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A NOTE ON ALGEBRAIC INTEGERS WITH PRESCRIBED FACTORIZATION PROPERTIES IN SHORT INTERVALS

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Dedicated to Professor Bogdan Bojarski
on the occasion of his 75-th birthday.

Abstract: We study the distribution of algebraic integers with prescribed factorization properties in short intervals and prove that for a large class of such numbers from a fixed algebraic number field K with a non-trivial class group, every interval of the form $(x, x+x^\theta)$ with a fixed $\theta > 1/2$ contains absolute value of the norm of such algebraic integer provided $x \geqslant x_0$. The constant x_0 effectively depends on K and θ.

Keywords: Factorization in algebraic number fields, short intervals, unique factorization.

1. Introduction and statement of results.

In a recent paper [2] we proved the following theorem.

Theorem 1.1. *Let K be an algebraic number field with the class number $h \geqslant 3$ and let $\theta > 1/2$ be a real number. Then there exists an effectively computable constant $x_0(K,\theta)$ such that for all $x \geqslant x_0(K,\theta)$ there exists an irreducible algebraic integer α in K satisfying*

$$x < |N_{K/\mathbb{Q}}(\alpha)| < x + x^\theta.$$

This result admits the following generalization. Let us denote by $H(K)$, $\mathcal{O}_K$, $\mathfrak{a}$ and $\mathfrak{p}$ the classgroup of K, its ring of integers, a generic ideal of $\mathcal{O}_K$ and a generic prime ideal of $\mathcal{O}_K$ respectively. Moreover, let us call a set $\mathcal{A}$ of ideals *regular* if there exist distinct ideal classes $X_1, \ldots, X_m \in H(K)$ and non-negative integers $c_1, \ldots, c_m$ such that

$$\{\mathfrak{a} \subset \mathcal{O}_K : \Omega_{X_j}(\mathfrak{a}) = c_j\ ,\ j = 1, \ldots, m\} \subset \mathcal{A},$$

where as usual for every $X \in H(K)$ we write

$$\Omega_X(\mathfrak{a}) = \sum_{\substack{\mathfrak{p}^k || \mathfrak{a} \\ \mathfrak{p} \in X}} k.$$

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The least upper bound for the sums $\sum_{j=1}^{m} c_j$, with c_j's as above is called the *size* of $\mathcal{A}$ and denoted by $s(A)$. The size of a regular set of ideals is always an integer or infinity. The latter holds for instance for the set of all ideals or for the set of all ideals from a fixed ideal class.

Theorem 1.2. *Let $\mathcal{A}$ be a regular set of ideals of size at least 3. Then for every $\theta > 1/2$ there exists an effectively computable constant $x_0(K,\theta)$ such that for all $x \geqslant x_0(K,\theta)$ there exists an ideal $\mathfrak{a} \in \mathcal{A}$ whose norm belongs to the interval $(x, x+x^\theta)$.*

Observe that the set of principal ideals generated by irreducible integers is a regular set of size $D(K)$, the Davenport constant of K, see [3], Chapter 9. Since $D(K) \geqslant 3$ if $h \geqslant 3$, Theorem 1.1 is a consequence of Theorem 1.2. Observe moreover that in this case the set of ideals under consideration is of a finite size, and hence the size of its complement is infinite. Therefore every interval of the form $(x, x+x^\theta)$, $\theta > 1/2$, $x \geqslant x_0(\theta, K)$ contains also an absolute value of the norm of an algebraic integer from K which is not irreducible.

In a similar way we can prove analogous results for other sets of algebraic integers with prescribed factorization properties such as elements with a unique factorization into irreducible factors, elements without the unique factorization property, but with all factorizations into irreducibles of the same length, elements having exactly k different lengths of factorizations, and many similar. As a sample we formulate a result concerning the unique factorization case.

Theorem 1.3. *For every algebraic number field K and every real number $\theta > 1/2$ there exists an effectively computable constant $x_0(K,\theta)$ such that every interval of the form $(x, x+x^\theta)$, $x \geqslant x_0(K,\theta)$, contains an absolute value of the norm of an algebraic integer from K having unique factorization into irreducible factors. If $h \geqslant 5$, $h \neq 8$, every such interval, possibly with a larger $x_0(K,\theta)$, contains an absolute value of the norm of an algebraic integer from K which has a unique factorization and is neither irreducible nor has a prime element divisor.*

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2. Proof of Theorem 1.2.

We shall be very brief as the proof follows that of [2] closely. For an ideal class $X \in H(K)$, a real number N, and complex $s = \sigma + it$ we set

$$S_N(s,X) = \sum_{\substack{\mathfrak{p}\in X \\ N \leqslant N(\mathfrak{p}) \leqslant 2N}} \frac{1}{N(\mathfrak{p})^s}.$$

Moreover for $\sigma > 1$ let

$$F(s,X) = \sum_{\mathfrak{p}\in X} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^s}.$$

Suppose that distinct ideal classes $X_1, \dots, X_m \in H(K)$ and positive integers $c_1, \dots, c_m$ are such that

$$\{\mathfrak{a} \subset \mathcal{O}_K : \Omega_{X_j}(\mathfrak{a}) = c_j,\ j = 1, \dots, m\} \subset \mathcal{A}$$

and

$$\lambda := \sum_{j=1}^{m} c_j \geqslant 3.$$

We define numbers d_j, $j = 1, \dots, m$ as follows. If $m = 1$ we put $d_1 = c_1 - 2$. Otherwise we define

$$d_j = \begin{cases} c_j & \text{if } j \leqslant m-2 \\ c_j - 1 & \text{if } j = m-1 \text{ or } j = m. \end{cases}$$

