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ON FINITE ENERGY SOLUTIONS FOR NONHOMOGENEOUS p-HARMONIC EQUATIONS

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Dedicated to Professor Bogdan Bojarski
on the occasion of his 75th birthday.

Abstract: Let Ω be an open subset of $\mathbb{R}^n$, we establish regularity results for solutions of some degenerate nonhomogeneous equations of the type

$$\operatorname{div}\langle A(x)Du, Du\rangle^{\frac{p-2}{2}} A(x)Du = \operatorname{div}F \quad \text{in} \quad \Omega \tag{1}$$

where $p \geqslant 2$. The nonnegative function $\mathcal{K}(x)$, which measures the degree of degeneracy of ellipticity bounds, lies in the exponential class, i.e. $\exp(\lambda\mathcal{K}(x))$ is integrable for some $\lambda > 0$. Under this assumption, the gradient of a finite energy solution of (1) lies in the Orlicz-Zygmund class $L^p \log^{-1} L(\Omega)$. Our results states that the gradient of such solution is more regular provided λ is sufficiently large and the datum $F = F(x)$ belongs to a suitable Orlicz-Zygmund class.

Keywords: Elliptic Equations, Mappings with Finite Distortion, Orlicz-Zygmund classes

1. Introduction

Let Ω be an open bounded subset of $\mathbb{R}^n$, we consider the nonhomogeneous equation

$$\operatorname{div}\mathcal{A}(x, Du) = \operatorname{div}F \tag{1.1}$$

for $u : \Omega \to \mathbb{R}^n$, where $F = F(x)$ is a field in $L^q \log^\alpha L(\Omega, \mathbb{R}^{n\times n})$, $\alpha \geqslant 0$, $q \geqslant 2$. As in the familiar model of p-harmonic operator, we suppose that

$$\mathcal{A}(x, \xi) = \langle A(x)\xi, \xi\rangle^{\frac{p-2}{2}} A(x)\xi \tag{1.2}$$

where $pq = p + q$ and $A(x) : \Omega \times \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ is a symmetric positive definite linear transformation on $\mathbb{R}^{n\times n}$ such that

$$m(x)|\xi|^2 \leqslant \langle A(x)\xi, \xi\rangle \leqslant M(x)|\xi|^2 \tag{1.3}$$

for almost every $x \in \Omega$ and all $\xi \in \mathbb{R}^{n\times n}$, and $0 < m(x) \leqslant M(x) < \infty$ a.e.. Throughout this paper we deal with weak solutions of (1.1) having finite energy. We say that a function $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ has finite energy provided

$$\langle A(x)Du, Du\rangle^{\frac{p}{2}} \in L^1_{\mathrm{loc}}(\Omega).$$

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In the following we set

Definition 1.1. *A function* $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ *with finite energy is a weak (local) solution of equation* (1.1) *if*

$$\int_\Omega \mathcal{A}(x, Du) D\varphi \, \mathrm{d}x = \int_\Omega F \; D\varphi \, \mathrm{d}x$$

for every function $\varphi \in C_0^\infty(\Omega)$.

When the ratio $M(x)/m(x)$ is bounded, these equations are strictly related to the theory of quasiregular mappings. Indeed, as B. Bojarki and T. Iwaniec stated in the fundamental paper [BI], the components of a quasiregular mappings solve equations of the type (1.1).

In recent years, in connection with the study of mappings with finite distortion (see [IM], [IS]), also equations at (1.1) in which the ellipticity bounds at (1.3) degenerates, have been considered; we are dealing here with genuine nonisotropic equations where the ratio $M(x)/m(x)$ is not bounded.
In this case, in order to achieve satisfactory estimates one must control the degree of degeneracy.

Following the framework of the theory of mappings with finite distortion, we can state the ellipticity bounds in (1.3) equivalently as

$$\frac{1}{p}\,|\xi|^p + \frac{1}{q}\,|\mathcal{A}(x,\xi)|^q \leqslant \mathcal{K}(x)\langle \mathcal{A}(x,\xi), \xi\rangle, \tag{1.4}$$

where the function $\mathcal{K} = \mathcal{K}(x) \geqslant 1$ depends on the ellipticity bounds $m(x)$ and $M(x)$ at (1.3). In the sequel we will refer to (1.4) as the "distortion inequality" and we will call $\mathcal{K}(x)$ the "distortion function" of equation (1.1).
In this paper we will be interested in the distortion $\mathcal{K}$ of the exponential class $EXP(\Omega)$. Precisely, we shall assume that $\mathcal{K}$ satisfies

$$\int_\Omega e^{\lambda\mathcal{K}} \, \mathrm{d}x < \infty. \tag{1.5}$$

Without loss of generality (see Section 2) we can assume that $\mathcal{K}$ admits a BMO-majorant. Precisely we can majorize $\mathcal{K}$ point-wise by a function $K(x) \in BMO(\mathbb{R}^n)$. We can also ensure a bound for the BMO-norm of $K(x)$ in terms of the exponent λ, i.e.

