{1}{2}},$$
$$J = (MN)^{\frac{\eta}{4}} \quad \text{and} \quad L = (MN)^{3\eta} \quad (\eta > 0),$$

so that we have

$$S(\alpha) = \frac{\Phi(\alpha)}{N(\alpha)} X \int_{\mathbb{C}} \hat{W}(z^2)dz - \frac{1}{4}\hat{W}(0) \sum_{d|\alpha, N(d)\leqslant X_2} \mu(d) - \frac{1}{4} \sum_{d|\alpha, N(d)>X_2} \mu(d) \frac{X}{N(d)} \int_{\mathbb{C}} \hat{W}(z^2)dz + \frac{1}{4} \sum_{0<N(l)\leqslant L} \sum_{d|\alpha, X_1<N(d)\leqslant X_2} \mu(d) \frac{X}{N(d)} \hat{\varphi}\left(\frac{\bar{x}l}{\bar{d}}\right) + O_A(XJ^{-A}). \tag{4.5}$$

We sum the first part on the right of (4.5) over B and apply Lemma 3.7, then the corresponding contribution to Σ_3 is

$$S_1 = \frac{\pi\sqrt{M}}{16}\int_0^\infty W(r^2)dr \sideset{}{^*}\sum_{N(b)\leqslant K}\sum_{(n_1,n_2)=1} a_{n_1\Delta}\overline{a_{n_2\Delta}}\frac{\tau(n_1n_2)\Phi(n_1n_2)}{N(b)^{1/2}N(n_1n_2)^{3/2}}$$
$$\times\left[\frac{b}{n_1n_2}\right]\sum_{e|\mu\Delta}\sum_{a=1,\mu}\frac{\mu(e)}{\sqrt{N(ea)}}\left[\frac{e\bar{a}}{n_1n_2}\right]$$
$$= \frac{\pi\sqrt{M}}{16}\int_0^\infty W(r^2)dr \sideset{}{^*}\sum_{\substack{b\in\mathcal{F}\\ N(b)\leqslant K}}\sum_{(n_1,n_2)=1} a_{n_1\Delta}\overline{a_{n_2\Delta}}\frac{\tau(n_1n_2)\Phi(n_1n_2)}{N(b)^{1/2}N(n_1n_2)^{3/2}}$$
$$\times\left[\frac{b}{n_1n_2}\right]2\left(1+\left[\frac{i}{n_1n_2}\right]\right)\sum_{e|\mu\Delta}\sum_{a=1,\mu}\frac{\mu(e)}{\sqrt{N(ea)}}\left[\frac{e\bar{a}}{n_1n_2}\right],$$

where $\mathcal{F}=\{z\in\mathbb{C}:\mathrm{Re}(z)>0,\mathrm{Im}(z)\geqslant 0\}$. We note that $\tau(n_1n_2)=N(n_1n_2)^{\frac{1}{2}}$ for $[\frac{i}{n_1n_2}]=1$ and

$$2\left(1+\left[\frac{i}{n_1n_2}\right]\right)\sum_{e|\mu\Delta}\sum_{a=1,\mu}\frac{\mu(e)}{\sqrt{N(ea)}}\left[\frac{e\bar{a}}{n_1n_2}\right]=4\left(1+\left[\frac{i}{n_1n_2}\right]\right)\kappa(\Delta,n_1n_2).$$

These show that

$$S_1=\frac{\pi}{4}\sideset{}{^*}\sum_{N(b)\leqslant K}\sqrt{\frac{M}{N(b)}}\sum_{(n_1,n_2)=1}a_{n_1\Delta}\overline{a_{n_2\Delta}}\frac{\Phi(n_1n_2)}{2N(n_1n_2)}$$
$$\times\kappa(\Delta,n_1n_2)\left[\frac{b}{n_1n_2}\right]\int_0^\infty W(r^2)dr=M_3.$$

Let S_2 denote the contribution for the second part on the right of (4.5) to $\Sigma_{3,B}$. Then we have

$$S_2\ll_\eta MN(\Delta)^\eta\sum_{N(d)\leqslant X_2}\sideset{}{^*}\sum_{N(b)\sim B}\left|\sum\sum_{\substack{(n_1,n_2)=1\\ d|n_1n_2}}a_{n_1\Delta}\overline{a_{n_2\Delta}}\frac{\tau(n_1n_2)}{N(n_1n_2)}\left[\frac{e\bar{a}b}{n_1n_2}\right]\right|$$
$$=MN(\Delta)^\eta\sum_{N(d)\leqslant X_2}\sideset{}{^*}\sum_{N(b)\sim B}$$
$$\times\left|\sum\sum_{\substack{(n_1,n_2)=1\\ d|n_1n_2}}a_{n_1\Delta}\frac{\tau(n_1)}{N(n_1)}\left[\frac{e\bar{a}}{n_1}\right]\overline{a_{n_2\Delta}}\frac{\tau(n_2)}{N(n_2)}\left[\frac{e\bar{a}}{n_2}\right]\left[\frac{n_1}{n_2}\right]\left[\frac{n_2}{n_1}\right]\left[\frac{b}{n_1n_2}\right]\right|$$

for some e and a with $1\leqslant N(e)\leqslant 2N(\Delta)$. We take $\Lambda=\{(1,1),(1,i),(i,1),(i,i)\}$ and write $\lambda=(\lambda_1,\lambda_2)$ for $\lambda\in\Lambda$. Then we obtain

$$S_2\ll_\eta MN(\Delta)^\eta\sum_{\lambda\in\Lambda}\sum_{N(d)\leqslant X_2}\sideset{}{^*}\sum_{N(b)\sim B}\left|\sum\sum_{\substack{(n_1,n_2)=1\\ d|n_1n_2}}a_{n_1}^{(1,\lambda)}\overline{b_{n_2}^{(1,\lambda)}}\left[\frac{b}{n_1n_2}\right]\right|$$

where

$$a_n^{(1,\lambda)} = \begin{cases} a_{n\Delta} \frac{\tau(n)}{N(n)} \left[\frac{e\bar{a}}{n}\right] & \text{if } n \equiv \lambda_1 \bmod 2\mathbb{Z}[i] \text{ and } \lambda_2 = 1, \\ a_{n\Delta} \frac{\tau(n)}{N(n)} \left[\frac{e\bar{a}}{n}\right] \left[\frac{i}{n}\right] & \text{if } n \equiv \lambda_1 \bmod 2\mathbb{Z}[i] \text{ and } \lambda_2 = i, \\ 0 & \text{otherwise,} \end{cases}$$

$$b_n^{(1,\lambda)} = \begin{cases} a_{n\Delta} \frac{\tau(n)}{N(n)} \left[\frac{e\bar{a}}{n}\right] & \text{if } n \equiv \lambda_2 \bmod 2\mathbb{Z}[i] \text{ and } \lambda_1 = 1, \\ a_{n\Delta} \frac{\tau(n)}{N(n)} \left[\frac{e\bar{a}}{n}\right] \left[\frac{i}{n}\right] & \text{if } n \equiv \lambda_2 \bmod 2\mathbb{Z}[i] \text{ and } \lambda_1 = i, \\ 0 & \text{otherwise.} \end{cases}$$