Let a real number x be sufficiently large, and let a be fixed in such a way that $1/2 - \varepsilon < a < 1/2$ $(0 < \varepsilon < (\theta - (1/2))/3)$. We set

$$N = x^{a/(\lambda-2)} \quad \text{and} \quad M = x^a$$

and consider the following subsidiary function

$$H_{\mathcal{A}}(s) = S_1(s) S_2(s) F(s, X_m),$$

where

$$S_1(s) = \prod_{j=1}^{m} S_N^{d_j}(s, X_j) \quad \text{and} \quad S_2(s) = S_M(s, X_{m_1}),$$

where $m_1 = \max(1, m-1)$. For $\sigma > 1$ we have

$$H_{\mathcal{A}}(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

where $a(n) \geqslant 0$ and $a(n) = 0$ if there is no ideal $\mathfrak{a} \in \mathcal{A}$ with $N(\mathfrak{a}) = n$. Hence to conclude the proof it suffices to show that

$$\sum_{x<n<x+y} a(n) > 0 \tag{2.1}$$

if $y \geqslant x^{(1/2)+3\varepsilon}$. This is done exactly in the same way as in [2]. We use the well known technique of detecting primes in a short interval, but instead of the zero density estimates we apply the following inequalities:

$$\sum_{\substack{\rho=\beta+i\gamma \\ \beta\geqslant\sigma, |\gamma|\leqslant T}} |S_j(\rho)|^2 \ll M^{2(1-\sigma)} L^3 \qquad (j = 1, 2) \tag{2.2}$$

where $0 < T \leqslant M$, derived from the classical mean value estimates for Dirichlet polynomials. The summation in (2.2) is over appropriate non-trivial zeros $\rho = \beta + i\gamma$ of all the Hecke L-functions associated to characters of the ideal class group of K. These estimates substitute the Density Hypothesis, and hence (2.1) follows if $y \geqslant x^{(1/2)+3\varepsilon}$, see the proof of Theorem 1.1 in [2] for details. The proof is complete.

3. Proof of Theorem 1.3.

Denote by E the unit class in $H(K)$ and observe that all numbers $\alpha \in \mathcal{O}_K$ with $\Omega_E(\alpha) = 3$ and $\Omega_X(\alpha) = 0$ for all other $X \in H(K)$, have unique factorization into irreducible factors. Therefore the set of principal ideals generated by such numbers is a regular set of size $\geqslant 3$. Hence the first assertion of Theorem 1.3 follows from Theorem 1.2.

The proof of the remaining part is slightly more involved. Let $\mathcal{A}(K)$ denote the set of principal ideals generated by integers having unique factorization but being neither irreducible nor products of prime elements. Moreover, let $\mathfrak{B}(K)$ denote the block semigroup of $H(K)$. We refer to [3], Chapter 9 for the definition of $\mathfrak{B}(K)$ and its principal properties as well as for the explanation of its role in factorization theory. If the class group $H(K)$ contains an element X of order $m \geqslant 5$ then $(X, X^2, X^{m-2}, X^{m-1})$ is an element of $\mathfrak{B}(K)$ with a unique factorization. Hence the set of ideals $\mathfrak{a}$ with $\Omega_{X^j}(\mathfrak{a}) = 1$ for $j = 1, 2, m-2, m-1$ and $\Omega_Y(\mathfrak{a}) = 0$ for all other $Y \in H(K)$ is a subset of $\mathcal{A}(K)$. If all ideal classes in $H(K)$ have order 2, then recalling that $h \geqslant 5$, $h \neq 8$, we see that $H(K)$ contains a subgroup of 16 elements of the form $\langle X_1\rangle \oplus \langle X_2\rangle \oplus \langle X_3\rangle \oplus \langle X_4\rangle$. Hence $(X_1, X_1X_2, X_2, X_3, X_3X_4, X_4)$ is an element of $\mathfrak{B}(K)$ with a unique factorization. Consequently the set of ideals $\mathfrak{a}$ with $\Omega_C(\mathfrak{a}) = 1$ for $C = X_1, X_1X_2, X_2, X_3, X_3X_4, X_4$ and $\Omega_C(\mathfrak{a}) = 0$ for all other $C \in H(K)$ is a subset of $\mathcal{A}(K)$. Finally, if all ideal classes in $H(K)$ have orders at most 4 and there exists a class X of order 3 or 4 then, since $h \geqslant 5$, $H(K)$ contains a subgroup of the form $\langle X\rangle \oplus \langle Y\rangle$, where $Y \in H(K)\backslash\{E\}$. Consequently $(X, X^{-1}, XY, X^{-1}Y^{-1})$ is an element of $\mathfrak{B}(K)$ with a unique factorization. We see that the set of ideals $\mathfrak{a}$ with $\Omega_C(\mathfrak{a}) = 1$ for $C = X, X^{-1}, XY, X^{-1}Y^{-1}$ and $\Omega_C(\mathfrak{a}) = 0$ for all other $C \in H(K)$ is a subset of $\mathcal{A}(K)$. Hence in all cases $\mathcal{A}(K)$ is a regular set of size $\geqslant 4$, and an application of Theorem 1.2 ends the proof.

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