$$\|K\|_{BMO} \leqslant \frac{c(n)}{\lambda}. \tag{1.6}$$

By assumptions (1.2), (1.4) and (1.5), the gradient of a weak solution lies locally in the Orlicz-Zygmund space $L^p \log^{-1} L(\Omega, \mathbb{R}^n)$.

Our goal here is to show that these solutions, under suitable assumptions on F, are more regular.
More precisely, our main result (Theorem 4.1) shows that if u is a weak solution of (1.1) and the BMO-norm of K at (1.6) is small enough then

$$|KF| \in L^q \log^\alpha L_{\mathrm{loc}}(\Omega) \Rightarrow |Du| \in L^p \log^\alpha L_{\mathrm{loc}}(\Omega).$$

In particular, when the distortion function $\mathcal{K}$ is bounded, for every weak solution of (1.1) having the gradient locally p-integrable, we get

$$|F| \in L^q \log^\alpha L_{\mathrm{loc}}(\Omega) \Rightarrow |Du| \in L^p \log^\alpha L_{\mathrm{loc}}(\Omega)$$

This extends a result of [AIKM], [IKM], [IO] related to the case $p = n$. In the setting of Lebesgue spaces, under the assumption that $\mathcal{K}$ is bounded, higher integrability results for the gradient of a solution of (1.1) have been proved in [I].
The main tool to prove our result is the solvability of the Dirichlet problem for equation at (1.1) in a cube in $\mathbb{R}^n$ (Theorem 3.3).

2. General Notations and Preliminary Results

In the sequel $\Omega \subset \mathbb{R}^n$ will be a domain and Q a cube in $\mathbb{R}^n$. Following the notations in the Introduction, we consider the nonhomogeneous equation

$$\operatorname{div} \mathcal{A}(x, Du) = \operatorname{div} F \qquad \text{in } \Omega$$

where $F : \Omega \to \mathbb{R}^{n\times n}$ is a vector field and

$$\mathcal{A}(x, \xi) = \langle A(x)\xi, \xi\rangle^{\frac{p-2}{2}} A(x)\xi$$

for almost every $x \in \Omega$ and all $\xi \in \mathbb{R}^{n\times n}$, with $A(x)$ satisfying (1.3).
By using (1.4), elementary algebraic analysis reveals that $\mathcal{A}(x, \xi)$ satisfies the following growth monotonicity conditions

(i) $|\xi|^p \leqslant c\mathcal{K}(x)\langle \mathcal{A}(x, \xi), \xi\rangle$
(ii) $|\mathcal{A}(x, \xi)|^q \leqslant c\mathcal{K}(x)\langle \mathcal{A}(x, \xi), \xi\rangle$
(iii) $\mathcal{K}^{-1}(x)|\xi - \eta|^p \leqslant c\langle \mathcal{A}(x, \xi) - \mathcal{A}(x, \eta), \xi - \eta\rangle$
(iv) $|\mathcal{A}(x, \xi) - \mathcal{A}(x, \eta)| \leqslant c\mathcal{K}^{p-1}|\xi - \eta|(|\xi|^{p-2} + |\eta|^{p-2})$

for almost every $x \in \Omega$ and all $\xi, \eta \in \mathbb{R}^{n\times n}$ and $c = c(p, q)$. The function $\mathcal{K}(x)$ lies in $EXP(\Omega)$, i.e.

$$\int_\Omega e^{\lambda\mathcal{K}} \mathrm{d}x < \infty \tag{2.1}$$

for some $\lambda > 0$.

Definition 2.1. *Let g be a locally integrable function on $\mathbb{R}^n$; g is said to be a* function of bounded mean oscillation on $\mathbb{R}^n$, *briefly $g \in BMO(\mathbb{R}^n)$, iff*

$$\|g\|_{BMO} := \sup_Q \fint_Q |g(y) - g_Q| \mathrm{d}y \tag{2.2}$$

is finite, where the supremum extends over all cubes Q in $\mathbb{R}^n$ with edges parallel to coordinate axes and $g_Q = \frac{1}{|Q|}\int_Q |g| \mathrm{d}x = \fint |g| \mathrm{d}x$.

