Functiones et Approximatio
40.2 (2009), 165–173

REFLECTION PRINCIPLE FOR QUASIMINIMIZERS

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Dedicated to Professor Bogdan Bojarski
on the occasion of his 75th birthday

Abstract: It is shown that the reflection principle holds for K–quasiminimizers in $\mathbb{R}^n, n \geqslant 2$, provided that $K \in [1, 2)$. For $n = 1$ the principle holds for all $K \geqslant 1$ and an example shows that K is not preserved in the reflection process. A local integrability result up to the boundary is proved for the derivative of a quasiminimizer in $\mathbb{R}^n, n \geqslant 1$; the result is needed for the reflection principle.

Keywords: quasilinear elliptic equation, reflection principle, the Sobolev space, quasiminimizer.

1. Introduction

Reflection principle is well-known for solutions of the degenerate quasilinear elliptic equation

$$\nabla \cdot A(x, \nabla u) = 0 \tag{1}$$

where $A(x, h) \cdot h \approx |h|^p, p > 1$, see [M]. More precisely, let H^+ be the upper half space in $\mathbb{R}^n, n \geqslant 1$, $P(x) = (x_1, ..., x_{n-1}, -x_n)$ the reflection in ∂H^+ and Ω an open set in H^+ such that there is an open set C in ∂H^+ with $C \subset \partial\Omega$. If now $u \in C(\Omega \cup C) \cap W^{1,p}_{\text{loc}}(\Omega)$ is a weak solution of (1) in Ω, then u can be extended by reflection to a solution of (1) in $\Omega^* = \Omega \cup C \cup P(\Omega)$ provided that $u|C = 0$. Here $W^{1,p}(\Omega)$ stands for the first order Sobolev space of $L^p(\Omega)$–functions whose first order distributional partial derivatives belong to $L^p(\Omega)$ and $W^{1,p}_{\text{loc}}(\Omega)$ is the corresponding local space. The reflection process also involves the reflection of the operator A to $P\Omega$. For the p–harmonic operator $A(x, h) = |h|^{p-2}h$, i.e. for the p–harmonic equation

$$\nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0, \tag{2}$$

the operator A remains invariant and hence the solutions to (2), the p–harmonic functions, satisfy the usual reflection principle. For the precise assumptions on A see [HKM].

2000 Mathematics Subject Classification: primary: 31B25, secondary: 35J65

Let $\Omega \subset \mathbb{R}^n$ be an open set and $K \geqslant 1$. A function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is called a K–*quasiminimizer* if

$$\int_{\Omega_1} |\nabla u|^p \, dx \leqslant K \int_{\Omega_1} |\nabla v|^p \, dx \tag{3}$$

for all open sets $\Omega_1 \subset\subset \Omega$ and for all functions v such that $u - v \in W^{1,p}_0(\Omega_1)$. The space $W^{1,p}_0(\Omega_1)$ is the subspace of $W^{1,p}(\Omega_1)$ whose functions have zero boundary values, i.e. they can be approximated by compactly supported functions in $W^{1,p}(\Omega_1)$. A function u is called a *quasiminimizer* if u is a K–quasiminimizer for some $K \geqslant 1$. If $K = 1$, then u is a *minimizer* and a solution of the p–harmonic equation (2) in Ω. For the theory of quasiminimizers we refer to [GG] and [KiM].

The proof for the reflection principle for solutions of the equation (1) breaks down to two separate problems. The first step is to show that the function u belongs to $W^{1,p}(U \cap H^+)$ for some neighborhood U of each point $x \in \partial H^+ \cap C$ and the second step is to show that the reflected u is a solution of the reflected equation in Ω^*. In Section 2 we consider the first step for quasiminimizers in $\mathbb{R}^n, n \geqslant 1$. For $n = 1$ a stronger form of this result can be found in [MS].

In Section 3 we show that the reflection principle applies to K–quasi-minimizers whenever $K \in [1, 2)$ in $\mathbb{R}^n, n \geqslant 1$. Our proof gives a rather rough estimate for the quasiminimizing constant of the reflected function but in Section 5 we show, by an example, that the reflection process does not preserve any $K > 1$.

