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HOMOTOPY MINIMAL PERIODS OF HOLOMORPHIC MAPS ON SURFACES

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Dedicated to Professor Bogdan Bojarski
on the occasion of his 75th birthday

Abstract: In this paper we study the minimal periods on a holomorphic map which are preserved by any of its deformation considering separately the case of continuous and holomorphic homotopy. A complete description of the set of such minimal periods for holomorphic self-map of a compact Riemann surface is given. It shows that a nature of answer depends on the geometry of the surface distinguishing the parabolic case of the Riemann sphere, elliptic case of tori and the hyperbolic case of a surface of genus $\geqslant 2$.

Keywords: Set of periods, periodic points, holomorphic maps, homotopy, Riemann surfaces

1. Introduction and statement of the results

The set of minimal periods $\mathrm{Per}(f)$ is one of the classical invariant in the study the of dynamical properties of a map f. But it is not stable in general, i.e. this set usually changes if, for example, we perturb the map, in particular is not preserved by a homotopy of the map. *It* is difficult to analyze this set by tools of algebraic topology. To avoid this difficulty the set of homotopy minimal periods, i.e. minimal periods which are preserved by any homotopy, has been studied by many authors (cf. [1], [15], [12], and [13, Chapt. VI] for an exposition of known results). On the other hand it is known that holomorphic maps of compact complex manifolds have many periodic points and large sets of minimal periods (see [2],[8]). Consequently, it is a natural to ask which minimal periods of a holomorphic map f are preserved by a holomorphic homotopy of f, and which of them are preserved by any continuous deformation of f.

In this note we answer this question for the holomorphic self–maps of a compact Riemann surface, i.e. compact two dimensional oriented manifold with a complex structure. The answer is complete for the case of the Riemann sphere and complex tori. For holomorphic maps of a hyperbolic surface our description only does not

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round off with a final conclusion for the maximal period of a holomorphic map, i.e. a conformal automorphism when this automorphism is not free. Proofs are elementary and essentially based on already known facts. On the other hand as a consequence of our approach we get also elementary proofs of statements on dynamics of surface homeomorphisms which originally were studied by more complicated tools ([3], [10]).

First we introduce the notation. Let X be a complex, closed manifold. We define the following sets. Let $\mathrm{Map}(X;X)$, or simply $\mathrm{Map}(X)$, denote the set of all continuous self maps of X, $\mathcal{H}ol(X;X)$ or shortly $\mathcal{H}ol(X)$ denote the set of all holomorphic self–maps of X, and $[X,X]$ denote the set of all homotopy classes of self maps of X, i.e. $[X,X] = \mathrm{Map}(X)/\sim$, where $\sim$ is the homotopy equivalence relation. In fact there is a bijection of $[X,X]$ with $\pi_0(\mathrm{Map}(X))$. Let next $[X,X]_{\mathcal{H}}$ denote the set of all homotopy holomorphic classes of holomorphic self maps of X.

Let $f,g \in \mathcal{H}ol(X)$ then $f \approx g$ if there exists a continuous one–parameter family $h_t \in \mathcal{H}ol(X)$ for all $t \in [0,1]$ such that $h_0 = f$ and $h_1 = g$. $[X,X]_{\mathcal{H}} = \mathcal{H}ol(X)/\approx$ denotes the set of all homotopy holomorphic classes of holomorphic self maps of X. In fact there is a bijection of $[X,X]_{\mathcal{H}}$ with $\pi_0(\mathcal{H}ol(X))$.

Let $\mathrm{Per}(f)$ be the set of minimal periods of a map f. For $f \in \mathcal{H}ol(X)$ we define

$$\mathrm{HPer}(f) = \bigcap_{g\sim f} \mathrm{Per}(g), \tag{1}$$

the set of *the homotopy minimal periods* (shortly *homotopy periods*) of f.

We define

$$\mathrm{HPer}^{\mathcal{H}}(f) = \bigcap_{\substack{g\sim f\\ g\in\mathcal{H}ol(X)}} \mathrm{Per}(g), \tag{2}$$

the set of *homotopy holomorphic minimal periods* (shortly *homotopy holomorphic periods*) of f. We note that here the homotopy is through continuous maps. Of course from the definitions it follows that

$$\mathrm{HPer}(f) \subset \mathrm{HPer}^{\mathcal{H}}(f),$$

because the intersection is taken over a smaller family in the definition of $\mathrm{HPer}^{\mathcal{H}}(f)$.

We remark that in general we have

$$\mathrm{HPer}(f) \subsetneq \mathrm{HPer}^{\mathcal{H}}(f). \tag{3}$$

Take $X = \mathbb{C}P(1) = \mathbb{S}^2 = \mathbb{C}_\infty$. It is known that every holomorphic self map of $\mathbb{S}^2$ is a rational function. Then (3) is satisfied for any holomorphic self map f of $\mathbb{S}^2$ of degree $\geqslant 2$. Indeed we will show in Theorem 1 that $\mathrm{HPer}(f) = \{1\}$. On the other hand from the Baker's Theorem (see [2] or [8]) it follows that $\mathrm{Per}(f) \supset \mathbb{N}\setminus\{2,3\}$ for any such map. Consequently

$$\{1\} = \mathrm{Per}(f) \subsetneq \mathbb{N}\setminus\{2,3\} \subset \mathrm{HPer}^{\mathcal{H}}(f).$$

We define

$$\mathrm{HPer}_{\mathcal{H}}(f) = \bigcap_{\substack{g \approx f \\ g \in \mathcal{H}ol(X)}} \mathrm{Per}(g), \tag{4}$$

the set of *holomorphic homotopic minimal periods* (shortly *holomorphic homotopy periods*) of f. We note that here the homotopy is through holomorphic maps. Again from the definitions it follows that

$$\mathrm{HPer}^{\mathcal{H}}(f) \subset \mathrm{HPer}_{\mathcal{H}}(f).$$

Indeed $f \approx g$ implies that $f \sim g$. Consequently for a given $f \in \mathcal{H}ol(X)$ the intersection in this last definition is over a smaller family than in the previous definition, where the second equality only has meaning if $g \in \mathcal{H}ol(X)$.

Our main results can be summarized in the next six theorems (Theorems 1, 2, 4, 7, 8 and 10).

Theorem 1. *Let $\mathbb{S}^2$ be the Riemann sphere and let $f : \mathbb{S}^2 \to \mathbb{S}^2$ be a holomorphic map. Then* $\mathrm{HPer}(f) = \{1\}$.

Theorem 1 is proved in Section 2.

Now we turn to the description of $\mathrm{HPer}^{\mathcal{H}}(f)$ and $\mathrm{HPer}_{\mathcal{H}}(f)$ for $f \in \mathcal{H}ol(\mathbb{S}^2)$. It is known that every $f \in \mathcal{H}ol(\mathbb{S}^2)$ is a rational function, i.e.

$$f(x) = R(z) = P(z)/Q(z),$$

with $P, Q \in \mathbb{C}[z]$ polynomials of degrees d_P and d_Q respectively (see Section 2.2 of [2]). The *degree* of the rational function $R(z)$ is defined as $d(R) = \max\{d_P, d_Q\}$. Of course if $R(z)$ is constant, then its degree is zero. Always we will assume that P and Q are relatively prime, i.e. that P and Q do not have common zeros.

A rational function $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a $d(R)$–fold covering map. Consequently its topological degree satisfies $\deg(R) = d(R)$, for more details see again Section 2.2 of [2]. This means that for all $w \in \mathbb{C}_\infty$ the equation $R(z) = w$ has exactly $d(R)$ solutions.