Modulo constant functions, $\|\cdot\|_{BMO}$ at (2.2) is a norm and BMO is a Banach space. Functions in BMO are in $L^p_{\mathrm{loc}}(\mathbb{R}^n)$, for any finite $p > 1$; in fact they are locally exponentially integrable, as shown by the well-known John-Nirenberg lemma: there exists a constant $\Theta = \Theta(n)$ such that for every $g \in BMO(\mathbb{R}^n)$ and every cube Q, we have

$$\fint_Q \exp\left(\Theta \frac{|g(x) - g_Q|}{\|g\|_{BMO}}\right) \mathrm{d}x \leqslant 2.$$

Clearly, bounded functions have bounded mean oscillation. On the contrary, BMO-functions need not be bounded. The usual example is $x \to \log|x|$.

Of particular importance in our applications will be the $L^p \log^\alpha L(\Omega)$ space, that is the Orlicz space $L^\Phi(\Omega)$ generated by $\Phi(t) = t^p \log^\alpha(e + t)$ at least for sufficiently large values of t, equipped by the Luxemburg norm, i.e. the space of all measurable functions f on Ω such that

$$\|f\|_\Phi := \inf\left\{\lambda > 0 : \int_\Omega \Phi\left(\frac{|f|}{\lambda}\right) \mathrm{d}x \leqslant 1\right\}. \tag{2.3}$$

For $p \geqslant 1$ and $\alpha \geqslant 0$ the non-linear functional

$$|f|_{p,\alpha} = \left[|f|^p \log^\alpha \left(e + \frac{|f|}{\|f\|_p}\right)\right]^{1/p}$$

is comparable with the Luxemburg norm at (2.3); for more details see [RR]. Before going on, we state the following Sobolev-Poincaré type inequality (see [IS], [IS1])

Lemma 2.2. *For each matrix field* $B \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^{n\times n})$ *with* $\operatorname{div} B \in L^s(\Omega, \mathbb{R}^{n\times n})$, $1 < s < n$, *and* $\varphi \in C_0^\infty(\Omega)$, *there exists a divergence free matrix field* $B_0 \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^{n\times n})$ *such that*

$$\|D(\varphi B - B_0)\|_s \leqslant c(n,s) \|\, |\nabla\varphi|\, |B|\, \|_s.$$

As B_0 is obtained via Riesz transform of φB it may be concluded that if $B \in L^{s'}_{\mathrm{loc}}(\Omega, \mathbb{R}^{n\times n})$ for some other exponent $1 < s' < +\infty$ then $B_0 \in L^{s'}(\mathbb{R}^n, \mathbb{R}^{n\times n})$. Hence if Ω is a cube centered in $x_0 \in \mathbb{R}^n$, then for $x \in \mathbb{R}^n - 2\Omega$ we have the pointwise inequality

$$|D(\varphi B - B_0)(x)| \leqslant c(n) \frac{\operatorname{diam}\Omega}{|x - x_0|^{n+1}} \||\varphi|\, |B|\, \|_{L^1(\mathbb{R}^n)} \tag{2.4}$$

In order to achieve satisfactory a priori estimates for equations defined at (1.1) in $\mathbb{R}^n$ a.e. we assume that

$$e^{\lambda\mathcal{K}} \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \qquad \text{and} \qquad e^{\lambda\mathcal{K}} \in L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n). \tag{2.5}$$

This condition certainly holds if the matrix field $\mathcal{A}(x,\xi)$ has compact support. In the sequel, as we announced in the Introduction, we shall replace the distortion function $\mathcal{K}$ by a BMO-majorant.
Precisely, we shall majorise $\mathcal{K}(x)$ point-wise as

$$\mathcal{K}(x) \leqslant K(x) \qquad \text{a.e. in } \Omega \tag{2.6}$$

where $K = K(x) \geqslant 1$ lies in $BMO(\mathbb{R}^n)$.
With the aid of (2.5) we can also ensure the uniform BMO-bound of K at (1.6). Moreover, we have the following global exponential integrability property of K