We decompose the sum over d into subintervals $(D, 2D]$ with $D = 2^{-k}X_2$ and apply Lemma 3.3 to each. Then we have

$$S_2 \ll_\eta (MN)^{3\eta} \frac{M}{N} N(\Delta)(D_1D_2)^{\frac{1}{2}} \mathcal{B}\Big(B, \frac{N}{D_1N(\Delta)}\Big)^{\frac{1}{2}} \mathcal{B}\Big(B, \frac{N}{D_2N(\Delta)}\Big)^{\frac{1}{2}} \sum_n |a_n|^2$$

for some values of D_1 and D_2. There are $O(\log K) = O((MN)^\eta)$ ranges for B, so that the total contribution to Σ_3 is

$$\ll_\eta (MN)^{4\eta} \frac{M}{N} N(\Delta)\sqrt{D_1D_2} \mathcal{B}\Big(B, \frac{N}{D_1N(\Delta)}\Big)^{\frac{1}{2}} \mathcal{B}\Big(B, \frac{N}{D_2N(\Delta)}\Big)^{\frac{1}{2}} \sum_n |a_n|^2$$

for certain values of D_1, D_2 and B. Since

$$D_1D_2 \ll X_2 = (MN)^\eta \frac{\sqrt{N(e)}}{N(\Delta)} N(MB)^{-\frac{1}{2}} \ll (MN)^\eta N(MB)^{-\frac{1}{2}},$$

we obtain that $1 \ll (MN)^\eta m_0$. Thus the contribution to Σ_3 is

$$\ll_\eta (MN)^{5\eta} N(\Delta) \frac{M}{N} m_0 \sqrt{D_1D_2} \mathcal{B}\Big(B, \frac{N}{D_1N(\Delta)}\Big)^{\frac{1}{2}} \mathcal{B}\Big(B, \frac{N}{D_2N(\Delta)}\Big)^{\frac{1}{2}} \sum_n |a_n|^2.$$

Let us put S_3 to be the contribution for the third part on the right of (4.5) to $\Sigma_{3,B}$. Then we obtain

$$S_3 \ll_\eta \frac{M^{\frac{1}{2}}}{B^{\frac{1}{2}}} N(\Delta)^\eta \sum_{N(d)>X_2} \frac{1}{N(d)} \sideset{}{^*}\sum_{N(b)\sim B} \Bigg| \sum_{\substack{(n_1,n_2)=1 \\ d|n_1n_2}}\sum a_{n_1\Delta}\overline{a_{n_2\Delta}} \frac{\tau(n_1n_2)}{\sqrt{N(n_1n_2)}} \left[\frac{e\bar{a}b}{n_1n_2}\right] \Bigg|$$

for some e and a with $1 \leqslant N(e) \leqslant 2N(\Delta)$. We decompose the range for d into dyadic ranges $(D, 2D]$ with $X_2 \leqslant D \leqslant 2N^2$. Then

$$S_3 \ll_\eta \frac{M^{\frac{1}{2}}}{B^{\frac{1}{2}}} N(\Delta)^\eta \sum_{\lambda\in\Lambda}\sum_D \frac{1}{D} \sum_{N(d)\sim D} \sideset{}{^*}\sum_{N(b)\sim B} \Bigg| \sum_{\substack{(n_1,n_2)=1 \\ d|n_1n_2}}\sum a_{n_1}^{(2,\lambda)} \overline{b_{n_2}^{(2,\lambda)}} \left[\frac{b}{n_1n_2}\right] \Bigg|,$$

where

$$a_n^{(2,\lambda)} = \begin{cases} a_{n\Delta}\dfrac{\tau(n)}{\sqrt{N(n)}}\left[\frac{e\bar{a}}{n}\right] & \text{if } n \equiv \lambda_1 \bmod 2\mathbb{Z}[i] \text{ and } \lambda_2 = 1, \\ a_{n\Delta}\dfrac{\tau(n)}{\sqrt{N(n)}}\left[\frac{e\bar{a}}{n}\right]\left[\frac{i}{n}\right] & \text{if } n \equiv \lambda_1 \bmod 2\mathbb{Z}[i] \text{ and } \lambda_2 = i, \\ 0 & \text{otherwise}, \end{cases}$$

$$b_n^{(2,\lambda)} = \begin{cases} a_{n\Delta}\dfrac{\tau(n)}{\sqrt{N(n)}}\left[\frac{e\bar{a}}{n}\right] & \text{if } n \equiv \lambda_2 \bmod 2\mathbb{Z}[i] \text{ and } \lambda_1 = 1, \\ a_{n\Delta}\dfrac{\tau(n)}{\sqrt{N(n)}}\left[\frac{e\bar{a}}{n}\right]\left[\frac{i}{n}\right] & \text{if } n \equiv \lambda_2 \bmod 2\mathbb{Z}[i] \text{ and } \lambda_1 = i, \\ 0 & \text{otherwise}. \end{cases}$$

Applying Lemma 3.3 to each, we have

$$S_3 \ll_\eta (MN)^{2\eta}\frac{M^{\frac12}}{B^{\frac12}}\sum_D \frac{\sqrt{D_1D_2}}{D}\mathcal{B}\Big(B, \frac{N}{D_1N(\Delta)}\Big)^{\frac12}\mathcal{B}\Big(B, \frac{N}{D_2N(\Delta)}\Big)^{\frac12}\sum_n |a_n|^2.$$

Since

$$\frac{1}{D} \ll \frac{1}{D_1D_2\log^2(2MN)} \ll \frac{1}{D_1D_2},$$

we obtain

$$S_3 \ll_\eta (MN)^{2\eta}\log N\sqrt{\frac{M}{BD_1D_2}}\mathcal{B}\Big(B, \frac{N}{D_1N(\Delta)}\Big)^{\frac12}\mathcal{B}\Big(B, \frac{N}{D_2N(\Delta)}\Big)^{\frac12}\sum_n |a_n|^2.$$