We consider the reflection principle for quasiminimizers in the one dimensional case in Sections 4 and 5. In this case it turns out that the upper bound 2 for K is not needed. This remains an open problem in $\mathbb{R}^n, n \geqslant 2$.

2. Integrability of the derivative up to the boundary

The following theorem is a slightly extended counterpart of the local result which holds for A–harmonic functions, see [M, Lemma 2.8]. We use a version of the concept of relative p–capacity, see [HKM, p. 144]. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $E \subset \overline{\Omega}$. For a compact set K and $s > 0$ we denote by $K^s(\Omega)$ the open set $\{x \in \Omega : d(x, K) < s\}$. We say that E is of *zero p–capacity relative to* Ω if for all $\varepsilon > 0$ and for all compact sets $K \subset E$ there exists a function $\varphi \in C(K \cup \Omega) \cap W^{1,p}(\Omega)$ such that $\varphi|\Omega \setminus K^\varepsilon(\Omega) = 0$, $\varphi|K = 1$ and

$$\int_{\Omega} |\nabla \varphi|^p \, dx < \varepsilon.$$

It is easy to see that if $E \subset \Omega$ is of p–capacity zero, then E is of p–capacity zero relative to Ω but the converse is not true as simple examples show.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and C an open subset of $\partial\Omega$. Suppose that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a K–quasiminimizer in Ω and $v \in W^{1,p}(\Omega)$. If u and v are bounded and*

$$\lim_{y \to x, y \in \Omega} u(y) = \lim_{y \to x, y \in \Omega} v(y) \tag{4}$$

for all $x \in C$ except in a set E of p–capacity zero relative to Ω, then each point $x \in C$ has an open neighborhood U such that $u \in W^{1,p}(U \cap \Omega)$.

Proof. Let E be the set of points in C where (4) does not hold. For each $i = 1, 2, \cdots$ the set

$$\Big\{x \in C : \liminf_{y \to x,\, y \in \Omega} u(y) + 1/i \leqslant \limsup_{y \to x,\, y \in \Omega} u(y)\Big\}$$

is a closed set in C and hence the sets S_u and S_v where the limits

$$\lim_{y \to x,\, y \in \Omega} u(y), \ \lim_{y \to x,\, y \in \Omega} v(y)$$

do not exist are countable unions of compact sets. On the other hand the sets

$$E^+ = \Big\{x \in C : \limsup_{y \to x,\, y \in \Omega} v(y) < \liminf_{y \to x,\, y \in \Omega} u(y)\Big\},$$
$$E^- = \Big\{x \in C : \limsup_{y \to x,\, y \in \Omega} u(y) < \liminf_{y \to x,\, y \in \Omega} v(y)\Big\}$$

are open in C and hence also countable unions of compact sets. Since $E = S_u \cup S_v \cup E^+ \cup E^-$, the set E is of the form $\cup F_i$ where $F_i \subset \partial C$, $i = 1, 2, \cdots$, are compact sets of zero p–capacity relative to Ω.

Since u and v are bounded, $|u|, |v| \leqslant M$ for some $M > 0$. Let $\varepsilon > 0$ and for each $i = 1, 2, \ldots$ choose functions $\varphi_i \in C(F_i \cup \Omega) \cap W^{1,p}(\Omega)$ such that $\varphi_i | \Omega \setminus F_i^{\varepsilon/i}(\Omega) = 0$ and

$$\|\nabla \varphi_i\|_{L^p(\Omega)} \leqslant \frac{\varepsilon}{2^i} \tag{5}$$

with $0 \leqslant \varphi_i \leqslant 2M$ and $\varphi_i = 2M$ on F_i. This is possible since F_i is a compact set of zero p–capacity relative to Ω.