Theorem 2. *Let let $f : \mathbb{S}^2 \to \mathbb{S}^2$ be a holomorphic map. Then*

$$\mathrm{HPer}^{\mathcal{H}}(f) = \mathrm{HPer}_{\mathcal{H}}(f) = \begin{cases} \{1\} & \textit{if } \deg(f) = 0, 1, \\ \mathbb{N} \setminus \{2, 3\} & \textit{if } \deg(f) = 2, \\ \mathbb{N} \setminus \{2\} & \textit{if } \deg(f) = 3, 4, \\ \mathbb{N} & \textit{if } \deg(f) \geqslant 5. \end{cases}$$

Theorem 2 is proved in Section 3.

Now we turn to study the periods of the holomorphic self maps of the 2–dimensional torus $\mathbb{T}^2$ with an arbitrary complex structure. First we recall the notion of linearization of a self map of the torus (cf [13]).

Suppose that the torus $\mathbb{T}^2$ is defined as $\mathbb{R}^2/\Gamma$, where Γ is a discrete subgroup of $\mathbb{R}^2$ isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. For a continuous map $f : \mathbb{T}^2 \to \mathbb{T}^2$ let $f_\sharp : \pi_1(\mathbb{T}^2) \to \pi_1(\mathbb{T}^2)$

with $\pi_1(\mathbb{T}^2) = \mathbb{Z}^2$ be the induced homomorphism. A choice of a basis in $\Gamma \equiv \mathbb{Z}^2$ defines an integral 2×2 matrix of $f_\sharp$, denoted by A_f, and called the linearization matrix of f. The matrix A_f is unique up to conjugation by an unimodular matrix (a choice of a basis).

The linear map induced by matrix A_f preserves the lattice $\Gamma \equiv \mathbb{Z}^2 \subset \mathbb{R}^2$. Thus induces a homomorphism $[A_f] : \mathbb{T}^2 \to \mathbb{T}^2$ defined as the factor of the homomorphism (linear map) $A_f : \mathbb{R}^2 \to \mathbb{R}^2$. By definition A_f depends only on the class $[f]$ in $\mathrm{Map}(\mathbb{T}^2)$ (see [13] for more details).

It is known that the topological torus $\mathbb{T}^2$ posses infinitely many non-equivalent holomorphic structures. In more detail, let γ^1, γ^2 be linearly independent (over $\mathbb{R}$) vectors of $\mathbb{C}$ and $\Gamma \subset \mathbb{C}$ be the lattice $\Gamma = \mathbb{Z}\gamma^1 \oplus \mathbb{Z}\gamma^2$. Then Γ is a discrete subgroup of $\mathbb{C}$ and the quotient space $\mathbb{C}/\Gamma$ is a one dimensional (over $\mathbb{C}$ complex manifold which is obviously homeomorphic to $\mathbb{T}^2$. $p : \mathbb{C} \to \mathbb{C}/\Gamma$ is a covering map which is holomorphic by the definition of the complex structure in $\mathbb{C}/\Gamma$.

Furthermore the tori $\mathbb{C}/\Gamma_1$ and $\mathbb{C}/\Gamma_1$ are equivalent as complex manifolds iff Γ_1 and Γ_2 are conjugated over $\mathbb{C}$, i.e. there exists a $\mathbb{C}$-linear automorphism $F : \mathbb{C} \to \mathbb{C}$ such that $\Gamma_2 = F\Gamma_1F^{-1}$. Note also that $F \in \mathrm{Aut}(\mathbb{C}) = \mathbb{C} \setminus \{0\}$ is a $\mathbb{R}$-linear map of $\mathbb{R}^2$ which commutes with $\imath$ thus can be identified with a 2×2 real matrix of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, $a, b \in \mathbb{R}$ (see [9] III for a complete exposition).

Lemma 3. *Let $\Gamma = \mathbb{Z}\gamma^1 \oplus \mathbb{Z}\gamma^2 \subset \mathbb{C}$ be a lattice in $\mathbb{C}$ and $f : \mathbb{C}/\Gamma = \mathbb{T}^2 \to \mathbb{T}^2 = \mathbb{C}/\Gamma$ be a holomorphic map. Then its linearization is a $\mathbb{C}$-linear map of $\mathbb{C} = \mathbb{R}^2$ which preserves Γ, i.e. in the basis γ^1, γ^2 (over $\mathbb{R}$) of $\mathbb{C}$ it is an integral 2×2 matrix $A_f \in \mathcal{M}_{2\times 2}(\mathbb{Z}) \cap \mathcal{M}_{1\times 1}(\mathbb{C})$.*

Lemma 3 is proved in Section 4.

Theorem 4. *Let $\mathbb{T}^2$ be a Riemann surface of genus 1, and let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a holomorphic map and $A_f \in \mathcal{M}_{2\times 2}(\mathbb{Z}) \cap \mathcal{M}_{1\times 1}(\mathbb{C})$ its linearization matrix. Then*

$$\mathrm{HPer}_{\mathcal{H}}(f) = \mathrm{HPer}^{\mathcal{H}}(f) = \mathrm{HPer}(f) = \mathrm{HPer}([A_f]).$$

Moreover we have the following mutually disjoint cases:

(*E*) $\mathrm{HPer}(f) = \emptyset$ *if and only if* $A_f = \mathrm{Id}$.

(*F*) $\mathrm{HPer}(f) \neq \emptyset$ *is finite if and only if*

(*a*) $A_f = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ *and* $\mathrm{HPer}(f) = \{1\}$,

(*b*) $A_f = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ *and* $\mathrm{HPer}(f) = \{1\}$,

(*c*) $A_f = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ *or* $A_f = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ *and* $\mathrm{HPer}(f) = \{1, 2\}$.

(*G*) *For the remaining A_f the set* $\mathrm{HPer}(f)$ *is infinite and is equal to* $\mathbb{N}$ *with two exceptions, called* special cases (*see [15, 12]*) *:*

(a) $A_f = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$ *and* $\mathrm{HPer}(f) = \mathbb{N} \setminus \{2\}$,

(b) $A_f = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$ *or* $A_f = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$ *and* $\mathrm{HPer}(f) = \mathbb{N} \setminus \{2, 3\}$.

Theorem 4 is proved in Section 4.

Remark 1. Since the statement of Theorem 4 does not depend on a complex structure on the torus, we will not point out it in next.

Observe that opposite to the case of sphere $\mathbb{S}^2$ the torus $\mathbb{T}^2$ has infinitely many different complex structures. On the other side opposite to the case $X = \mathbb{S}^2$, where there are many holomorphic maps (all rational), in the case $X = \mathbb{T}^2$ we have very few holomorphic maps (affine maps). But in the latter case $\mathrm{HPer}_{\mathcal{H}}(f) = \mathrm{HPer}(f)$, consequently a holomorphic representant in $[f]$ determines the complex form of A_f and so of $\mathrm{HPer}(f)$.

Moreover in the next proposition we show that $\mathrm{P}(f) = \mathrm{HPer}(f)$ for a holomorphic map of $\mathbb{T}^2$.

Proposition 5. *Let* $f : \mathbb{T}^2 \to \mathbb{T}^2$ *be a holomorphic map. Then*

$$\mathrm{HPer}(f) \ = \mathrm{Per}(f)\,.$$

As a direct consequence of Proposition 5 we get the following Šarkovsky type result.

Corollary 6. *Let* $\mathbb{T}^2$ *be the torus with any complex structure and* $f : \mathbb{T}^2 \to \mathbb{T}^2$ *be a holomorphic map. If* f *has a periodic point of period* $n \geqslant 3$ *then for every continuous deformation* g *of* f

$$\mathbb{N} \setminus \{1, 2\} \subset \mathrm{Per}(g) = \mathrm{HPer}(g)\,.$$

If f *has a periodic point of period* $n \geqslant 4$ *then for every continuous deformation* g *of* f

$$\mathrm{Per}(g) = \mathrm{HPer}(g) = \mathbb{N}\,.$$

We also postpone the proof of Proposition 5 and Corollary 6 to Section 4.