Hence we see that the total contribution to Σ_3 is

$$\ll_\eta (MN)^{3\eta}\sqrt{\frac{M}{BD_1D_2}}\mathcal{B}\Big(B, \frac{N}{D_1N(\Delta)}\Big)^{\frac12}\mathcal{B}\Big(B, \frac{N}{D_2N(\Delta)}\Big)^{\frac12}\sum_n |a_n|^2$$

for some B with $\frac12 \leqslant B \leqslant K$. Since

$$D_1D_2 \gg \frac{X_2}{\log^2(2MN)} = \frac{(MN)^\eta\sqrt{N(e)}N}{N(\Delta)(MB)^{\frac12}\log^2(2MN)} \gg_\eta NN(\Delta)^{-1}(MB)^{-\frac12},$$

we have

$$\begin{aligned}\sqrt{\frac{M}{BD_1D_2}} &\ll \frac{M}{N}\sqrt{D_1D_2}\min\Big\{N(\Delta), \frac{N/\sqrt{MB}}{D_1D_2}\Big\} \\ &\ll N(\Delta)\frac{M}{N}\sqrt{D_1D_2}\min\Big\{1, \frac{N/\sqrt{MB}}{D_1D_2}\Big\}.\end{aligned}$$

Thus the contribution to Σ_3 becomes

$$\ll_\eta (MN)^{3\eta}N(\Delta)\frac{M}{N}m_0\sqrt{D_1D_2}\mathcal{B}\Big(B, \frac{N}{D_1N(\Delta)}\Big)^{\frac12}\mathcal{B}\Big(B, \frac{N}{D_2N(\Delta)}\Big)^{\frac12}\sum_n |a_n|^2.$$

Let S_4 denote the contribution for the forth part on the right of (4.5) to $\Sigma_{3,B}$. Then we obtain

$$S_4 \ll M^{\frac{1}{2}} B^{-\frac{1}{2}} L N(\Delta)^\eta \sum_{X_1 < N(d) \leqslant X_2} \frac{1}{N(d)} \sideset{}{^*}\sum_{N(b)\sim B} \\ \times \Bigg| \sum_{\substack{(n_1,n_2)=1\\ d|n_1n_2}}\sum a_{n_1\Delta}\overline{a_{n_2\Delta}} \frac{\tau(n_1n_2)}{\sqrt{N(n_1n_2)}} \left[\frac{e\overline{ab}}{n_1n_2}\right] \hat{\varphi}\left(\frac{l}{d}\frac{N(en_1n_2)^{\frac{1}{4}}}{M^{\frac{1}{4}}(\overline{ab})^{\frac{1}{2}}}\right)\Bigg|$$

for some a, e and l with $1 \leqslant N(e) \leqslant 2N(\Delta)$ and $N(l) \leqslant L$. We will remove the factor $\hat{\varphi}$ by means of Lemma 3.5 with $\rho(x,y) = \hat{\varphi}(x+iy)$. We first decompose the sum over d into the parts depending on the signs of real and imaginary parts of $ld^{-1}(\overline{ab})^{-1/2}$. We treat the part that both are positive. Using the formula

$$\hat{\varphi}\left(\frac{l}{d}\frac{N(en_1n_2)^{\frac{1}{4}}}{M^{\frac{1}{4}}(\overline{ab})^{\frac{1}{2}}}\right) = \frac{1}{(2\pi i)^2}\int_{(\sigma_2)}\int_{(\sigma_1)} \rho_{1,1}(s_1,s_2)\left(\mathrm{Re}\,\frac{l}{d(\overline{ab})^{\frac{1}{2}}}\right)^{-s_1} \\ \times \left(\mathrm{Im}\,\frac{l}{d(\overline{ab})^{\frac{1}{2}}}\right)^{-s_2}\left(\frac{N(en_1n_2)}{M}\right)^{-\frac{1}{4}(s_1+s_2)} ds_1 ds_2,$$

the corresponding part S_4' is bounded by

$$\ll M^{\frac{1}{2}+\frac{1}{4}(\sigma_1+\sigma_2)} B^{-\frac{1}{2}+\sigma_1+\sigma_2} X_2^{\sigma_1+\sigma_2} L N(\Delta)^\eta \int_{\mathbb{R}^2} |\rho_{1,1}(\sigma_1+it_1,\sigma_2+it_2)| \\ \times \sum_{X_1<N(d)\leqslant X_2} N(d)^{-1} S_\lambda(\sigma_1+it_1,\sigma_2+it_2,d)dt_1dt_2$$

for some choice of λ, where

$$S_\lambda(s_1,s_2,d) = \sideset{}{^*}\sum_{N(b)\sim B}\Bigg|\sum_{\substack{(n_1,n_2)=1\\ d|n_1n_2}}\sum a_{n_1}^{(2,\lambda)}\overline{b_{n_2}^{(2,\lambda)}}\left[\frac{b}{n_1n_2}\right] N(n_1n_2)^{-\frac{1}{4}(s_1+s_2)}\Bigg|.$$

Splitting the range for d into intervals $D < d \leqslant 2D$ and applying Lemma 3.3 to each, we obtain an estimate

$$S_4' \ll_{\sigma_1,\sigma_2} (M^{\frac{1}{4}}N^{-\frac{1}{2}}BX_2N(\Delta)^{\frac{1}{2}})^{\sigma_1+\sigma_2}(MN)^{2\eta}L(\log X_2) \\ \times \sqrt{\frac{M}{BD_1D_2}}\mathcal{B}\left(B,\frac{N}{D_1N(\Delta)}\right)^{\frac{1}{2}}\mathcal{B}\left(B,\frac{N}{D_2N(\Delta)}\right)^{\frac{1}{2}}\sum_n |a_n|^2$$

for some D_1 and D_2, and any $\sigma_1,\sigma_2>0$. Hence

$$\sum_B S_4' \ll (MN^2)^{\sigma_1+\sigma_2}(MN)^{3\eta}L\sqrt{\frac{M}{BD_1D_2}} \\ \times \mathcal{B}\left(B,\frac{N}{D_1N(\Delta)}\right)^{\frac{1}{2}}\mathcal{B}\left(B,\frac{N}{D_2N(\Delta)}\right)^{\frac{1}{2}}\sum_n |a_n|^2$$

for certain value of B. The condition for D_1 and D_2 in Lemma 3.3 shows that

$$D_1D_2 \gg \frac{X_1}{\log^2(2MN)} \gg (MN)^{-\eta}NN(\Delta)^{-1}(MB)^{-\frac{1}{2}}$$

and

$$\sqrt{\frac{M}{BD_1D_2}} \ll (MN)^{\eta}N(\Delta)\frac{M}{N}\sqrt{D_1D_2}\min\left\{1, \frac{N/\sqrt{MB}}{D_1D_2}\right\}.$$