Fix $x_o \in C$ and let $r = d(x_o, \partial\Omega \setminus C)/2$. Choose a Lipschitz cut–off function Ψ^+such that $0 \leqslant \Psi^+ \leqslant 2M$, $\Psi^+ = 0$ in $B(x_o, r/2)$ and $\Psi^+ = 2M$ on $\mathbb{R}^n \setminus B(x_o, r)$. The function

$$w_o = \Psi^+ + \sum_i \varphi_i$$

is continuous in Ω since each point $y \in \Omega$ has a neighborhood where only a finite number of the functions φ_i are non–zero. Moreover, from (5) it follows

$$\|\nabla w_o\|_{L^p(\Omega)} \leqslant \|\nabla \Psi^+\|_{L^p(\Omega)} + \sum_i \|\nabla \varphi_i\|_{L^p(\Omega)} \leqslant \|\nabla \Psi^+\|_{L^p(\Omega)} + \varepsilon. \tag{6}$$

Write $U_\varepsilon^+ = \{y \in \Omega \cap B(x_o, r) : u(y) > v(y) + w_o(y) + \varepsilon\}$. Now $\overline{U_\varepsilon^+}$ is compact in Ω. To see this let $x_i \in U_\varepsilon^+$ and $x_i \to x \in C$. Then

$$\liminf_{i \to \infty} u(x_i) \geqslant v(x) + \varepsilon$$

provided that $x \in C \setminus E$ and

$$\liminf_{i\to\infty} u(x_i) \geqslant \liminf_{i\to\infty} v(x_i) + 2M + \varepsilon \geqslant M + \varepsilon$$

provided that $x \in C \cap E$. Hence $\overline{U}_\varepsilon^+$ does not meet C. It easily follows that $\overline{U}_\varepsilon^+$ cannot meet $\partial B(x_o, r)$ because $w_o \geqslant \Psi \geqslant 2M$ there.

Since $\overline{U}_\varepsilon^+$ is compact in Ω and $u - (v + w_o + \varepsilon) \in W_o^{1,p}(U_\varepsilon^+)$, we can use the quasiminimizing property of u although U_ε^+ need not be open in Ω, see [KiM, Lemma 3.2]. This yields

$$\begin{aligned} \|\nabla u\|_{L^p(U_\varepsilon^+)} &\leqslant K^{1/p} \|\nabla (v + w_o)\|_{L^p(\Omega_\varepsilon^+)} \\ &\leqslant K^{1/p} \big(\|\nabla v\|_{L^p(\Omega_\varepsilon^+)} + \|\nabla w_o\|_{L^p(\Omega)} \big) \\ &\leqslant K^{1/p} \big(\|\nabla v\|_{L^p(\Omega_\varepsilon^+)} + \|\nabla \Psi^+\|_{L^p(\Omega)} + \varepsilon \big) < \infty. \end{aligned} \tag{7}$$

Letting $\varepsilon \to 0$ we obtain from the Lebesgue monotone convergence theorem that

$$\|\nabla u\|_{L^p(U^+)} \leqslant K^{1/p} \big(\|\nabla v\|_{L^p(U^+)} + \|\nabla \Psi^+\|_{L^p(\Omega)} \big) \tag{8}$$

where $U^+ = \{ y \in \Omega \cap B(x_o, r) : u(y) > v(y) + \Psi^+(y) \}$.

Setting $\Psi^- = -\Psi^+$ and using $-w_o$ instead of w_o we obtain a similar estimate

$$\|\nabla u\|_{L^p(U^-)} \leqslant K^{1/p} \big(\|\nabla v\|_{L^p(U^-)} + \|\nabla \Psi^-\|_{L^p(\Omega)} \big) \tag{9}$$

where $U^- = \{ y \in \Omega \cap B(x_o, r) : u(y) < v(y) + \Psi^-(y) \}$. Since

$$\{ y \in \Omega \cap B(x_o, r/2) : u(y) \neq v(y) \} \subset U^+ \cup U^-$$

and $\nabla u = \nabla v$ a.e. in the set $\{u(y) = v(y)\}$, we obtain from (8) and (9)

$$\|\nabla u\|_{L^p(\Omega \cap B(x_o, r/2))} \leqslant 2K^{1/p} \big(\|\nabla v\|_{L^p(\Omega)} + \|\nabla \Psi^+\|_{L^p(\Omega)} \big) < \infty. \tag{10}$$