Let $f : X \to X$ be a self-map of a compact manifold and $L(f) \in \mathbb{Z}$ be its Lefschetz number (see [13] for the definition). The celebrated *Lefschetz Fixed Point Theorem* says that if $L(f) \neq 0$, then f has a fixed point. Therefore if $L(f^m) \neq 0$ then f has a periodic orbit of period a divisor of m, i.e. not necessarily of minimal period m. The converse of the theorem is not true (see for instance [4]).

Although the Lefschetz numbers $L(f^m)$ contain information about the fixed points of f^m, they cannot be used to study the existence of periodic points of a given period. To deal with this problem the *periodic Lefschetz numbers* were introduced in [6] and in the context of a holomorphic map in [16]. Some authors call the periodic Lefschetz numbers as the *Dold multplicities*, see [13] for an exposition of known results and a literature. The Lefschetz m-periodic number is defined as

$$l_m(f) = \sum_{r|m} \mu(r) L(f^{m/r}),$$

where μ is the Möbius function (cf. [18]). By the Möbius inversion formula (see for example [18]),

$$L(f^m) = \sum_{r|m} l_r(f).$$

Suppose now that X is a surface of genus $\mathrm{g}(X) \geqslant 2$.

Theorem 7. *Let X be a Riemann surface of genus $\mathrm{g}(X) \geqslant 2$. Then every holomorphic map $f : X \to X$ is invertible in the class of holomorphic maps, f has a finite order $m \geqslant 1$, and f belongs to the finite group* $\mathrm{Aut}(X)$ *of conformal automorphisms of X. Moreover we have*

$$\mathrm{HPer}_{\mathcal{H}}(f) = \mathrm{HPer}^{\mathcal{H}}(f) = \mathrm{Per}(f),$$

and

(*i*) $m \in \mathrm{Per}(f)$,
(*ii*) $k|m$, $k < m$ *belongs to* $\mathrm{Per}(f)$ *if and only if there exists a prime p dividing k such that $L(f^k) > L(f^{k/p})$ or equivalently $l_k(f) \neq 0$.*

Theorem 7 is proved in Section 5.

Theorem 8. *Let X be a Riemann surface of genus $\mathrm{g}(X) \geqslant 2$. Let $h \in \mathrm{Aut}(X)$ and m be the order of h. Then the following statements hold.*

(*i*) *If h acts free then* $\mathrm{HPer}(f) = \mathrm{HPer}^{\mathcal{H}}(f) = \mathrm{Per}(f) = \{m\}$.
(*ii*) *If h does not act freely then for every $k|m$, $k < m$, we have that $k \in \mathrm{HPer}(h)$ if and only if $k \in \mathrm{Per}(h)$.*

Theorem 8 is also proved in Section 5.

Boyland [3] and also Hart and Keppelmann [10] worked with the general case of a homeomorphism f of order $m > 1$ of a compact Riemann surface of genus $g \geqslant 2$. They studied the more general problem of the dynamics of f [3], or of the reducibility of the Nielsen classes for the powers of f^k when k divides m [10]. However they do not provide any more information for the case discussed here which we got by elementary considerations. But due to a classical fact on homeomorphisms of hyperbolic compact surfaces a part of their results is a consequence of what we present here. We have (see [11])

Theorem 9 (Hurwitz). *Given a finite group of homeomorphisms G of a compact topological surface X of genus $g \geqslant 2$, there is a structure of a Riemann surface on X, i.e. a complex analytical structure on X, in which G is a group of conformal automorphisms.*

Consequently due to Theorem 9 we have a lot of information about homeomorphisms of finite order which are known in the classical theory of conformal automorphisms (see [9, 19] for more information). We would like to point out the Sierakowski paper [19]. For a given conformal map, he has shown how find out all appearing minimal periods in an effective algebraic way.

Remark 2. If f and h are two finite order preserving orientation homeomorphisms of a compact surface X of genus $\geqslant 2$, then f is an automorphism in a complex structure and so is g due to the Hurwitz theorem. But from this theorem does not follow that they are conformal, i.e. holomorphic maps, in the same complex analytic structure of X, thus belong to the same group of $\mathrm{Aut}(X)$.

Let $f : X \to X$ be a homeomorphism of a compact manifold. We define

$$\mathrm{HPer}_{\mathrm{Homeo}}(f) = \bigcap_{g \sim f} \mathrm{Per}(g), \tag{5}$$

where $g : X \to X$ is a homeomorphism of X. Note that by the definition we have $\mathrm{HPer}(f) \subset \mathrm{HPer}_{\mathrm{Homeo}}(f)$.

Theorem 10. *Suppose that X is a Riemann surface of genus $\geqslant 2$. Let $f : X \to X$ be a holomorphic map, i.e. o conformal map of finite order $m > 1$. Then*

- (a) *For every $k|m$, $k < m$ we have that $k \in \mathrm{HPer}_{\mathrm{Homeo}}(f)$ if and only if $k \in \mathrm{HPer}(f) = \mathrm{Per}(f)$.*
- (b) *Always $m \in \mathrm{HPer}_{\mathrm{Homeo}}(f)$.*

Theorem 10 is also proved in Section 5.

2. Proof of Theorem 1

Theorem 1 is a consequence of the following proposition.

Proposition 11. *Let $d \geqslant 2$ and let $f : \mathbb{S}^d \to \mathbb{S}^d$ be a continuous map. Then*

$$\mathrm{HPer}(f) = \begin{cases} \{1\} & \textit{if } \deg(f) \neq -1, \\ \emptyset & \textit{if } \deg(f) = -1, \end{cases}$$

for d even; and

$$\mathrm{HPer}(f) = \begin{cases} \{1\} & \textit{if } \deg(f) \neq 1, \\ \emptyset & \textit{if } \deg(f) = 1, \end{cases}$$

for d odd.

Proof. Since $d \geqslant 2$ the sphere $\mathbb{S}^d$ can be represented as $\mathbb{S}^d = \sum \mathbb{S}^{d-1}$, $d-1 \geqslant 1$, where

$$\sum \mathbb{S}^{d-1} = \mathbb{S}^{d-1} \times I / (\mathbb{S}^{d-1} \times \{0\} \cup \mathbb{S}^{d-1} \times \{1\})$$

and $I = [0, 1]$. Denote by (z, t) the coordinates in $\mathbb{S}^d = \sum \mathbb{S}^{d-1}$.

Let r be the degree of f. Then let $\varphi_r : \mathbb{S}^{d-1} \to \mathbb{S}^{d-1}$ be any map of $\deg(\varphi_r) = r$ (for $d-1 = 1$ we can take $\varphi_r(z) = z^r$).