Thus

$$\sum_B S_4' \ll (MN^2)^{\sigma_1+\sigma_2}(MN)^{7\eta}N(\Delta)\frac{M}{N}m_0\sqrt{D_1D_2} \times \mathcal{B}\left(B, \frac{N}{D_1N(\Delta)}\right)^{\frac{1}{2}}\mathcal{B}\left(B, \frac{N}{D_2N(\Delta)}\right)^{\frac{1}{2}}\sum_n |a_n|^2.$$

When we choose η, σ_1 and σ_2 sufficiently small compared with ε, this estimate is included in the error terms in Lemma 4.2. For the other parts we take similar arguments to obtain estimates included in the error terms.

Finally, for the error term on the right of (4.5) we obtain that it makes a negligible contribution to Σ_3, providing that we choose A sufficiently large compared with ε. This completes the proof of Lemma 4.2. ■

Our result for Σ_4 is as follows.

Lemma 4.3. *Let $\varepsilon > 0$ be given. Let $N(\Delta) < N$ and $0 < K \leqslant M^{1-\varepsilon}$, and put $q = q(n_1, n_2) = n_1n_2/(n_1, n_2)^2$. Then we have*

$$\Sigma_4 = M_4 + E_4 \sum |a_n|^2,$$

where

$$M_4 = \frac{\pi}{4} \sideset{}{^*}\sum_{\substack{N(v)\leqslant K \\ (v,\mu\Delta)=1}} \sqrt{\frac{M}{N(v)}} \sum\sum_{(n_1,n_2)=\Delta} a_{n_1}\overline{a_{n_2}}\frac{\Phi(q\Delta)}{2N(q\Delta)}\left[\frac{v}{q}\right]\int_0^\infty W(r^2)dr$$

and

$$E_4 \ll_\varepsilon 1 + (MN)^\varepsilon\sqrt{\frac{M}{BD_1D_2}}\mathcal{B}\left(B, \frac{N}{D_1N(\Delta)}\right)^{\frac{1}{2}}\mathcal{B}\left(B, \frac{N}{D_2N(\Delta)}\right)^{\frac{1}{2}}$$

for certain values of B, D_1, D_2 satisfying that $1 \ll B \ll K$, $D_1, D_2 \gg (MN)^{-\varepsilon}$ and

$$D_1D_2 \gg (MN)^{-\varepsilon}N(\Delta)^{-1}M^{\frac{1}{2}}B^{-\frac{1}{2}}.$$

Proof. Put $m = u^2 v$ with v squarefree. Then we have

$$\begin{aligned}\Sigma_4 &= \frac{1}{4} \sum_{(n_1,n_2)=\Delta}\sum a_{n_1}\overline{a_{n_2}} \sum_{N(v)\leqslant K}^{*} \sum_{u\in\mathbb{Z}[i]}^{\text{odd}} W\left(\frac{N(u^2v)}{M}\right)\left[\frac{u^2v}{q\Delta^2}\right] \\ &= \frac{1}{4} \sum_{(n_1,n_2)=\Delta}\sum a_{n_1}\overline{a_{n_2}} \sum_{\substack{N(v)\leqslant K\\(v,\Delta)=1}}^{*} \left[\frac{v}{q}\right] S(v,n_1,n_2)\end{aligned}$$

where

$$S(v,n_1,n_2) = \sum_{(u,\mu q\Delta)=1} W\left(\frac{N(u^2)}{M/N(v)}\right).$$

We split the range for v into intervals $B < N(v) \leqslant 2B$. For each range we apply Lemma 3.6 to $S(v,n_1,n_2)$ by taking

$$\begin{aligned}&\varphi(z) = W(N(z^2)), \quad x = (M/N(v))^{\frac{1}{4}}, \quad X = (M/N(v))^{\frac{1}{2}} \\ &X_1 = (MN)^{-\eta} M^{\frac{1}{2}} B^{-\frac{1}{2}}, \quad X_2 = (MN)^{\eta} M^{\frac{1}{2}} B^{-\frac{1}{2}}, \\ &J = (MN)^{\frac{\eta}{2}} \quad \text{and} \quad L = (MN)^{3\eta}.\end{aligned}$$

Then we have

$$\begin{aligned}&S(v,n_1,n_2) \\ &\quad = \frac{\Phi(\mu q\Delta)}{N(\mu q\Delta)}\sqrt{\frac{M}{N(v)}}\int_{\mathbb{C}} W(N(z^2))dz - \frac{1}{4}W(0)\sum_{d\mid\mu q\Delta, N(d)\leqslant X_2}\mu(d) \\ &\quad\quad - \frac{1}{4}\sum_{d\mid\mu q\Delta, N(d)>X_2}\frac{\mu(d)}{N(d)}\sqrt{\frac{M}{N(v)}}\int_{\mathbb{C}} W(N(z^2))dz \qquad (4.6) \\ &\quad\quad + \frac{1}{4}\sum_{0<N(l)\leqslant L}\sum_{\substack{d\mid\mu q\Delta\\ X_1<N(d)\leqslant X_2}}\frac{\mu(d)}{N(d)}\sqrt{\frac{M}{N(v)}}\hat{\varphi}\left(\frac{M^{\frac{1}{4}} l}{N(v)^{\frac{1}{4}}\bar{d}}\right) + O_A(XJ^{-A}).\end{aligned}$$

The first part on the right of (4.6) is equal to

$$\frac{\pi\Phi(q\Delta)}{2N(q\Delta)}\sqrt{\frac{M}{N(v)}}\int_0^{\infty} W(r^2)dr.$$

Taking the summation over B, n_1, n_2 and v, the total value becomes M_4.

The second part vanishes, since $W(0) = 0$.