Since u is bounded, this shows that $u \in W^{1,p}(\Omega \cap B(x_o, r/2)$ and $U = \Omega \cap B(x_o, r/2)$ is the required neighborhood of x_o. The theorem follows. ■

Remark 2.2. The boundedness of u cannot be dispensed in Theorem 2.1 for $n > 1$; the Poisson kernel in the unit ball gives a typical counterexample in the case $p = 2$.

The next theorem is a global version of Theorem 2.3.

Theorem 2.3. *Suppose that u is a bounded K–quasiminimizer in a bounded open set Ω of $\mathbb{R}^n$ and that $v \in W^{1,p}(\Omega)$ is bounded. If*

$$\lim_{y \to x, y \in \Omega} u(y) = \lim_{y \to x, y \in \Omega} v(y) \tag{11}$$

for all $x \in \partial\Omega$ except in a set E of p–capacity zero relative to Ω, then $u \in W^{1,p}(\Omega)$ and

$$\int_\Omega |\nabla u|^p \, dx \leqslant K \int_\Omega |\nabla v|^p \, dx. \tag{12}$$

Moreover, $u - v \in W_0^{1,p}(\Omega)$ provided that the set E is of p–capacity zero.

Proof. The proof is similar to that of Theorem 2.1 except that the function Ψ is not needed. The estimate (12) then follows from (10); the constant 2 can be omitted in this case. If the set E is of p–capacity zero, then [HKM, Theorem 4.5] yields $u - v \in W_0^{1,p}(\Omega)$. ■

3. Reflection principle for quasiminimizers

In this section we show that K–quasiminimizers can be reflected if K satisfies $1 \leqslant K < 2$. It remains an open question if an upper bound for K is needed in $\mathbb{R}^n, n \geqslant 2$.

If u is a K–quasiminimizer in an open set $\Omega \subset \mathbb{R}^n$, then, after a redefinition on a set of measure zero, u is locally Hölder continuous in Ω. Thus we can always assume that a quasiminimizer is continuous. We denote by $P : \mathbb{R}^n \to \mathbb{R}^n$ the reflection $P(x) = (x_1, x_2, \cdots, -x_n)$ in ∂H^+.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set in the upper half space H^+ and let $C \subset \partial\Omega \cap \partial H^+$ be open in ∂H^+. Suppose that $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\Omega \cup C)$ is a K–quasiminimizer in Ω with $u|C = 0$ and define $u(x) = -u(Px), x \in P\Omega$. Then for $1 \leqslant K < 2$ the reflected function u is a K'–quasiminimizer,*

$$K' = \frac{K}{2-K},$$

in $\Omega^* = \Omega \cup C \cup P\Omega$.

Proof. Theorem 2.1 together with the continuity of u in $\Omega \cup C$ and the ACL^p characterization of the Sobolev functions, see e.g. [HKM, p. 260], implies that the function u belongs to $C(\Omega^*) \cap W^{1,p}_{\mathrm{loc}}(\Omega^*)$. Let Ω_1 be an open subset of Ω^* with $\overline{\Omega}_1$ compact in Ω^* and let v satisfy $v - u \in W_0^{1,p}(\Omega_1)$. We need to show

$$\int_{\Omega_1} |\nabla u|^p \, dx \leqslant K' \int_{\Omega_1} |\nabla v|^p \, dx. \tag{13}$$