Let next $\eta : I \to I$ be a continuous map such that $\eta(0) = 0$, $\eta(1) = 1$ and $\eta(t) > t$. We consider the Shub map $h : \mathbb{S}^d \to \mathbb{S}^d$ defined by

$$h = h_{r,\eta}([z,t]) := [\varphi_r(z), \eta(t)].$$

Note that h is continuous, $\deg(h) = r$, $k \sim f$ and that all the periodic points of h are the two fixed points $\{[\mathbb{S}^{d-1} \times \{0\}], [\mathbb{S}^{d-1} \times \{1\}]\}$. Consequently

$$\mathrm{HPer}(h) \subset \mathrm{Per}(h) = \{1\}. \tag{6}$$

If d is even and $\deg(f) = -1$ then $f \sim -\mathrm{Id}$. But $\mathrm{Per}(-\mathrm{Id}) = \{2\}$, therefore

$$\mathrm{HPer}(h) \subset \{1\} \cap \{2\} = \emptyset.$$

Analogously if d is odd and $\deg(f) = 1$ then $f \sim \mathrm{Id}$. But then $\mathrm{Id} \sim \varphi$ with $\mathrm{Fix}(\varphi) = \emptyset$. Note that φ can be taken as the one–time map of a flow φ_t defined by nonzero vector field on $\mathbb{S}^d$. Such a vector field exists by the Poincaré–Hopf Theorem see for more details [17]. Consequently $1 \notin \mathrm{HPer}(f)$ which gives $\mathrm{HPer}(f) = \emptyset$ taking into account (6). This completes the proof of the proposition. ■

Proof of Theorem 1. Since $d = 2$ and $\deg(f) \geqslant 0$ for any holomorphic map, the statement of the theorem follows from Proposition 11. ■

Remark 3. We remark that Proposition 11 is a very special case of a general fact which can be concluded from Jezierski–Wecken Theorem for periodic points, however only in dimension $d \geqslant 3$ (see [13] for more details on the Jezierski–Wecken theorem for periodic points). This fact states that if $f : X \to X$ is a self map of a simply connected compact piecewise–linear manifold of dimension $d \geqslant 3$, then

$$\mathrm{HPer}(f) = \begin{cases} \{1\} & \text{if } L(f) \neq 1, \\ \emptyset & \text{if } L(f) = 1, \end{cases}$$

where $L(f)$ is the Lefschetz number of f.

3. Proof of Theorem 2

To prove Theorem 2 we begin with a well-know fact, which proof we include for a convenience of the reader.

Proposition 12. *Any two rational self maps of $S^2 = \mathbb{C}_\infty$ are holomorphically homotopic if and only if they are homotopic. Consequently,* $\mathrm{HPer}^{\mathcal{H}}(f) = \mathrm{HPer}_{\mathcal{H}}(f)$.

Proof. We have to show that two homotopic rational maps are holomorphically homotopic. Since $\deg(f)$ determines the homotopy class of a self-map of a sphere, it is enough to show that every map rational map f of $\deg(f) = d(f) = d$ is holomorphically homotopic to $z \mapsto z^d$. Let $f(z) = P(z)/Q(z)$, where P, Q are polynomial of $d(P) \leqslant d$, $d(Q) \leqslant d$ relatively prime.

The case $d(f) = 0$ is obvious, since then P, Q are constant thus holomorphically homotopic.

Suppose that $d(f) \geqslant 1$. We assume first that $d(P) = d$ and $d(Q) \leqslant d$.

If $d(Q) = 0$ then $Q = c$ is a constant, and $f(z) = P(z)/c = P'(z)$, where $P'(z) = a'_d z^+ \cdots + a'_1 z + a'_0$ is a polynomial of degree d. A holomorphic homotopy $P'_t(z) = a'_d z^d + (1-t)(a'_{d-1} z^{d-1} + \ldots + a'_0)$ shows that $f(z) \approx az^d$, $a \neq 0$. But obviously $az^d \approx z^d$ which shows the statement in this case.

So we can assume that $1 \leqslant d(Q) \leqslant d$. To short notation put $\tilde{d} = d(Q)$.

Next observe that we can assume that $Q(0) \neq 0$ deforming holomorphically Q (thus f) if necessary. Of course, if $P(0) = 0$ then $Q(0) \neq 0$, as they are relatively prime. Let $P(z) = a \prod_{i=1}^{d} (z - z_i)$, $a \neq 0$, and $Q(z) = \tilde{a} z^k \prod_{j=1}^{d'} (z - \tilde{z}_j)$, $\tilde{a} \neq 0$, $z_j \neq 0$ for $1 \leqslant j \leqslant d'$, $k + d' = \tilde{d}$, $k > 0$.

Next, let $c \in \mathbb{C}$ be a constant such that we have $[0, c] \cap \{z_1, \ldots, z_d\} = \emptyset$.

It is geometrically obvious that there exist pathes $\sigma_j(t) : [0,1] \to \mathbb{C}$, $1 \leqslant j \leqslant d'$, such that:

$$\sigma_j(0) = \tilde{z}_j, \quad \sigma_j(1) = c \quad \text{for every } 1 \leqslant j \leqslant d',$$

$$\sigma_i(t) \cap \sigma_j(t) = \{c\} \quad \text{for } i \neq j,$$

and

$$\{z_1, \ldots, z_d\} \cap \left(\bigcup_{j=1}^{d'} \sigma_j(t) \right) = \emptyset, \qquad [0, c] \cap \left(\bigcup_{j=1}^{d'} \sigma_j(t) \right) = \{c\}.$$

Put $Q_t(z) = \tilde{a}((1-t)z - tc)^k \prod_{j=1}^{d'} (z - \sigma_j(t))$. Note that $\forall\ t \in [0,1]$ $Q_t(z)$ has not a common zero with $P(z)$ and $h_t(z) = P(z)/Q_t(z)$ gives a holomorphic homotopy to a map of the form $\tilde{c}\, P(z)$. The case $1 \leqslant d \leqslant \tilde{d}$ follows by the same argument. ■

Proof of Theorem 2. . Suppose that $\deg(f) = d(f) = 0$. Since f is holomorphic, then f is constant. Since $\mathrm{Per}(\text{constant}) = \{1\}$, it follows that

$$\mathrm{HPer}^{\mathcal{H}}(f) = \mathrm{HPer}_{\mathcal{H}}(f) = \mathrm{Per}(\text{constant}) = \{1\}.$$

Next assume that $\deg(f) = d(f) = 1$. Then $f = (az + b)/(cz + d)$ with $ad - bc \neq 0$ is a Möbius map. We claim that every Möbius map f is holomorphically homotopic to a Möebius map h such that $\mathrm{Per}(h) = \{1\}$.

From Proposition 12 it follows that $f \approx z \approx az$ with any $a \in \mathbb{C} \neq 0$. Taking $h(z) = az$ we see that the equation $h^n(z) = z$, i.e. $a^n z = z$ has only two solutions $z = 0, \infty$ if a is not a root of unity. In other words $\mathrm{P}(h) = \{0, \infty\}$ and these two points are fixed points. This shows that $\mathrm{HPer}_{\mathcal{H}}(f) \subset \mathrm{Per}(h) = \{1\}$.

On the other hand every Möbius transformation has at least two fixed points (cf. [2]) which shows that $\{1\} \in \mathrm{HPer}\, f_{\mathcal{H}}(f)$ and completes the argument for $d(f) = 1$.

Now we shall consider the case $\deg(f) \geqslant 2$. From the Baker Theorem (see [2] or [8]) it follows that

$$\begin{array}{lll} \mathbb{N} \setminus \{2,3\} \subset \mathrm{Per}(f) & \text{if } d(f) = 2, & \\ \mathbb{N} \setminus \{2\} \subset \mathrm{Per}(f) & \text{if } d(f) = 3, 4, & \text{and} \\ \mathbb{N} = \mathrm{HPer}^{\mathcal{H}}(f) & \text{if } d(f) > 4, & \end{array}$$

for every rational map of $\mathbb{S}^2$.