The total contribution for the third part to Σ_4 becomes

$$\begin{aligned}
S &= -\frac{1}{16}\sum_{B}\sum_{N(d)>X_2}\frac{\mu(d)}{N(d)}\sideset{}{^*}\sum_{\substack{N(v)\sim B\\(v,\Delta)=1}}\sqrt{\frac{M}{N(v)}}\\
&\quad\times\sum\sum_{\substack{(n_1,n_2)=\Delta\\d|\mu q\Delta}} a_{n_1}\overline{a_{n_2}}\left[\frac{v}{q}\right]\int_{\mathbb{C}}W(N(z^2))dz\\
&\ll M^\eta\sum_{N(d)>X_2}\frac{1}{N(d)}\sideset{}{^*}\sum_{\substack{N(v)\sim B\\(v,\Delta)=1}}\sqrt{\frac{M}{B}}\left|\sum\sum_{\substack{(n_1,n_2)=\Delta\\d|\mu q\Delta}} a_{n_1}\overline{a_{n_2}}\left[\frac{v}{q}\right]\right|\\
&\ll M^\eta\sum_{N(d)>X_2}\frac{1}{N(d)}\sideset{}{^*}\sum_{\substack{N(v)\sim B\\(v,\Delta)=1}}\sqrt{\frac{M}{B}}\left|\sum\sum_{\substack{(n_1,n_2)=1\\d|\mu n_1n_2\Delta}} a_{n_1\Delta}\overline{a_{n_2\Delta}}\left[\frac{v}{n_1n_2}\right]\right|
\end{aligned}$$

for certain value of B. We write $d'=(d,\mu\Delta)$ and $e=d/d'$. Then we have

$$S\ll_\eta (MN)^\eta\sqrt{\frac{M}{B}}\sum_{N(e)>X_2/2N(\Delta)}\frac{1}{N(e)}\sideset{}{^*}\sum_{N(v)\sim B}\left|\sum\sum_{\substack{(n_1,n_2)=1\\e|n_1n_2}} a_{n_1\Delta}\overline{a_{n_2\Delta}}\left[\frac{v}{n_1n_2}\right]\right|.$$

We decompose the sum over e into subintervals $D<N(e)\leqslant 2D$ and apply Lemma 3.3 to each. This yields

$$S\ll_\eta (KN)^{3\eta}\sqrt{\frac{M}{BD_1D_2}}\mathcal{B}\left(B,\frac{N}{D_1N(\Delta)}\right)^{\frac{1}{2}}\mathcal{B}\left(B,\frac{N}{D_2N(\Delta)}\right)^{\frac{1}{2}}\sum_n|a_n|^2$$

for certain values of D_1 and D_2 satisfying that

$$\frac{1}{\log(2MN)}\ll D_i\ll\frac{N^2}{\log(2MN)}$$

for $i=1,2$, and

$$\frac{(MN)^\eta M^{\frac{1}{2}}}{N(\Delta)B^{\frac{1}{2}}\log^2(2MN)}\ll D_1D_2\ll\frac{N^2}{\log^2(2MN)}.$$

Let T denote the contribution of the forth part on the right of (4.6) to Σ_4.

Then

$$T = \frac{1}{16} \sum_{B} \sideset{}{^*}\sum_{\substack{N(v)\sim B\\(v,\Delta)=1}} \sum_{0<N(l)\leqslant L} \sum_{X_1<N(d)\leqslant X_2} \frac{\mu(d)}{N(d)} \sqrt{\frac{M}{N(v)}} \hat{\varphi}\left(\frac{M^{\frac{1}{4}} l}{N(v)^{\frac{1}{4}} \bar{d}}\right)$$

$$\times \sum\sum_{\substack{(n_1,n_2)=\Delta\\ d|\mu q\Delta}} a_{n_1}\overline{a_{n_2}} \left[\frac{v}{q}\right]$$

$$\ll (MN)^{3\eta} \sqrt{\frac{M}{B}} \sum_{B} \sum_{X_1<N(d)\leqslant X_2} \frac{1}{N(d)} \sideset{}{^*}\sum_{\substack{N(v)\sim B\\(v,\Delta)=1}} \left| \sum\sum_{\substack{(n_1,n_2)=\Delta\\ d|\mu q\Delta}} a_{n_1}\overline{a_{n_2}} \left[\frac{v}{q}\right] \right|.$$

We write $d' = (d, \mu\Delta)$ and $e = d/d'$ to obtain that

$$T \ll_\eta (MN)^{4\eta} \sqrt{\frac{M}{B}} \sum_{\frac{X_1}{2N(\Delta)}<N(e)\leqslant X_2} \frac{1}{N(e)} \sideset{}{^*}\sum_{N(v)\sim B} \left| \sum\sum_{\substack{(n_1,n_2)=1\\ e|n_1n_2}} a_{n_1\Delta}\overline{a_{n_2\Delta}} \left[\frac{v}{q}\right] \right|.$$

We decompose the sum over e into subintervals $(D, 2D]$ and apply Lemma 3.3 to each, so that

$$T \ll_\eta (MN)^{6\eta} \left(\frac{M}{BD_1D_2}\right)^{\frac{1}{2}} \mathcal{B}\left(B, \frac{N}{D_1N(\Delta)}\right)^{\frac{1}{2}} \mathcal{B}\left(B, \frac{N}{D_2N(\Delta)}\right)^{\frac{1}{2}} \sum_n |a_n|^2$$

for certain values of D_1 and D_2 satisfying that

$$\frac{1}{\log(2MN)} \ll D_i \ll \frac{(MN)^\eta M^{\frac{1}{2}}}{B^{\frac{1}{2}} \log(2MN)}$$

for $i = 1, 2$, and

$$\frac{M^{\frac{1}{2}}}{N(\Delta)B^{\frac{1}{2}}(MN)^\eta \log^2(2MN)} \ll D_1D_2 \ll \frac{(MN)^\eta M^{\frac{1}{2}}}{B^{\frac{1}{2}} \log^2(2MN)}.$$

Finally, the error term of (4.6) makes a negligible contribution to Σ_3, providing that we choose A sufficiently large compared with ε. The proof of Lemma 4.3 is complete. ■

We shall investigate a bound for the difference $M_3 - M_4$.