Continue the function v to $\Omega^* \setminus \Omega_1$ as u. Then the function v belongs to $W^{1,p}_{\mathrm{loc}}(\Omega^*)$. Set $v^*(x) = -v(Px), x \in P\Omega_1$. Now $v^* - u \in W_0^{1,p}(P\Omega_1)$ and we can continue v^* to $\Omega^* \setminus P\Omega_1$ as u. Again the function v^* belongs to $W^{1,p}_{\mathrm{loc}}(\Omega^*)$. Write $w = (v + v^*)/2$ and $\Omega_1^* = \Omega_1 \cup P\Omega_1$. The open set Ω_1^* is symmetric with respect to ∂H^+ and for each $x \in \partial H^+$

$$w(x) = (v(x) + v^*(x))/2 = (v(x) - v(Px))/2 = 0.$$

This means that $w - u$ belongs to $W_0^{1,p}(\Omega_1^* \cap H^+)$ and to $W_0^{1,p}(\Omega_1^* \cap PH^+)$ since we can take v to be the quasicontinuous version of v in $W^{1,p}_{\mathrm{loc}}(\Omega^*)$, see [HKM, Chapter 4], and such a function $\varphi \in W^{1,p}_{\mathrm{loc}}(\Omega^*)$ belongs to $W_0^{1,p}(\Omega')$ if and only if $\varphi(x) = 0$ p–quasieverywhere in $\partial\Omega'$ provided that $\overline{\Omega'}$ is compact in Ω^*. Note also that $w = u$ in $\partial\Omega_1^* \setminus \partial H^+$. Now we can use the quasiminimizing property of u in $\Omega_1^* \cap H^+$ and in $\Omega_1^* \cap PH^+$, see also (12), to obtain

$$\int_{\Omega_1^*} |\nabla u|^p \, dx \leqslant K \int_{\Omega_1^*} |\nabla w|^p \, dx. \tag{14}$$

Since

$$\int_{\Omega_1} |\nabla u|^p \, dx = \int_{\Omega_1^*} |\nabla u|^p \, dx - \int_{\Omega_1^* \setminus \Omega_1} |\nabla u|^p \, dx$$

and since the convexity of $t \to |t|^p$ and (14) yield

$$\int_{\Omega_1^*} |\nabla w|^p \, dx \leqslant \frac{1}{2} \int_{\Omega_1^*} |\nabla v|^p \, dx + \frac{1}{2} \int_{\Omega_1^*} |\nabla v^*|^p \, dx,$$

we obtain

$$\int_{\Omega_1} |\nabla u|^p \, dx \leqslant \frac{K}{2} \int_{\Omega_1^*} |\nabla v|^p \, dx + \frac{K}{2} \int_{\Omega_1^*} |\nabla v^*|^p \, dx - \int_{\Omega_1^* \setminus \Omega_1} |\nabla u|^p \, dx. \tag{15}$$

On the other hand

$$\int_{\Omega_1^*} |\nabla v|^p \, dx = \int_{\Omega_1} |\nabla v|^p \, dx + \int_{\Omega_1^* \setminus \Omega_1} |\nabla u|^p \, dx \tag{16}$$

and by symmetry

$$\int_{\Omega_1^*} |\nabla v^*|^p \, dx = \int_{\Omega_1^*} |\nabla v|^p \, dx. \tag{17}$$

Putting (15)–(17) together we obtain

$$\begin{aligned} \int_{\Omega_1} |\nabla u|^p \, dx &\leqslant K \int_{\Omega_1} |\nabla v|^p \, dx + K \int_{\Omega_1^* \setminus \Omega_1} |\nabla u|^p \, dx - \int_{\Omega_1^* \setminus \Omega_1} |\nabla u|^p \, dx \\ &\leqslant K \int_{\Omega_1} |\nabla v|^p \, dx + (K-1) \int_{\Omega_1} |\nabla u|^p \, dx \end{aligned}$$

because by the symmetry properties of u and Ω_1^*

$$\int_{\Omega_1^* \setminus \Omega_1} |\nabla u|^p \, dx \leqslant \int_{\Omega_1} |\nabla u|^p \, dx.$$

If now $K < 2$, then this yields (13) as required and the proof is complete. ■

Remark 3.2. If $K = 1$ in Theorem 3.1, i.e. we consider minimizers, then Theorem 3.1 gives the usual reflection principle. Note that the proof is based on the Dirichlet principle only and thus it can be applied to a great variety of situations.