$$e^{\lambda(K-K_0)} - 1 \in L^1(\mathbb{R}^n)$$

for some $K_0 \in L^\infty(\mathbb{R}^n)$, such that

$$1 \leqslant K_0(x) \leqslant K(x).$$

More details about such majorisation can be found in [IS]; we just remark that the following "type" Hölder inequality (see Lemma 5.1 of [IMMP]) holds: if $\alpha \geqslant 1$, $f \in L^p \log^\alpha L$ and $K(x) \in BMO(\mathbb{R}^n)$ is the function at (2.6) then

$$\|Kf\|_{L^p \log^{\alpha-p}} \leqslant \frac{c}{\lambda}\|f\|_{L^p \log^\alpha L} + c[K]\|f\|_{L^p \log^{\alpha-1} L}$$

with $c = c(\alpha)$ and

$$[K] = \|K_0\|_\infty + \frac{1}{\lambda}\int_{\mathbb{R}^n} [e^{\lambda(K-K_0)} - 1]\mathrm{d}x.$$

Under assumptions (2.5) the following a priori estimate holds (see Theorem 13.1 of [IMMP])

Theorem 2.3. *For any $\alpha \geqslant 0$ there exists $\lambda_\alpha \geqslant 1$ such that, whenever K satisfies* (1.6) *with $\lambda \geqslant \lambda_\alpha$, we have*

$$\|\, |Du|^p + |\mathcal{A}(x,Du)|^q \,\|_{L \log^\alpha L(\mathbb{R}^n)} \leqslant c\|KF\|^q_{L^q \log^\alpha L(\mathbb{R}^n)} \tag{2.7}$$

with $c = c(K,n,\alpha)$.

3. A priori estimates

We start by proving a local a priori estimate for solutions of equation (1.1).

Proposition 3.1. *Let u be a solution of* (1.1) *with $|Du| \in L^p \log^\alpha L_{\mathrm{loc}}(\Omega)$ and $|\mathcal{A}(x,Du)| \in L^q \log^\alpha L_{\mathrm{loc}}(\Omega)$. For any $\alpha \geqslant 0$ there exists a critical exponent $\lambda_\alpha \geqslant 1$ such that whenever K satisfies* (1.6) *with $\lambda \geqslant \lambda_\alpha$ and $|KF| \in L^q \log^\alpha L(\Omega)$, then*

$$\begin{aligned} &\|\, |Du|^p + |\mathcal{A}(x,Du)|^q \,\|_{L \log^\alpha L(Q)} \leqslant \\ &\qquad \leqslant c\left(\|\, |Du|^p + |\mathcal{A}(x,Du)|^q \,\|_{L^{\frac{n}{n+1}}(2Q)} + \|KF\|^q_{L^q \log^\alpha L(2Q)}\right) \end{aligned} \tag{3.1}$$

for any cube $Q \subset 2Q \subset \Omega$, with $c = c(Q,K)$.

Proof of Proposition 3.1 . Fix a cube $Q \subset 2Q \subset \Omega$ and a cut-off function $\varphi \in C_0^\infty(2Q)$, such that $0 \leqslant \varphi \leqslant 1$ with $\varphi = 1$ on Q. To have shorter notation we introduce the matrix field

$$B(x) = \mathcal{A}(x, Du) - F$$

and we define

$$H(x) = \varphi^p B(x) = \mathcal{A}(x, \varphi^q Du) - \varphi^p F \tag{3.2}$$

Equation (1.1) and assumptions (1.2), (1.4) and (1.6) yield

$$\operatorname{div} H = (\mathcal{A}(x, Du) - F) D\varphi^p \in L^q \log^\alpha L(\Omega)$$

Applying divergence operator to (3.2) we obtain

$$\operatorname{div}(\mathcal{A}(x, \varphi^q Du) - \varphi^p F) = \operatorname{div}(H(x) - H_0) \tag{3.3}$$

where H_0 can be any divergence free matrix field. We use Lemma 2.2 to find H_0 such that

$$\|D(H - H_0)\|_s \leqslant c\|\operatorname{div} H\|_s = \|D\varphi^p(\mathcal{A}(x, Du) - F)\|_s \tag{3.4}$$

for $1 < s \leqslant q$. If we decompose $\varphi^q Du = g + Dw$ with $w = \varphi^q(u - u_Q)$ and $g = -uD\varphi^q$, by (3.3) we obtain that w, whose gradient lies in $L^q \log^\alpha L$, is a solution of the equation

$$\operatorname{div} \mathcal{A}(x, Dw) = \operatorname{div}\ G \quad \text{in } \mathbb{R}^n,$$

with

$$G = [\mathcal{A}(x, Dw) - \mathcal{A}(x, g(x) + Dw)] + [H(x) - H_0] + \varphi^p F.$$