Lemma 4.4. *Let* $1 \leqslant N(\Delta) \leqslant N$ *and* $0 < K \leqslant M$. *Then we have*

$$M_3 - M_4 \ll M^{\frac{1}{2}} K^{-\frac{1}{2}} (MN)^\varepsilon \mathcal{B}(KN(\Delta)^2(MN)^\varepsilon, N(MN)^\varepsilon) \sum_n |a_n|^2.$$

Proof. We begin to consider M_3. Put

$$\alpha(w) = \frac{1}{4} \sum_{\substack{w=be, e|\Delta \\ N(b)\leqslant K \\ b\ odd\ squarefree}} \mu(e).$$

Then we have

$$\sideset{}{^*}\sum_{N(b)\leqslant K} \frac{\kappa(\Delta,q)}{\sqrt{N(b)}} \left[\frac{b}{q}\right] = \sum_{w} \frac{\alpha(w)}{\sqrt{N(w)}} \left[\frac{w}{q}\right]$$

where the sum is over odd Gaussian integers w satisfying that $N(w) \leqslant KN(\Delta)$ and $w = rs^2$ with r odd squarefree and $s|\Delta$. Hence we obtain

$$M_3 = \frac{\pi}{4}\sqrt{M} \sum_{(n_1,n_2)=\Delta} a_{n_1}\overline{a_{n_2}} \frac{\Phi(q)}{2N(q)} \sum_{w} \frac{\alpha(w)}{\sqrt{N(w)}} \left[\frac{w}{q}\right] \int_0^\infty W(r^2)dr.$$

Next we consider M_4. Since $(q, \Delta) = 1$, we have

$$\frac{\Phi(q\Delta)}{2N(q\Delta)} = \frac{\Phi(q)}{2N(q)} \frac{\Phi(\Delta)}{N(\Delta)}$$

and

$$\frac{\Phi(\Delta)}{N(\Delta)} = \frac{1}{4} \sum_{u|\Delta} \frac{\mu(u)}{\sqrt{N(u^2)}} \left[\frac{u^2}{q}\right].$$

Hence we have

$$\sideset{}{^*}\sum_{\substack{N(v)\leqslant K \\ (v,\mu\Delta)=1}} \frac{1}{\sqrt{N(v)}} \frac{\Phi(\Delta)}{N(\Delta)} \left[\frac{v}{q}\right] = \sum_{w} \frac{\beta(w)}{\sqrt{N(w)}} \left[\frac{w}{q}\right]$$

where

$$\beta(w) = \frac{1}{4} \sum_{\substack{vu^2=w, u|\Delta \\ N(v)\leqslant K \\ (v,\mu\Delta)=1 \\ v\ squarefree}} \mu(u).$$

Thus

$$M_4 = \frac{\pi}{4}\sqrt{M} \sum_{(n_1,n_2)=\Delta} a_{n_1}\overline{a_{n_2}} \frac{\Phi(q)}{2N(q)} \sum_{w} \frac{\beta(w)}{\sqrt{N(w)}} \left[\frac{w}{q}\right] \int_0^\infty W(r^2)dr$$

where the sum is taken over w such that w is odd, $N(w) \leqslant KN(\Delta)^2$ and $w = rs^2$ with r odd squarefree and $s|\Delta$.

Let w be an odd integer with $N(w) \leqslant K$. Then we have

$$\alpha(w) = \frac{1}{4} \sum_{e|(w,\Delta)} \mu(e)\mu^2(w/e)$$

and

$$\beta(w)=\frac{1}{4}\sum_{\substack{w=vu^2,u\mid\Delta\\(v,\Delta)=1}}\mu(u)\mu^2(w/u^2).$$

We note that the two sums on the right are multiplicative in w, and therefore $\gamma(w)=\alpha(w)-\beta(w)=0$ in this case. Also we can check $\gamma(w)\ll d(w)$ for any w with $N(w)\in(K,KN(\Delta)^2]$. Hence we have

$$\begin{aligned}M_3-M_4&\\ &\ll M^{\frac12}K^{-\frac12}(MN)^{\eta}\sum_{K<N(w)\leqslant KN(\Delta)^2}\Bigg|\mathop{\sum\sum}_{(n_1,n_2)=\Delta}a_{n_1}\overline{a_{n_2}}\frac{\Phi(q)}{N(q)}\left[\frac{w}{q}\right]\Bigg|\\ &\ll M^{\frac12}K^{-\frac12}(MN)^{\eta}\sum_{s\mid\Delta}\ \sideset{}{^*}\sum_{KN(\Delta)^{-2}<N(r)\leqslant KN(\Delta)^2}\Bigg|\mathop{\sum\sum}_{(n_1,n_2)=\Delta}a_{n_1}\overline{a_{n_2}}\frac{\Phi(q)}{N(q)}\left[\frac{rs^2}{q}\right]\Bigg|.\end{aligned}$$

Let us put

$$a_n^{(3)}=\begin{cases}\frac{\Phi(n)}{N(n)}a_{n\Delta} & \text{if } (n,s)=1,\\ 0 & \text{otherwise.}\end{cases}$$

Then, decomposing the range for r into subintervals $(R,2R]$, we obtain

$$M_3-M_4\ll M^{\frac12}K^{-\frac12}(MN)^{2\eta}\sideset{}{^*}\sum_{N(r)\sim R}\Bigg|\mathop{\sum\sum}_{(m_1,m_2)=1}a_{m_1}^{(3)}\overline{a_{m_2}^{(3)}}\left[\frac{r}{m_1m_2}\right]\Bigg|$$

for certain values of s and R. We apply Lemma 3.3 with $D=\frac12$ to show that

$$M_3-M_4\ll M^{\frac12}K^{-\frac12}(MN)^{3\eta}\mathcal{B}(R,N')\sum|a_n|^2$$

for some $N'\ll N(MN)^{\eta}$. Finally Lemma 3.2 allows us to replace R by $KN(\Delta)^2(MN)^{2\eta}$, and N' by $N(MN)^{2\eta}$. We choose η sufficiently small compared with ε to complete the proof of Lemma 4.4. ■

We shall combine Lemma 3.1, 4.2, 4.3 and 4.4 to give the following bound for $\mathcal{C}(M,N,K,\Delta)$.

Lemma 4.5. *Let c be a real number >1. Let $\xi\in(1,c]$ be given. Assume that*

$$\mathcal{B}(M,N)\ll_{\varepsilon}(MN)^{\varepsilon}(M+N^{\xi}) \tag{4.7}$$

for any $\varepsilon>0$ and $1\leqslant N\leqslant M$. Then we have

$$\mathcal{C}(M,N,K,\Delta)\ll_{\varepsilon}N(\Delta)^{2c}(MN)^{\varepsilon}(M+N+M^{\frac12}K^{\xi-\frac12}) \tag{4.8}$$

for $1\leqslant N(\Delta)<N$ and $N^2M^{-1}(MN)^{\varepsilon}\leqslant K\leqslant M(MN)^{-\varepsilon}$.