4. Reflection principle for $n = 1$

On the real line quasiminimizers form a much simpler class of functions than in $\mathbb{R}^n, n \geqslant 2$. In [GG] it was shown that $f : (a, b) \to \mathbb{R}$ is a K–quasiminimizer if and only if f satisfies a reverse type inequality

$$\int_c^d |f'|^p \, dx \leqslant K \frac{|f(d) - f(c)|^p}{|d - c|^{p-1}} \tag{18}$$

in each subinterval $[c, d]$ of (a, b). Every quasiminimizer is locally absolutely continuous and locally Hölder continuous in (a, b). In [MS] it was shown that a K–quasiminimizer $f : (a, b) \to \mathbb{R}$ always has a continuous extension to a or to b provided that $a > -\infty$ or $b < \infty$, respectively. Moreover, f then satisfies (18) in all intervals $[c, d] \subset [a, b]$. This means that the first step, Theorem 2.1, holds for quasiminimizers in the one dimensional case without any assumptions on boundary values. Note also that a K–quasiminimizer is a monotone function and strictly monotone if not constant, see [GG]. For additional properties of one dimensional quasiminimizers see [MS].

Below we consider reflection principle in the case $n = 1$ and show, by an example, that the quasiminimizing constant K is not preserved in the reflection process. This is rather surprising since quasiminimizers tend to be more flexible than minimizers. Let $f : (a, b) \to \mathbb{R}$ be a K–quasiminimizer and $a > -\infty$. As noted above, the limit

$$f(a) = \lim_{x \to a+} f(x)$$

exists. Define $f^* : (2a - b, b) \to \mathbb{R}$ as

$$f^*(x) = \begin{cases} f(x), & x \in [a, b), \\ 2f(a) - f(2a - x), & x \in (2a - b, a). \end{cases}$$

If $a = 0$ and $f(a) = 0$, then $f^*(x) = -f(-x), x \in (-b, 0)$ and thus f^* is the function obtained from f by the usual reflection.

Theorem 4.1. *Suppose that $f : (a, b) \to \mathbb{R}$ is a K–quasiminimizer, and $a > -\infty$. Then the function f^* is a K_o–quasiminimizer in $(2a - b, b)$ with $K_o = 2^p K$.*

Proof. We may assume that $a = 0$, $f(0) = 0$ and f is increasing. Let $[c, d] \subset (-b, b)$. We need to show

$$\int_c^d |f^{*\prime}|^p \, dx \leqslant K_o \frac{|f^*(d) - f^*(c)|^p}{|d - c|^{p-1}}. \tag{19}$$

This is clear if $[c, d] \subset (-b, 0]$ or $[c, d] \subset [0, b)$. Hence we may assume that $0 \in (c, d)$. Now either $|c| \geqslant d$ or $|c| \leqslant d$ and, by symmetry, we may assume $|c| \leqslant d$. Then

$$\int_c^0 |f^{*\prime}(x)|^p \, dx = \int_0^{|c|} |f'(x)|^p \, dx$$

and we obtain

$$\begin{aligned}\int_c^d |f^{*\prime}(x)|^p \, dx &= 2\int_0^{|c|} |f'(x)|^p \, dx + \int_{|c|}^d |f'(x)|^p \, dx \\ &\leqslant 2\int_0^d |f'(x)|^p \, dx \leqslant 2K\frac{|f(d)-f(0)|^p}{|d-0|^{p-1}}.\end{aligned}$$

On the other hand

$$|f(d)-f(0)|^p = |f(d)|^p \leqslant (f(d)+f(-c))^p = (f^*(d)-f^*(c))^p$$

and

$$d^{p-1} = (2d)^{p-1}/2^{p-1} \geqslant (d-c)^{p-1}/2^{p-1}$$

because $2d \geqslant d+|c| = d-c$. Hence the three previous inequalities yield (19) as required and the proof is complete. ■

For $K \geqslant 1$ set $K'' = K/(2-K)$ if $K < 2-2^{-p}$ and $K'' = 2^pK$ if $K \geqslant 2-2^{-p}$. Now Theorems 3.1 and 4.1 give the following improvement for the quasiminimizing constant of the function f^*.