Notice that $KG \in L^q \log^\alpha L(\mathbb{R}^n)$. for any α. Indeed, by (3.4) and Sobolev Imbedding Theorem $H - H_0 \in L^{\frac{ns}{n-s}}$. Thus $H - H_0 \in L^r_{\mathrm{loc}}$ for every $1 < r \leqslant \frac{nq}{n-q}$ and $K(H - H_0)$ lies in $L^q \log^\alpha L$ as K is in BMO. By noting that $\mathcal{A}(x, Dw) \in L^q \log^\alpha L(\Omega)$, applying the apriori estimate in $\mathbb{R}^n$ we get

$$\begin{aligned}
\|\mathcal{A}(x, Dw)\|^q_{L^q \log^\alpha L(\mathbb{R}^n)} + \|Dw\|^p_{L^p \log^\alpha L(\mathbb{R}^n)} &\leqslant c\|KG\|^q_{L^q \log^\alpha L(\mathbb{R}^n)} \\
&\leqslant c[\|K(x)|\mathcal{A}(x, Dw) - \mathcal{A}(x, g(x) + Dw)|\,\|^q_{L^q \log^\alpha L(\mathbb{R}^n)} \\
&\quad + \|K(x)(H(x) - H_0)\|^q_{L^q \log^\alpha L(\mathbb{R}^n)} + \|K\varphi^p F\|^q_{L^q \log^\alpha L(\mathbb{R}^n)}].
\end{aligned}$$

Here and in what follows $c = c(n, K)$.
By structure condition (iv) we can replace the last term to get

$$\begin{aligned}
&\|\mathcal{A}(x, Dw)\|^q_{L^q \log^\alpha L(\mathbb{R}^n)} + \|Dw\|^p_{L^p \log^\alpha L(\mathbb{R}^n)} \\
&\qquad \leqslant c\Big[\|K(x)\, K(x)^{p-1}|g(x)|\,(|g(x)|^{p-2} + |Dw|^{p-2})\|^q_{L^q \log^\alpha L(\Omega)} \\
&\qquad\quad + \|K(x)\,(H(x) - H_0)\|^q_{L^q \log^\alpha L(\mathbb{R}^n)} + \|\varphi^p KF\|^q_{L^q \log^\alpha L(\Omega)}\Big].
\end{aligned}$$

Applying Young's inequality we obtain

$$\|\mathcal{A}(x,Dw)\|^q_{L^q\log^\alpha L(\mathbb{R}^n)} + \|Dw\|^p_{L^p\log^\alpha L(\mathbb{R}^n)} \leqslant c\Big[\|K(x)^p|g(x)|\,\|^p_{L^p\log^\alpha L(\Omega)} + \|K(x)\,(H(x)-H_0)\|^q_{L^q\log^\alpha L(\mathbb{R}^n)} + \|\varphi^p KF\|^q_{L^q\log^\alpha L(\Omega)}\Big]. \tag{3.5}$$

By Hölder inequality, since $K \in BMO$

$$\|\,K(x)^p|g(x)|\,\|_{L^p\log^\alpha L(\Omega)} \leqslant c\,\|g\|_{L^{\frac{np}{n-p+1}}(2Q)} \leqslant c\|Du\|_{L^{\frac{np}{n+1}}(2Q)}.$$

In order to estimate the second term in the right hand side of (3.5), we fix a function $\eta \in C_0^\infty(5Q)$, $0 \leqslant \eta \leqslant 1$, which equals 1 on the cube $4Q$. Then, denoting $\mathscr{H}(x) = H(x) - H_0$, we can write

$$\|K(x)\mathscr{H}(x)\|_{L^q\log^\alpha L(\Omega)} \leqslant \|\eta K(x)\mathscr{H}(x)\|_{L^q\log^\alpha L(5Q)} + \|(1-\eta)K(x)\mathscr{H}(x)\|_{L^q\log^\alpha L(\mathbb{R}^n-4Q)}$$