Moreover there are known rational maps h of degree 2 such that $2 \notin \mathrm{Per}(h)$ or $3 \notin \mathrm{Per}(h)$, or of degree 3 or 4 such that $2 \notin \mathrm{Per}(h)$, see again [2]. This means that

$$\begin{array}{lll} \mathrm{HPer}^{\mathcal{H}}(f) \subset \mathbb{N} \setminus \{2,3\} & \text{if } d = 2, & \text{and} \\ \mathrm{HPer}^{\mathcal{H}}(f) = \mathbb{N} \setminus \{2\} & \text{if } d = 3, 4. & \end{array}$$

The above inclusions show the statement for the discussed case which completes the proof of Theorem 2. ■

Remark 4. We must say that Proposition 12 is a special case of a more general result of G. Segal [20]. He proved the following. Let F_d denote the space of all rational mappings of $\mathbb{S}^2$ of degree d and Map_d the space of all continuous mappings of $\mathbb{S}^2$ of degree d. Then the natural inclusion $\iota : F_d \subset \mathrm{Map}_d$ is a homotopy equivalence up to dimension d, i.e. $\pi_j(\iota)$ is an isomorphism for $0 \leqslant j \leqslant d$. In particular $\pi_0(\iota) : \pi_0(F_d) \equiv \pi_0(\mathrm{Map}_d)$, which is exactly the statement of Proposition 12. On the other hand, a proof of this general theorem is long and uses difficult arguments of algebraic topology.

4. Proof of Theorem 4

First we prove Lemma 3.

Proof of Lemma 3. It is known that every holomorphic map $f : \mathbb{C}/\Gamma = \mathbb{T}^2 \to \mathbb{T}^2 = \mathbb{C}/\Gamma$ is of the form $f = \varphi + c_0$, where $\varphi : \mathbb{T}^2 \to \mathbb{T}^2$ is a homomorphism of the complex Lie group $G = \mathbb{C}/\Gamma = \mathbb{T}^2$ and c_0 is a translation by $c_0 \in \mathbb{T}^2$, see Proposition III. 11.5 of page 136 in [9]. ■

Remark 5. Roughly speaking this follows from the fact that a lift $\tilde{f} : \mathbb{C} \to \mathbb{C}$ of a holomorphic map $f : \mathbb{T}^2 \to \mathbb{T}^2$ is a holomorphic map which preserves the lattice Γ. Consequently it is homotopic to a linear (here over $\mathbb{C}$) map of $\mathbb{C}$. Next, one can show that two homotopic holomorphic maps to a complex torus are equal up to a translation by a constant.

The fact that $\varphi : \mathbb{T}^2 \to \mathbb{T}^2$ is a homomorphism of the complex Lie group $G = \mathbb{T}^2$ means that φ has a lift to a homomorphism $\Phi : T_eG = \mathbb{C} \to \mathbb{C} = T_eG$ which is a $\mathbb{C}$–linear map. Since $\Phi(0) = 0$, and $\mathbb{C}$ is contractible, Φ is unique.

Observe that $f \approx \varphi$ by the homotopy $f_t := \varphi + tc_0$. Consequently $A_f = A_\varphi$. On the other hand A_φ is equal to the matrix of Φ, because $\Phi(\Gamma) \subset \Gamma$, $\Gamma \equiv \mathbb{Z}^2$, as φ is the factor of Φ. This shows that A_f is $\mathbb{C}$–linear and completes the proof of the lemma.

Before proving Theorem 4 we need the next result.

Lemma 13. *If $f, g \in \mathcal{H}ol(\mathbb{T}^2)$ and $f \sim g$ then $f \approx g$.*

Proof. Note that $f \sim g$ if and only if $f_\sharp = g_\sharp$ for the map induced on $\pi_1(\mathbb{T}^2) = \mathbb{Z}^2$.

Let $f = \varphi + c_0$ and $g = \psi + \bar{c}_0$ where $\varphi = \Phi/\Gamma$ and $\psi = \Psi/\Gamma$ are the factors of the homomorphisms ($\mathbb{C}$–linear maps) of Φ and Ψ respectively. We showed that $f_\sharp = \Phi$ and $g_\sharp = \Psi$, consequently $f = \varphi + c_0$ and $g = \varphi + \bar{c}_0$. Now $f_t = \varphi + (1-t)c_0 + t\bar{c}_0$ gives the required holomorphic homotopy. ■

Proof of Theorem 4. Since $A_f = A \in \mathcal{M}_{2\times 2}(\mathbb{Z}) \cap \mathcal{M}_{1\times 1}(\mathbb{C})$ it follows that

$$A = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}, \tag{7}$$

where $\alpha, \beta \in \mathbb{Z}$.

Now we can use the classification of all possible homotopy periods of the continuous self maps of $\mathbb{T}^2$ given in [1]. However we will refer to a reformulation of this theorem presented in Theorem 1.3 of [12] because it is stated in the terms that we use here, this way of formulating the results was first introduced in [15].

The cases (E), (F) and (G) are distinguished by the spectrum $\sigma(A)$ of characteristic polynomial $\chi_A(\lambda)$ of A.

Let $\chi_A(\lambda) = \lambda^2 - a\lambda + b$ be the characteristic polynomial. Note that $a = \operatorname{trace}(A)$ and $b = \det(A)$. But here additionally

$$\begin{array}{ll} a = 2\alpha \in 2\mathbb{Z} & a \text{ is even, and} \\ b = \alpha^2 + \beta^2 \geqslant 0 & \text{is a sum of natural squares.} \end{array} \tag{8}$$

Case (E). The set $\mathrm{HPer}(f)$ is empty if and only if $1 \in \sigma(A)$. But every eigenspace of a $\mathbb{C}$–linear map is a $\mathbb{C}$–linear subspace, so $E_{\lambda=1} = \mathbb{C}$, i.e. $A = \mathrm{Id}$. It also can be derived from the condition $a + b + 1 = 0$ of Theorem 1.3 of [12] by a direct computation and using (8).

Case (F). It occurs if $\sigma(A) \subset \{0\} \cup \mathcal{U}$, where $\mathcal{U}$ denotes the set of all roots of unity.

If $0 \in \sigma(A)$ then $A = 0$ by the same argument as above, and $f = c_0$ is then a constant map. This gives $\mathrm{HPer}(f) = \{1\}$. So we are left with the cases when $\sigma(A)$ consists only of roots of unity (two conjugated), i.e. $b = \det(A) = \lambda\bar{\lambda} = 1$, $\chi_A(\lambda) = \lambda^2 - 2\alpha\lambda + 1$ with $\alpha \in \mathbb{Z}$ and $\alpha^2 + \beta^2 = b = 1$. The last has solutions $\alpha = \pm 1$ and $\beta = 0$, or $\alpha = 0$ and $\beta = \pm 1$. Note that the case $\alpha = 1$ and $\beta = 0$ reduces to the case (E).

The case $\alpha = -1$ and $\beta = 0$, i.e. $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ provides $\mathrm{HPer}(f) = \{1\}$, as it follows from [1] when $a = -2$ and $b = 1$.

The cases $\alpha = 0$ and $\beta = 1$, and $\alpha = 0$ and $\beta = -1$ correspond to the matrices $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, respectively. We have $\mathcal{H}\,\mathrm{Per}(f) = \{1, 2\}$, because $a = 0$ and $b = 1$. This completes the case (F).

Case (G). For proving this last case when $\mathrm{HPer}(f)$ is infinite we will also check which cases of (G) in Theorem 1.3 of [12] can be represented by a matrix A satisfying (7). Since $a = 2\alpha$ and $b = \alpha^2 + \beta^2$ we consider the following subcases.

Subcase $a = -2$ and $b = 2$. Then $\alpha = -1$ and $\beta = \pm 1$ which yields to $A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$ or $A = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$, and then $\mathrm{HPer}(f) = \mathbb{N} \setminus \{2, 3\}$.