Corollary 4.2. *Let f and f^* be as in Theorem 4.1. Then f^* is a K''–quasiminimizer in $(2a-b,b)$.*

5. Example

We present a simple example which shows that the quasiminimizing constant K is not preserved in the reflection process.

For simplicity we consider the case $p=2$ only. Let $f:[0,\infty)\to\mathbb{R}$ be the function $f(x)=x^\alpha, \alpha>1/2, \alpha\neq 1$. Note that the function f with $\alpha\leqslant 1/2$ is not a quasiminimizer with exponent $p=2$ in $[0,\infty)$ because f' does not belong to $L^2([0,b])$ for any $b>0$.

Lemma 5.1. *The function f is a K–quasiminimizer with $K=\alpha^2/(2\alpha-1)$ and f is not a K'–quasiminimizer for any $K'<K$.*

Proof. Let $[a,b]\subset[0,\infty)$. Now

$$\int_a^b f'^2 \, dx = \frac{\alpha^2}{2\alpha-1}\big(b^{2\alpha-1}-a^{2\alpha-1}\big)$$

and the inequality $\big(b^{1/2}a^{\alpha-1/2}-a^{1/2}b^{\alpha-1/2}\big)^2 \geqslant 0$ yields

$$b^{2\alpha-1}-a^{2\alpha-1} \leqslant \frac{(b^\alpha-a^\alpha)^2}{b-a}.$$

Hence we obtain

$$\frac{\alpha^2}{2\alpha-1}\big(b^{2\alpha-1}-a^{2\alpha-1}\big) \leqslant K\frac{(b^\alpha-a^\alpha)^2}{b-a}$$

and since $f(a) = a^\alpha$ and $f(b) = b^\alpha$, we see from (18) that f is a K–quasiminimizer as required.

That K is the best possible quasiminimizing constant for f follows from the fact that for $a = 0$ the above inequalities are equalities. The lemma follows. ■

Next we consider the function f^* obtained from f by reflection. Now $f^* \colon \mathbb{R} \to \mathbb{R}$ and both $f^*|(-\infty, 0]$ and $f^*|[0,\infty)$ are K–quasiminimizers, $K = \alpha^2/(2\alpha - 1)$.

Lemma 5.2. *The function f^* is not a K–quasiminimizer.*

Proof. Consider an interval $[a, b]$ where $a < 0 < b$ and $|a| < b$. Now

$$\int_a^b |f^{*\prime}|^2\, dx = 2\int_0^{|a|} f'^2\, dx + \int_{|a|}^b f'^2\, dx = \frac{\alpha^2}{2\alpha-1}\big(b^{2\alpha-1} + |a|^{2\alpha-1}\big)$$

and if f^* is a K–quasiminimizer, then from the above equality we obtain

$$\frac{\alpha^2}{2\alpha-1}\big(b^{2\alpha-1} + |a|^{2\alpha-1}\big) \leqslant K\frac{(|a|^\alpha + b^\alpha)^2}{b + |a|}.$$

This yields

$$(b + |a|)\big(|a|^{2\alpha-1} + b^{2\alpha-1}\big) \leqslant b^{2\alpha} + 2b^\alpha |a|^\alpha + |a|^{2\alpha}$$

which is equivalent to

$$(b|a|)\big(|a|^{2\alpha-2} + b^{2\alpha-2}\big) \leqslant 2b^\alpha |a|^\alpha.$$

But a simple computation shows that the last inequality implies $(|a|^{\alpha-1} - b^{\alpha-1})^2 \leqslant 0$ which is not true because $|a| \neq b$ and $\alpha \neq 1$. The lemma follows. ■

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Received: 11 April 2008; **revised:** 25 June 2008