Then, by Hölder inequality and relation (2.4) and (3.4) we have

$$\begin{aligned}\|K(x)\mathscr{H}(x)\|_{L^q\log^\alpha L(\mathbb{R}^n)} &\leqslant c\,\|\mathscr{H}(x)\|_{L^{\frac{nq}{n-q+1}}(2Q)} \\ &\quad + \left\|\frac{\operatorname{diam} Q}{|x-x_0|}\right\|_{L^q\log^\alpha L(\mathbb{R}^n-4Q)} \int_{\mathbb{R}^n} |D\varphi^p||\mathcal{A}(x,Du)-F| \\ &\leqslant c\left(\||D\varphi^p|\mathcal{A}(x,Du)\|^q_{L^{\frac{nq}{n+1}}(2Q)} + \|\,|D\varphi^p|F\|_{L^{\frac{nq}{n+1}}(2Q)}\right) \\ &\leqslant c\,\|\mathcal{A}(x,Du)\|^q_{L^{\frac{nq}{n+1}}(2Q)} + \|F\|_{L^q\log^\alpha L(2Q)}\end{aligned} \tag{3.6}$$

with $c = c(n,K,\Omega)$.
Since $\varphi = 1$ on the cube Q, we get by (3.5) and (3.6)

$$\begin{aligned}&\|\mathcal{A}(x,Du)\|^q_{L^q\log^\alpha L(Q)} + \|Du\|^p_{L^p\log^\alpha L(Q)} \\ &\quad\leqslant c(Q,K)\left(\|Du\|^p_{L^{\frac{np}{n+1}}(2Q)} + \|\mathcal{A}(x,Du)\|^q_{L^{\frac{nq}{n+1}}(2Q)}\right) + \|KF\|^q_{L^q\log^\alpha L(2Q)},\end{aligned}$$

for any cube $Q \subset 2Q \subset \Omega$, which concludes the proof. ∎

Now, let us consider the problem

$$\begin{cases} \operatorname{div}\mathcal{A}(x,Du) = \operatorname{div} F & \text{in } Q_0 \\ u = 0 & \text{on } \partial Q_0 \end{cases} \tag{3.7}$$

where Q_0 is a cube in $\mathbb{R}^n$ and $\mathcal{A}(x,\xi)$ is defined in Section 2.
Assume that $\mathcal{K}(x)$ lies in $EXP(Q_0)$, i.e.

$$\int_{Q_0} e^{\lambda\mathcal{K}}\mathrm{d}x < \infty \qquad \text{for some } \lambda > 0. \tag{3.8}$$

We study solutions of (3.7) with the gradient in $L^p \log^\alpha L(Q_0)$ vanishing on ∂Q_0 in the sense of distributions. for a solution of problem (3.7) the following a priori estimate holds

Proposition 3.2. *Let u be a solution of (3.7). For any $\alpha \geqslant 0$ there exists a critical exponent $\lambda_\alpha \geqslant 1$ such that, whenever K satisfies (1.6) with $\lambda > \lambda_\alpha$ and $|KF| \in L^q \log^\alpha L(Q_0)$, we have*

$$\|Du\|^p_{L^p \log^\alpha L(Q_0)} + \|\mathcal{A}(x, Du)\|^q_{L^q \log^\alpha L(Q_0)} \leqslant c\|KF\|^q_{L^q \log^\alpha L(Q_0)} \tag{3.9}$$

where $c = c(K, n, Q_0)$.

Proof. We can argue as in Lemma 3.1 of [M]. We can reflect u, F and the coefficients of the matrix $A(x)$ across the face of Q_0. New cubes emerge in this process and we continue reflecting infinitely times. In this way we extend the p-harmonic equation

$$\operatorname{div} \mathcal{A}(x, Du) = \operatorname{div} F \qquad \text{in } \mathbb{R}^n \tag{3.10}$$

and at the end we look at u as a local solution of the extended equation in the double cube $2Q_0$ (for more details see [IS]).
By Proposition 3.1 we get

$$\begin{aligned}&\|Du\|^p_{L^p \log^\alpha L(Q_0)} + \|\mathcal{A}(x, Du)\|^q_{L^q \log^\alpha L(Q_0)} \\ &\qquad \leqslant c\left(\|Du\|^p_{L^{\frac{np}{n+1}}(2Q_0)} + \|A(x, Du)\|^q_{L^{\frac{nq}{n+1}}(2Q_0)} + \|KF\|^q_{L^q \log^\alpha L(2Q_0)}\right)\end{aligned}$$

with $c = c(Q_0, K)$.
Now, since $K \in BMO$ and the norms over $2Q_0$ are controlled by those over Q_0, condition (ii) of Section 2 and Hölder inequality, yield

$$\begin{aligned}\|A(x, Du)\|^q_{L^{\frac{nq}{n+1}}(2Q_0)} &\leqslant \|K^{p-1}|Du|^{p-1}\|^q_{L^{\frac{nq}{n+1}}(2Q_0)} = \|K\, Du\|^p_{L^{\frac{np}{n+1}}(2Q_0)} \\ &\leqslant \|K\, Du\|^p_{L^{\frac{np}{n+1}}(Q_0)} \leqslant c(K)\|K^{-1/p}|Du|\,\|^p_p.\end{aligned} \tag{3.11}$$