Subcase $a \neq 0$ and $a + b + 1 = 0$. Substituting $a = 2\alpha$ and $b = \alpha^2 + \beta^2$ we get that $(\alpha + 1)^2 + \beta^2 = 0$. Hence $\alpha = -1$ and $\beta = 0$, which gives $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and reduces to case (F). Consequently this subcase does not provide new $A's$.

Subcase $a + b = 0$. Once more substituting $a = 2\alpha$ and $b = \alpha^2 + \beta^2$ we obtain $(\alpha + 1)^2 + \beta^2 = 1$. Solving this equation in α we get $\alpha = 0$, $\alpha = -1$, or $\alpha = -2$.

If $\alpha = 0$ then $\beta = 0$, consequently $A = 0$, already discussed in case (F). If $\alpha = -1$ then $\beta = \pm 1$ again discussed in case (F). Finally if $\alpha = -2$ and $\beta = 0$ we have $A = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$, and then $\mathrm{HPer}(f) = \mathbb{N} \setminus \{2\}$.

Subcase $a + b + 2 = 0$. This is the last case for (G) listed in Theorem 1.3 of [12]. Then $(\alpha + 1)^2 + \beta^2 + 1 = 0$ which has no real solutions. This completes the proof of Theorem 4. ∎

Proof of Proposition 5. In this proof we use the notion of Nielsen number and its basic properties presented in the next section. Also we need an information on the Nielsen numbers of self–maps of tori contained in [13, 4.3.3] but (see also [15]).

Let x be a fixed point of f of minimal period n. The point x is an isolated point of $\mathrm{Fix}(f^n)$, because otherwise f is the identity on $\mathbb{T}^2$ (see the proof of Lemma 3). Since $\mathrm{ind}(f^n, x) = \mathrm{sgn}\det(I - Df(x)) = 1 > 0$, the Nielsen class $[x] \in \mathcal{N}(f)$ of x is essential.

Any holomorphic map $f : \mathbb{T}^2 \to \mathbb{T}^2$ is of the form $f = [A] + [c]$, where $A : \mathbb{C} \to A$, $c \in \mathbb{C}$ and A is a complex matrix preserving a lattice $\Gamma \simeq \mathbb{Z} \oplus \mathbb{Z}$ (see [9]) as we already used in the proof of Lemma 3. Next observe that $f^n([x]) = [A]^n[x] + [\tilde{c}] = [A^n x] + [\tilde{c}_n]$, where $\tilde{c}_n = [A^{n-1}c + A^{n-2}c + \cdots + c]$. Now we can modify slightly the arguments of proof of [13, Th. 4.3.14] to get the following fact.

For every map $f : \mathbb{T}^2 \to \mathbb{T}^2$ of the form $f = [A] + [c]$, with A a complex matrix with integral entries, $A \neq 0, \mathrm{Id}$, and every $n \geqslant 1$ we have

1) f^n has exactly $\det(I - A^n)$ fixed points;
2) no two fixed points of f^n are Nielsen related (with respect to f^n); and
3) $N(f^n) = L(f^n) = \det(I - A^n) = \mathrm{Card}(\mathrm{Fix}(f^n))$.

In the proof of [13, Th. 4.3.14] such a statement is shown for a map of the form $[A] : \mathbb{T}^d \to \mathbb{T}^d$ of a d–dimensional torus, where A is any integral $d \times d$ matrix. Proofs of properties (2) and (3) are the same. We only comment the proof of (1). It is enough to consider the case $n = 1$ replacing A by A^n for the case of any n.

We have $\mathrm{Fix}(f) = \{[x] : [A][x]+[c] = [x]\}$ which is equal to $\{x \in \mathbb{C} : ((I-A)x = c \mod \Gamma$, i.e. the class $[x]$ is a fixed point if and only if $(I-A)x = c+v$ where $v \in \mathbb{Z}^2$ in a basis of Γ, or equivalently $x = (I-A)^{-1}c + w$ where $w = (I-A)^{-1}(v) \in (I-A)^{-1}(\Gamma) \subset \Gamma$. Note that $\Gamma' = (I-A)^{-1}(\mathbb{Z}^2)$ is a sub–lattice of Γ. Consequently, we have so many different fixed points as many distinct classes $[w]$ in Γ'/Γ. But $\mathrm{Card}(\Gamma'/\Gamma) = |\det(I-A)|$, by a geometric number theory (see [13, Th. 4.3.3] for the reference). Here $\det(I-A) > 0$, which proves (1).

Remind that for any map f and $k|n$, $k < n$, we have $\mathrm{Fix}(f^k) \subset \mathrm{Fix}(f^n)$. We can take any maximal such k, i.e. $k = n/p$, p a prime. If $[x]$ has the minimal period equal to n then there exists a prime p such that $[x] \in \mathrm{Fix}(f^n) \subset \mathrm{Fix}(f^{n/p})$, thus $\mathrm{Fix}(f^n) > \mathrm{Fix}(f^{n/p})$. Finally for a holomorphic map $f = [A] + [c]$ we have $N(f^n) = \mathrm{Fix}(f^n) > \mathrm{Fix}(f^{n/p}) = N(f^{n/p})$. But the latter implies that $n \in \mathrm{HPer}(f)$ as follows from the main of [15] (see [13, VI] for generalizations). ∎

Proof of Corollary 6. From Proposition 5 it follows that $\mathrm{Per}(f) = \mathrm{HPer}(f)$, and consequently we can use the tables listed in the statement of Theorem 4. But $3 \in \mathrm{HPer}(f)$ excludes the case (E) and all the cases of (F) of the statement of Theorem 4, leaving only the case (G) (a), i.e. $\mathrm{HPer}(f) = \mathbb{N} \setminus \{2\}$. Next, if $n \in \mathrm{HPer}(f)$ with $n \geqslant 4$ then $\mathrm{HPer}(f) = \mathbb{N}$ by the same theorem, which completes the proof. ∎

5. Proof of Theorem 7

Let X be a Riemann surface of genus $\mathrm{g}(X) \geqslant 2$ and $\mathrm{Aut}(X)$ be the group of conformal automorphisms of X. We have the following well known fact see for more details [9].

Proposition 14. *If $f \in \mathcal{H}ol(X)$, then $f \in \mathrm{Aut}(X)$.*

Lemma 15. *Let $f, h \in \mathrm{Aut}(X)$. If $f \approx h$, then $f = h$.*

Proof. The group $\mathrm{Aut}(X)$ is finite, consequently any curve $f_t : I \to \mathrm{Aut}(X)$ is constant, i.e. $\pi_0(\mathrm{Aut}(X)) = \mathrm{Aut}(X)$. ∎

Proposition 16. *Let $f, h \in \mathrm{Aut}(X)$. If $f \sim h$, then $f = h$.*

Proof. For $h \in \mathrm{Aut}(X)$ let $A_h : H_1(X,\mathbb{Z}) \to H_1(X,\mathbb{Z})$ be the induced homomorphism on the first homology group with integer coefficients which is isomorphic to $\mathbb{Z}^{2\mathrm{g}}$. Since $h \mapsto A_h$ is functorial, i.e. $A_{h_1 \cdot h_2} = A_{h_1} \cdot A_{h_2}$, we get a representation of $\mathrm{Aut}(X)$ in $GL(2g, \mathbb{Z})$. Since the homomorphism $h \mapsto A_h$ is a monomorphism, i.e. the representation is faithful (see page 270 of [9]), $A_h = A_f$ implies $f = h$. But $f \sim h$ implies $A_f = A_h$, so the proposition is proved. ∎

From Proposition 16 it follows immediately the next result because there is only one holomorphic map in each holomorphic class.