Proof. We first prove that, under the assumption (4.7),

$$\mathcal{B}(M,N) \ll_\varepsilon (MN)^\varepsilon (M + N^\xi)$$

and

$$\mathcal{B}(M,N) \ll_\varepsilon (MN)^\varepsilon (M^\xi + N)$$

for any positive real numbers M and N. Since these estimates are trivial when M or N is less than 1, we prove them for M, $N \geqslant 1$. The former inequality with $N \leqslant M$ immediately follows from the assumption (4.7). When $M \leqslant N$, we use Lemma 3.1 to obtain

$$\mathcal{B}(M,N) \leqslant 2\mathcal{B}(N,M) \ll_\varepsilon (MN)^\varepsilon (N + M^\xi) \ll (MN)^\varepsilon (M + N^\xi)$$

Moreover the later inequality follows from the former and Lemma 3.1.

In view of (4.3), (4.4), Lemma 4.2, 4.3 and 4.4, it is enough to estimate E_3, E_4 and $M_3 - M_4$ under the assumption (4.7) to obtain our result. We begin to investigate E_3. The assumption shows that

$$\mathcal{B}\left(B, \frac{N}{D_1 N(\Delta)}\right)^{\frac{1}{2}} \mathcal{B}\left(B, \frac{N}{D_2 N(\Delta)}\right)^{\frac{1}{2}} \ll (MN)^{3\varepsilon}\left(B^\xi + B^{\frac{\xi}{2}} N^{\frac{1}{2}} + N D_1^{-\frac{1}{2}} D_2^{-\frac{1}{2}}\right).$$

Since $m_0 \leqslant \frac{N/\sqrt{MB}}{D_1 D_2}$, we have

$$\frac{M}{N}\sqrt{D_1 D_2} m_0 B^\xi \ll_\varepsilon (MN)^\varepsilon M^{\frac{1}{2}} B^{\xi - \frac{1}{2}}.$$

Similarly, since $m_0 \leqslant \left(\frac{N/\sqrt{MB}}{D_1 D_2}\right)^{\frac{1}{2}}$, we obtain

$$\frac{M}{N}\sqrt{D_1 D_2} m_0 B^{\frac{\xi}{2}} N^{\frac{1}{2}} \leqslant M^{\frac{3}{4}} B^{\frac{\xi}{2} - \frac{1}{4}} = \left(M^{\frac{1}{2}} B^{\xi - \frac{1}{2}}\right)^{\frac{1}{2}} M^{\frac{1}{2}} \ll M^{\frac{1}{2}} B^{\xi - \frac{1}{2}} + M.$$

Finally, since $m_0 \leqslant 1$, we have

$$\frac{M}{N}\sqrt{D_1 D_2} m_0 N (D_1 D_2)^{-\frac{1}{2}} \leqslant M.$$

Thus we deduce

$$E_3 \ll (MN)^{5\varepsilon} N(\Delta)(M^{\frac{1}{2}} K^{\xi - \frac{1}{2}} + M).$$

We now consider E_4. By the assumption we have

$$\mathcal{B}\left(B, \frac{N}{D_1 N(\Delta)}\right)^{\frac{1}{2}} \mathcal{B}\left(B, \frac{N}{D_2 N(\Delta)}\right)^{\frac{1}{2}} \ll_\varepsilon (MN)^{3\varepsilon}\left(B^\xi + B^{\frac{\xi}{2}} N^{\frac{1}{2}} + N(D_1 D_2)^{-\frac{1}{2}}\right).$$

Since $D_1 D_2 \gg (MN)^{-2\varepsilon}$, we find that

$$\sqrt{\frac{M}{BD_1 D_2}} B^\xi \ll_\varepsilon (MN)^\varepsilon M^{\frac{1}{2}} B^{\xi - \frac{1}{2}}.$$

Since

$$D_1 D_2 \gg (MN)^{-\varepsilon} N(\Delta)^{-1} M^{\frac{1}{2}} B^{-\frac{1}{2}},$$

we obtain

$$\begin{aligned}\sqrt{\frac{M}{BD_1D_2}} B^{\frac{\xi}{2}} N^{\frac{1}{2}} &\ll_\varepsilon (MN)^{\frac{1}{2}\varepsilon} N(\Delta)^{\frac{1}{2}} M^{\frac{1}{4}} N^{\frac{1}{2}} B^{\frac{\xi}{2}-\frac{1}{4}} \\ &= (MN)^{\frac{1}{2}\varepsilon} N(\Delta)^{\frac{1}{2}} (MB^{\xi-\frac{1}{2}})^{\frac{1}{2}} N^{\frac{1}{2}} \\ &\ll (MN)^{\frac{1}{2}\varepsilon} N(\Delta)^{\frac{1}{2}} (M^{\frac{1}{2}} B^{\xi-\frac{1}{2}} + N)\end{aligned}$$

and

$$\sqrt{\frac{M}{BD_1D_2}} N(D_1D_2)^{-\frac{1}{2}} \ll_\varepsilon (MN)^\varepsilon N(\Delta) N.$$

Thus we have

$$E_4 \ll (MN)^{5\varepsilon} N(\Delta)(M^{\frac{1}{2}} K^{\xi-\frac{1}{2}} + N).$$

We finally consider $M_3 - M_4$. The assumption yields that

$$\begin{aligned}&M^{\frac{1}{2}} K^{-\frac{1}{2}} (MN)^\varepsilon \mathcal{B}(KN(\Delta)^2 (MN)^\varepsilon, N(MN)^\varepsilon) \\ &\qquad \ll_\varepsilon (MN)^{(4+\xi+\varepsilon)\varepsilon} M^{\frac{1}{2}} K^{-\frac{1}{2}} (K^\xi N(\Delta)^{2\xi} + N) \\ &\qquad = (MN)^{(4+\xi+\varepsilon)\varepsilon} (N(\Delta)^{2\xi} M^{\frac{1}{2}} K^{\xi-\frac{1}{2}} + M^{\frac{1}{2}} N K^{-\frac{1}{2}}).\end{aligned}$$

Since $M^{\frac{1}{2}} N K^{-\frac{1}{2}} < (MN)^{-\frac{1}{2}\varepsilon} M$, we obtain

$$M_3 - M_4 \ll_\varepsilon (MN)^{(4+\xi+\varepsilon)\varepsilon} (N(\Delta)^{2\xi} M^{\frac{1}{2}} K^{\xi-\frac{1}{2}} + M) \sum_{N(n)\sim N} |a_n|^2.$$

We combine the above estimates, so that

$$\mathcal{C}(M, N, K, \Delta) \ll_\varepsilon (MN)^{(4+\xi+\varepsilon)\varepsilon} N(\Delta)^{2\xi} (M^{\frac{1}{2}} K^{\xi-\frac{1}{2}} + M + N).$$

Thus Lemma 4.5 follows from $\xi \leqslant c$. ■

We use Lemma 4.1 and 4.5 to obtain a new bound for $\mathcal{B}(M, N)$ as follows.