Summarizing

$$\begin{aligned}&\|Du\|^p_{L^p \log^\alpha L(Q_0)} + \|\mathcal{A}(x, Du)\|^q_{L^q \log^\alpha L(Q_0)} \\ &\qquad\qquad \leqslant c(K)\|K^{-1/p}|Du|\,\|^p_p + \|KF\|^q_{L^q \log^\alpha L(Q_0)}.\end{aligned}$$

Now, using u as a test function in (3.7)

$$\begin{aligned}\int \frac{1}{K}\,|Du|^p &\leqslant \int_\Omega \langle A(x, Du), Du\rangle = \int_\Omega F\, Du \\ &= \int_\Omega \left(K^{1/p}F\right)\left((K^{-1/p}\, Du\right) \leqslant \|K^{1/p}F\|_q\, \|K^{-1/p}\, Du\|_p\end{aligned}$$

we deduce

$$\|K^{-1/p} Du\|_p^p \leqslant c(q)\|KF\|_q^q. \tag{3.12}$$

Combining (3.11) and (3.12) we finally have

$$\|Du\|^p_{L^p \log^\alpha L(Q_0)} + \|\mathcal{A}(x, Du)\|^q_{L^q \log^\alpha L(Q_0)} \leqslant c\,\|KF\|^q_{L^q \log^\alpha L(Q_0)}$$

with $c = c(n, K, Q_0)$. ∎

With the aid of Proposition 3.2 by following the same arguments of Theorem 3.2 of [M], we can state the following

Theorem 3.3. *For any $\alpha > 0$ there exists a critical exponent λ_α such that if K satisfies* (1.6) *with $\lambda > \lambda_\alpha$ and $|KF| \in L^q \log^\alpha L(\Omega)$, then the problem* (3.7) *admits a unique solution u with $|Du| \in L^p \log^\alpha L(\Omega)$. We also have the uniform bounds*

$$\begin{aligned} \|Du\|^p_{L^p \log^\alpha L} &\leqslant c\|KF\|^q_{L^q \log^\alpha L}, \\ \|\mathcal{A}(x, Du)\|_{L^q \log^\alpha L} &\leqslant c\|KF\|_{L^q \log^\alpha L} \end{aligned} \tag{3.13}$$

where $c = c(n, \alpha, K)$.

To prove our main statement, we also need a uniqueness result for solutions of problem (3.7) whose gradient lies in $L^p \log^{-1} L(\Omega, \mathbb{R}^{n\times n})$.

Proposition 3.4. *Let u and v be two solutions of problem* (3.7) *with the gradient in $L^p \log^{-1} L(\Omega, \mathbb{R}^{n\times n})$. If $\mathcal{A}(x, Du)$ and $\mathcal{A}(x, Dv)$ belong to $L^q \log^{-1} L(\Omega, \mathbb{R}^{n\times n})$ respectively, then $u = v$ a.e. in Ω.*

For the proof see [M, Proposition 3.3].

4. The Main Result

Now we are able to prove the main result.

Theorem 4.1. *Let u be a weak solution of equation* (1.1)*. For any $\alpha \geqslant 0$ there exists a critical exponent λ_α such that whenever* (2.1) *is satisfied for $\lambda > \lambda_\alpha$ then*

$$|KF| \in L^q \log^\alpha L_{\mathrm{loc}}(\Omega) \Longrightarrow |Du| \in L^p \log^\alpha L_{\mathrm{loc}}(\Omega)$$

and, for any cube $Q \subset 2Q \subset \Omega$,

$$\begin{aligned} &\int_Q |Du|^p \log^\alpha \left(e + \frac{|Du|}{\|Du\|_p}\right) \mathrm{d}x \\ &\qquad \leqslant c\left(\int_{2Q} \langle \mathcal{A}(x, Du), Du\rangle \mathrm{d}x + \|KF\|^q_{L^q \log^\alpha L(2Q)}\right) \end{aligned} \tag{4.1}$$

where $c = c(\alpha, Q)$.