Corollary 17. *For a holomorphic map $f : X \to X$ of surface of genus $\mathrm{g}(X) \geqslant 2$ we have that $\mathrm{HPer}_{\mathcal{H}}(f) = \mathrm{HPer}^{\mathcal{H}}(f) = \mathrm{Per}(f)$.*

The next lemma is also well known and it is a special case of a theorem of Jiang and Guo [14]. They proved it with essentially weaker assumption that h is a preserving orientation homeomorphism of X.

In next we use the notion of *fixed point index* $\mathrm{ind}(f,x) \in \mathbb{Z}$ of f at a fixed point x(see [13] for a definition).

Lemma 18. *Let* $h \in \mathrm{Aut}(X)$. *Then*

(a) $\mathrm{Fix}(h)$ *is either* X *or finite.*
(b) *If* $\mathrm{Fix}(h)$ *is finite, then* $\mathrm{ind}(f,x) = 1$ *for all* $x \in \mathrm{Fix}(h)$.
(c) $L(h)$ *is the cardinality of the set* $\mathrm{Fix}(h)$ *if* $h \neq \mathrm{Id}$, *and* $2 - 2\mathrm{g}$ *if* $h = \mathrm{Id}$.

Proof. Suppose that $\mathrm{Fix}(h) \neq X$. Since h is an a holomorphic homeomorphism, $\mathrm{Fix}(h)$ is a complex analytic submanifold, thus either X or a finite set, hence statement (a) is proved. For an isolated fixed point x $Dh(x)$ is a complex map, thus if 1 is an eigenvalue of $Dh(x)$ then $Dh(x) = \mathrm{id}$. But this is impossible, because the each curve at x, and consequently each vector v tangent to it, should is mapped by h into another curve, so the vector v is mapped into another vector by $h(x)$. By the fixed point index property, then we have $\mathrm{ind}(f,x) = \mathrm{sgn}\det(\mathrm{Id} - Dh(x)$.

Note that if 1 is not an eigenvalue of $Dh(x)$, then $\mathrm{ind}(h,x) = \mathrm{sgn}(\det(\mathrm{Id} - Dh(x)))$, but $\mathrm{Id} - Dh(x)$ is a complex linear map which gives $\det(\mathrm{Id} - Dh(x)) > 0$. So statement (b) follows.

Statement (c) follows from (a) and (b). ∎

While $L(h)$ denotes the Lefschetz number of $h \in \mathrm{Aut}(X)$, we denote by $N(h)$ the Nielsen number of h, see for instance [13] for a definition.

Lemma 19. *Let* $h \in \mathrm{Aut}(X)$, $h \neq \mathrm{Id}$. *Then* $L(h) = N(h) = \mathrm{Card}(\mathrm{Fix}(h))$.

Proof. This lemma is also a special case of a theorem of Jiang and Guo [14] who proved that $L(h) = N(h)$ for any preserving orientation homeomorphism $h : X \to X$. They showed every $x \in \mathrm{Fix}(h)$ is singleton and $[x]$ is an essential Nielsen class. This here follows from Lemma 18. Consequently since the Nielsen number is the cardinality of the essential Nielsen classes we have

$$N(h) = \mathrm{Card}(\mathrm{Fix}(h)) = L(h).$$ ∎

Now we describe $\mathrm{Per}(h)$ for $h \in \mathrm{Aut}(X)$. Let m be the order of h.

Suppose first that h acts free on X, i.e. for all $x \in X$ we have that $h^m(x) = x$ and there is not $x \in X$ and $k|m$, $k < m$ such that $h^k(x) = x$. The latter implies that $L(h^k) = 0$ for every $k|m$, $k < m$.

Remark 6. We remark that it is well-known, that there is a necessary condition on an order m of a finite cyclic group acting freely on a compact manifold X saying that $m|\chi(X)$. The latter is equal to $2 - 2\mathrm{g}$ for a Riemann surface of genus g.

On the other hand there exist elements of $\mathrm{Aut}(X)$ with this property, i.e. of order $m|\chi(X) = 2 - 2\mathrm{g}$, see [9, 5]. In this case we have $\mathrm{Per}(h) = \{m\}$.

Now we suppose that there exists $k < m$, $k|m$ such that $\mathrm{Fix}(h^k) \neq \emptyset$. Note that

$$l|k \quad \text{implies} \quad \mathrm{Fix}(h^l) \subset \mathrm{Fix}(h^k). \tag{9}$$

Consequently $k \in \mathrm{Per}(h)$ if and only if there exists $x \in \mathrm{Fix}(h^k) \setminus \bigcup_{l|k} \mathrm{Fix}(h^l)$. From (9) it is enough to take only $l = k/p$ with p prime, the largest proper divisors of k. Indeed

$$\mathrm{Fix}(h^k) \setminus \bigcup_{\substack{l|k \\ l<k}} \mathrm{Fix}(h^l) = \mathrm{Fix}(h^k) \setminus \bigcup_{p|k} \mathrm{Fix}(h^{k/p}).$$

It shows that $k \in \mathrm{Per}(h)$ if and only if there exists $p|k$ such that $L(h^k) > L(h^{k/p})$.

Next we observe that $h : \mathrm{Fix}(h^k) \to \mathrm{Fix}(h^k)$ is a map of a finite set, because $h^k(h(x)) = h(h^k(x)) = h(x)$. Thus $\mathrm{Fix}(h^k) = \sum_{l|k} \mathrm{Card}\ \mathrm{P}_l(h)$, where $\mathrm{P}_l(h)$ denotes the set of all periodic points of h of period l. From the Möebius formula and using Lemmas 18 and 19 we get

$$\begin{aligned} \mathrm{Card}\ \mathrm{P}_k(h) &= \sum_{l|k} \mu\left(\frac{k}{l}\right) \mathrm{Card}\ \mathrm{Fix}(h^l) \\ &= \sum_{l|k} \mu\left(\frac{k}{l}\right) L(h^l) = \sum_{l|k} \mu\left(\frac{k}{l}\right) N(h^l). \end{aligned}$$

Now in our case we show that $m \in \mathrm{Per}(h)$. It is enough to show that $\mathrm{P}_m(h) \neq \emptyset$. But for all $k|m$, $k < m$ we have that $\mathrm{Fix}(h^k)$ is empty or finite. This shows that

$$X \setminus \bigcup_{\substack{k|m \\ k<m}} \mathrm{Fix}(h^k)$$

is open and dense, thus it is not empty. ■

To discuss a description of $\mathrm{HPer}(f)$ we have to recall the notion of Nielsen–Jiang prime periodic number $NP_k(f)$.

Let X be a finite CW–complex (or compact ENR), and $f : X \to X$ a map. We define the equivalence relation $x \underset{f^k}{\sim} y$ if and only if $x, y \in \mathrm{Fix}(f^k)$ and they are Nielsen related with respect to f^k (cf. [13]). The equivalent classes $[x]_{f^k}$ of this relation are denoted by $\mathcal{N}(f^k)$. For $l|k$ and $x, y \in \mathrm{Fix}(f^l) \subset \mathrm{Fix}(f^l)$ if x, y are f^l related they are f^k related This means that we have a map $\mathcal{N}(f^l) \to \mathcal{N}(f^k)$ defined by $[x]_{f^l} \mapsto [x]_{f^k}$.

A class $[x] \in \mathcal{N}(f^k)$, or also its orbit with respect to the action of f^k is called *reducible* if $[x] \underset{f^k}{\sim} [y]$, where $[y] \in \mathcal{N}(f^l)$ for some $l|k$, $l < k$. Otherwise $[x]$ is called *irreducible*.