Lemma 4.6. *Under the same assumption in Lemma 4.5 we have*

$$\mathcal{B}(M, N) \ll_\varepsilon (MN)^\varepsilon (M + M^{1-\xi} N^{2\xi-1})$$

for any $\varepsilon > 0$.

Proof. When $N(MN)^\varepsilon \geqslant M$, we immediately obtain

$$\begin{aligned}\mathcal{B}(M, N) &\ll_\varepsilon (MN)^\varepsilon (M + M^{1-\xi} N^{2\xi-1} (M/N)^{\xi-1}) \\ &\ll_\varepsilon (MN)^{c\varepsilon} (M + M^{1-\xi} N^{2\xi-1}).\end{aligned}$$

We now consider the case $N(MN)^\varepsilon < M$. We can apply Lemma 4.5. Let $1 < L \leqslant N$ and $1 < D_0 \leqslant L$. Then, by inserting (4.8) into (4.2), we obtain that

$$\begin{aligned}\mathcal{B}(M,L,K) &\ll_\varepsilon (ML)^\varepsilon\Big\{\mathcal{B}(M,L',K) + D_0^{2c+1}(M+L+M^{\frac12}K^{\xi-\frac12})\Big\}\\ &\ll (MN)^\varepsilon\Big\{\mathcal{B}(M,L',K) + D_0^{2c+1}(M+M^{\frac12}K^{\xi-\frac12})\Big\}\end{aligned}$$

for certain value of $L' \leqslant L/D_0$. When we take $K = N^2M^{-1}(MN)^\varepsilon$, then we have

$$\mathcal{B}(M,L,K) \ll_\varepsilon (MN)^{(c+1)\varepsilon}(\mathcal{B}(M,L',K) + D_0^{2c+1}(M + M^{1-\xi}N^{2\xi-1})).$$

Let R be a large integer and set $D_0 = N^{1/R}$. Then we can choose positive real numbers $N_0 = N$, $N_1, N_2, \dots, N_{r_0}$ satisfying that $N_{r_0} \leqslant D_0 < N_{r_0-1}$, $N_r \leqslant D_0^{-r}N_0$ and

$$\mathcal{B}(M,N_{r-1},K) \ll_\varepsilon (MN)^{(c+1)\varepsilon}\{\mathcal{B}(M,N_r,K) + D_0^{2c+1}(M+M^{1-\xi}N_0^{2\xi-1})\} \quad (4.9)$$

for $r = 1, 2, \dots, r_0$. We remark that $r_0 \leqslant R$ from the choice of D_0. By using the inequality (4.9) repeatedly, we see that

$$\mathcal{B}(M,N,K) \ll_{R,\varepsilon} (MN)^{(c+1)R\varepsilon}\{\mathcal{B}(M,N_{r_0},K) + D_0^{2c+1}(M+M^{1-\xi}N^{2\xi-1})\}.$$

By the trivial estimate $\mathcal{B}(M,N_{r_0},K) \ll MN_{r_0} \leqslant MD_0$, we have

$$\begin{aligned}\mathcal{B}(M,N,K) &\ll_{R,\varepsilon} (MN)^{(c+1)R\varepsilon}D_0^{2c+1}(M+M^{1-\xi}N^{2\xi-1})\\ &= (MN)^{(c+1)R\varepsilon}N^{(2c+1)/R}(M+M^{1-\xi}N^{2\xi-1}).\end{aligned}$$

Hence we deduce that

$$\mathcal{B}(M,N,K) \ll_\varepsilon (MN)^\varepsilon(M+M^{1-\xi}N^{2\xi-1})$$

for any $\varepsilon > 0$. Since $\mathcal{B}(M,N) \ll \mathcal{B}(M,N,K)$ for $K \leqslant M/2$, Lemma 4.6 follows. ∎

Lemma 4.6 produces the following bound for $\mathcal{B}(M,N)$.

Lemma 4.7. *Under the same assumption in Lemma 4.5, we have*

$$\mathcal{B}(M,N) \ll_\varepsilon (MN)^\varepsilon(M + N^{(2\xi-1)/\xi})$$

for any $\varepsilon > 0$.

Proof. We use Lemma 4.6 to prove Lemma 4.7. When $N^{(2\xi-1)/\xi} \leqslant M$, we easily obtain

$$\begin{aligned}\mathcal{B}(M,N) &\ll_\varepsilon (MN)^\varepsilon(M+M^{1-\xi}N^{2\xi-1})\\ &\leqslant (MN)^\varepsilon(M+N^{(2\xi-1)/\xi}).\end{aligned}$$

When $N^{(2\xi-1)/\xi} > M$, Lemma 3.2 and Lemma 4.6 show that

$$\begin{aligned}\mathcal{B}(M,N) &\ll \mathcal{B}(N^{(2\xi-1)/\xi}(MN)^{\varepsilon}, N)\\ &\ll_{\varepsilon} (MN)^{\varepsilon}\big\{N^{(2\xi-1)/\xi}(MN)^{\varepsilon} + (N^{(2\xi-1)/\xi}(MN)^{\varepsilon})^{1-\xi}N^{2\xi-1}\big\}\\ &\leqslant 2(MN)^{2\varepsilon}N^{(2\xi-1)/\xi}.\end{aligned}$$

These conclude the result. ■

We now prove Theorem 1. By Lemma 2.3 we see that the assumption (4.7) in Lemma 4.5 is true if we take $\xi = 8$. Hence we can apply Lemma 4.7 where we note that $(2\xi-1)/\xi < \xi$ for $\xi > 1$. By repeating this procedure with respect to ξ we obtain the estimate (4.1). This completes the proof of Theorem 1.

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Address: Kazuhiro Onodera: Department of Mathematics, Tokyo Institute of Technology, Oh-okayama, Meguro-ku, Tokyo 152-8551, Japan;

E-mail: onodera@math.titech.ac.jp

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