In particular, for nonhomogeneous p-laplacian equation

$$\operatorname{div}|Du|^{p-2}Du = \operatorname{div} F \tag{4.2}$$

we have

Corollary 4.2. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a weak solution of (4.2). Then for any $\alpha > 0$*

$$|F| \in L^q \log^\alpha L_{\mathrm{loc}}(\Omega) \Longrightarrow |Du| \in L^p \log^\alpha L_{\mathrm{loc}}(\Omega).$$

Proof of Theorem 4.1. Following the same notations as in Proposition 3.1, we fix a cube $Q \subset 2Q \subset \Omega$ and a cut-off function $\varphi \in C_0^\infty(2Q)$ such that $0 \leqslant \varphi \leqslant 1$ with $\varphi = 1$ on Q. Let us consider the matrix field

$$B(x) = \mathcal{A}(x, Du) - F$$

and let us define

$$H(x) = \varphi^p B(x) = \mathcal{A}(x, \varphi^q Du) - \varphi^p F. \tag{4.3}$$

Equation (1.1) and relations (1.4) and (2.1) yield

$$\operatorname{div} H = (\mathcal{A}(x, Du) - F) D\varphi^p \ \in L^q \log^{-1} L(\Omega, \mathbb{R}^n).$$

Applying divergence operator in (4.3) we obtain

$$\operatorname{div}(\mathcal{A}(x, \varphi^q\, Du) - \varphi^p F) = \operatorname{div}(H(x) - H_0) \tag{4.4}$$

where H_0 can be any divergence free matrix field. We use Lemma 2.2 to find H_0 such that

$$\|D(H - H_0)\|_s \leqslant c \|\operatorname{div} H\|_s \quad \text{for every } 1 < s < q, \tag{4.5}$$

with $c = c(q, n)$. If we decompose $\varphi^q Du = g + Dw$ with $w = \varphi^q(u - u_Q)$ and $g = -uD\varphi^q$, by (4.4) we obtain that w, whose gradient lies in $L^q \log^{-1} L(2Q, \mathbb{R}^n)$, is a solution of the equation

$$\begin{cases} \operatorname{div} \mathcal{A}(x, Dw) = \operatorname{div} G & \text{in } 2Q, \\ w = 0 & \text{on } \partial(2Q) \end{cases} \tag{4.6}$$

where, by (4.4),

$$G = [\mathcal{A}(x, Dw) - \mathcal{A}(x, g(x) + Dw)] + [H(x) - H_0] + \varphi^p F.$$

Notice that $KG \in L^q \log^\alpha L(2Q, \mathbb{R}^n)$, for any $\alpha > 0$. Indeed by (4.5) and the Sobolev Imbedding Theorem $H - H_0 \in L^{\frac{ns}{n-s}}(\Omega, \mathbb{R}^n)$. Thus $H - H_0 \in L^r_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ for every $1 < r < \frac{nq}{n-q}$ and so $K(H - H_0)$ lies in $L^q \log^\alpha L(2Q, \mathbb{R}^n)$ for any $\alpha > 0$ as K is exponentially integrable.

Also $K|\mathcal{A}(x, Dw) - \mathcal{A}(x, g + Dw)| \in L^q \log^\alpha L(2Q)$, in fact, by the condition (iv) in Section 2 we have

$$|\mathcal{A}(x, Dw) - \mathcal{A}(x, g + Dw)| \leqslant K(x)^{p-1} |g(x)| \left(|g(x)|^{p-2} + |Dw|^{p-2}\right).$$

For a fixed $\alpha > 0$, let v be a solution of Problem (4.6) given by Theorem 3.3, then by definition of H, condition (iv) in Section 2 and (4.3) we get

$$\begin{aligned} |\mathcal{A}(x, Dw)| &\leqslant |\mathcal{A}(x, Dw) - \mathcal{A}(x, g + Dw)| + |\mathcal{A}(x, g + Dw)| \\ &\leqslant c\, K(x)^{p-1}|g(x)| \left(|g(x)|^{p-2} + |Dw|^{p-2}\right) + \varphi^p|\mathcal{A}(x, Du)|. \end{aligned}$$

So, as u is a weak solution and K is exponentially integrable, relation (1.4) implies that $\mathcal{A}(x, Dw) \in L^q \log^{-1} L(2Q, \mathbb{R}^n)$ and by Proposition 3.4, $Du = Dv$ a.e. in Q. Estimate (4.1) follows by Theorem 3.3. ■

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