By definition $N\mathrm{P}_k(f)$ is the cardinal of the classes $[x] \in \mathcal{N}(f^l)$ which are essential and irreducible. It is known that $NP_k(f)$ is a homotopy invariant and $NP_k(f) \leqslant \mathrm{Card}\ P_k(f)$ for every $k \geqslant 1$. The next result shows that a conformal automorphism of order m minimize the k–periodic points, $k < m$ in the homotopy class.

Lemma 20. *Let $h \in \mathrm{Aut}(X)$ and m be its order. Then for all $k|m$, $k < m$ we have that $NP_k(f) = \mathrm{Card}\ P_k(h)$.*

Proof. Let $l|k$, $l < k$ with $k < m$. Since $k < m$ we have that $\mathrm{Fix}(h^k)$ is finite or empty. By Lemma 19 $x \sim h^k y$ or $x \sim h^l y$ if and only if $x = y$, i.e. $[x] = \{x\}$ in $\mathcal{N}(h^k)$ or $\mathcal{N}(h^l)$ respectively. Consequently $[x] \in \mathcal{N}(h^k)$ is irreducible if and only if

$$x \in \mathcal{N}(h^k) \setminus \bigcup_{\substack{l|k \\ l<k}} \mathcal{N}(h^l) = \mathrm{Fix}(h^k) \setminus \bigcup_{\substack{l|k \\ l<k}} \mathrm{Fix}(h^l) = P_k(h).$$

Since every class is essential by Lemma 15 we get the statement. ■

Proof of Theorem 8. Since $\mathrm{Per}(f) \supset \mathrm{HPer}(f)$ we have that $\mathrm{HPer}(f) \subset \{m\}$.

For all $x \in X$ we have $h^m(x) = x$, and $x \sim_{h^m} x'$ for all $x, x' \in X$. Thus there is one Nielsen class of h^m. Since $L(h^k) = \chi(X) = 2-2\mathrm{g} \neq 0$, this class is essential.

On the other hand since $\mathrm{Fix}(h^k) = \emptyset$ for $k < m$, we get $\mathcal{N}(h^k) = \emptyset$ for $k < m$. Consequently this class $[x] \in \mathcal{N}(h^m)$ is irreducible, so $NP_m(h) = 1$. This gives

$$\mathrm{Card}\ P_m(f) \geqslant NP_m(f) = NP_m(h) = 1,$$

for every $f \sim h$.

Now we assume that there is $k < m$, $k|m$ such that $\mathrm{Fix}(h^k) \neq \emptyset$. By Lemma 19 for every $k \in \mathrm{Per}(h)$ we have that $NP_k(h) = \mathrm{Card}\ P_k(h) > 0$. Consequently for such k we have $\mathrm{Card}\ P_k(f) \geqslant NP_k(f) = NP_k(h) > 0$ for every map $f \sim h$. This shows that $k \in \mathrm{HPer}(h)$. ■

Proof of Theorem 10. The case (a) follows from the inclusions

$$\mathrm{Per}(f) \supset \mathrm{HPer}_{\mathrm{Homeo}}(f) \supset \mathrm{HPer}(f)$$

and Theorem 8 which says that $\mathrm{HPer}(f) \setminus \{m\} = \mathrm{Per}(f) \setminus \{m\}$.

For the case (b) note that in Theorem 8 we have already showed that $m \in \mathrm{HPer}(f)$ if f is a free homeomorphism of order m.

Let next h be a homeomorphism homotopic to a homeomorphism f of finite order m. First note that by the Epstein theorem if h homotopic to f, then h is isotopic to f (see [7]). Now we can use the Thurston theorem, which says that every preserving orientation homeomorphism of an orientable surface X of genus $\geqslant 2$ is isotopic to a homeomorphism g of X such that g is either (i) pseudo–Anosov, or (ii) of finite order, or (iii) reducible (see [21]). Since h is isotopic to f, the cases (i) and (iii) are excluded, i.e. h must have a finite order. To see that an order of h is equal to m we can once more use the argument of Proposition 16. Indeed h has the same order as $A_h = A_f = m$, since $h \sim f$, which ends the proof.

For a such h we have that

$$\bigcup_{\substack{k|m \\ k<m}} \mathrm{Fix}(h^k)$$

is a finite set, and consequently

$$P_m(h) = X \setminus \bigcup_{\substack{k|m \\ k<m}} \mathrm{Fix}(h^k)$$

is open and dense, thus non–empty. Consequently $m \in \mathrm{Per}(f)$, which shows that $m \in \mathrm{HPer}_{\mathrm{Homeo}}(f)$. ■

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References

[1] Ll. Alsedà, S. Baldwin, J. Llibre, R. Swanson, W. Szlenk, *Torus maps and Nielsen numbers*, Contemporary Math. **152** (1993), 1–7.

[2] A. Beardon, *Iteration of Rational Functions*, Graduate Texts in Mathematics **132**, Springer–Verlag, New York, 1991.

[3] P. Boyland, *Isotopy stability of dynamics on surfaces*, Geometry and topology in dynamics (Winston-Salem, NC, 1998/San Antonio, TX, 1999), 17–45, Contemp. Math., **246**, Amer. Math. Soc., Providence, RI, 1999.

[4] R. F. Brown, *The Lefschetz fixed point theorem*, Scott, Foresman and Company, Glenview, IL, 1971.

[5] C. Corrales, J. M. Gamboa, G. Gromadzki, *Automorphisms of Klein surfaces with fixed points*, Glagow Math. J. **41** (1999), 183–189.

[6] A. Dold, *Fixed point indices of iterated maps*, Invent. math. **74** (1983), 419–435.

[7] D. B. Epstein, *Curves on 2-manifolds and isotopies*, Acta Math. **115**, (1966), 83–107.

[8] N. Fagella, J. Llibre, *Periodic points of holomorphic maps via Lefschetz numbers*, Trans. Amer. Math. Soc. **352** (2000), 4711–4730.

[9] H. M. Farkas, K. Kra, *Riemann Surfaces*, Graduate Texts in Mathematics **71**, Springer–Verlag, New York, 1980.

[10] E. Hart, E. Keppelmann, *Nielsen periodic point theory for periodic maps on orientable surfaces*, Topology Appl. **153**, (2006), 1399–1420.

[11] B. von Kerejarto, Vorlesungen uber Topologie. I Flachentopologie, Springer Verlag, 1923.

[12] J. Jezierski, W. Marzantowicz, *Homotopy minimal periods for maps of three dimensional nilmanifolds*, Pacific J. Math. **209** (2003), 85–101.

[13] J. Jezierski, W. Marzantowicz, *Homotopy methods in topological fixed and periodic points theory*, Topological Fixed Point Theory and Its Applications **3**, Springer, Dordrecht, 2006.

[14] B. Jiang, J.H. Guo, *Fixed points of surface diffeomorphisms*, Pacific J. Math. **160** (1993), 67–89.

[15] B. Jiang, J. Llibre *Minimal sets of periods for torus maps*, Discrete and Continuous Dynamical Systems **4** (1998), 301–320.

[16] J. Llibre, *Lefschetz numbers for periodic points*, Contemporary Math. **152** (1993), 215–227.

[17] J. W. Milnor, *Topology from the differentiable viewpoint*, based on notes by David W. Weaver. Revised reprint of the 1965 original. Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1997.

[18] I. Niven, H.S. Zuckerman, *An introduction to the theory of numbers*, fourth edition, John Wiley & Sons, New York, 1980.

[19] M. Sierakowski, *Sets of periods for automorphisms of compact Riemann surfaces*, J. Pure Appl. Algebra **208** (2007), 561–574.

[20] G. Segal, *The topology of spaces of rational functions*, Acta Math. **143** (1979), 39–72.

[21] W. P. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc. **19** (1988), 417–431.

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