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LOCAL BOUNDEDNESS OF MINIMIZERS OF INTEGRAL FUNCTIONALS WITH (p,q)–GROWTH ON METRIC SPACES

Outi Elina Maasalo, Bianca Stroffolini, Anna Verde

Dedicated to Professor Bogdan Bojarski
on the occasion of his 75th birthday

Abstract: We study local boundedness of the local minimizers of functionals with (p,q)-growth on metric measure spaces equipped with a doubling measure and supporting a weak Poincaré inequality. The metric space is not required to be complete.

Keywords: doubling measure, metric spaces, minimizers, nonstandard growth, Poincaré inequality

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open set and let us consider the functional

$$\int_\Omega F(x, Du(x))dx, \tag{1.1}$$

where $F : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function satisfying the growth condition

$$c_1|z|^p \leqslant F(x,z) \leqslant c_2(1+|z|^q) \tag{1.2}$$

for $1 < p \leqslant q$. In the case $p = q$ it is well known that minimizers of the p–Dirichlet integral are locally Hölder continuous. One of the possible methods to prove this fact is to use the Moser iteration; see [20, 21]. The iteration shows that the minimizers satisfy the Harnack inequality, which in turn implies the Hölder continuity. Another approach, by De Giorgi, leads directly to the Hölder continuity of the minimizers, and the Harnack inequality can be obtained as a corollary; see, for example, [6].

The study of the case $p < q$, also called the nonstandard case, was initiated by Marcellini and his series of papers [16, 17, 18], where he studied regularity of solutions of elliptic equations and variational problems with (p,q)–growth conditions. For instance, he proved the local Hölder continuity of minimizers under an

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additional assumption on the partial derivatives of minimizers. Furthermore, in the case $n \geqslant 2$, he proved the local Lipschitz continuity for a weak solution that belongs to $W^{1,q}(\Omega)$. Local boundedness of the minimizers has been proved, for example, by Moscariello and Nania in [19].

In most of the papers that deal with non–standard growth conditions is assumed that p and q are close to each other. Indeed, there are counterexamples which show that minimizers of (1.1) may not be locally bounded if p and q are far apart; see, for example, [17]. On the other hand, the condition $q < p^\star = \frac{np}{n-p}$ ensures that the minimizers are locally bounded; see [19].

In the context of a metric measure space, where the measure is doubling and the space supports a weak Poincaré inequality, the p–Dirichlet integral has been widely studied. Kinnunen and Shanmugalingam, [14], adapted De Giorgi's method to this context and proved that quasiminimizers of the p–Dirichlet integral satisfy a weak Harnack inequality, the strong maximum principle, and are locally Hölder continuous. Björn and Marola, [3], applied the Moser iteration technique to show that a weak Harnack holds for quasiminimizers of the p–Dirichlet integral.

However, it seems to us that the case with non–standard growth conditions has not yet been studied in the metric setting. The purpose of this note is to prove local boundedness of the minimizers of variational integrals of the type (1.1), where the function F satisfies (1.2). Our approach is based on a De Giorgi type argument and we follow the ideas of [19].

Finally, we observe that the right–hand side of (1.2) is not needed in the proof of the existence of local minimizers of (1.1). Indeed, direct methods of the calculus of variations are also available in the metric space setting. The lower semicontinuity property of the variational integral can be achieved with the assumption that F is convex in the gradient variable. On the other hand, the left–hand side of (1.2) guarantees coercivity.

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2. Preliminaries

2.1. General context

Let (X, d) be a metric space equipped with a Borel regular measure μ. In a metric measure space the concept of an upper gradient serves as a substitute for the Sobolev gradient. Suppose that $1 \leqslant p < \infty$ and let u be a real–valued function in X. A nonnegative Borel measurable function g is said to be a p–weak *upper gradient* of u if we have

$$|u(\gamma(a)) - u(\gamma(b))| \leqslant \int_\gamma g\, ds \tag{2.1}$$

for p–almost every rectifiable path $\gamma\colon [a, b] \to X$. In other words (2.1) holds for all paths in X except for a family of paths which is of zero p–modulus. We recall also that a path γ is rectifiable if $\text{length}(\gamma) < \infty$. The reader may see, for example, [5], [11] and [22] for a discussion of upper gradients and [13] or [22] for the definition of p–modulus and discussion of paths.

The Sobolev space on a metric measure space, called the Newtonian space $N^{1,p}(X)$ can now be defined as a collection of equivalence classes of p–integrable functions with p–weak integrable upper gradients. The precise definition and further information can be found in various references, and we mention, for example, [22]. If E is a measurable set in X, the Newtonian space with zero boundary values $N_0^{1,p}(E)$ can be defined as the collection of functions in $N^{1,p}(X)$ that are zero outside E. For this and equivalent definitions see, for example, [1] or [22, 23].

For $1 \leqslant p < \infty$ every function u that has a p–integrable p–weak upper gradient has a minimal p–integrable p–weak upper gradient in X, denoted g_u, in the sense that for any other p–integrable p–weak upper gradient g of u we have $g_u \leqslant g$ μ–a.e. in X. Thus Newtonian functions can be used to study the p–Dirichlet integral

$$\int_\Omega g_u \, d\mu, \tag{2.2}$$

and moreover functionals of the type (1.1).

A reader not familiar with the metric space setting and Newtonian spaces may see, for example, [2], [5], [8, 9], [11, 12] or [22, 23] for details. Potential theory on noncomplete spaces has been studied in [3], and in parts of [4] and [14]. We present here the main assumptions and properties only briefly.

Our notation is standard. Throughout the paper we assume that the measure of every nonempty open set is positive and that the measure of every bounded set is finite. The measure μ is assumed to be doubling, in other words there exists a constant $c_d \geqslant 1$ such that

$$\mu(B(x,2r)) \leqslant c_d \mu(B(x,r))$$

for every x in X and $r > 0$. A space that carries a doubling measure is always doubling as a metric space, that is, each $B(x,r)$ can be covered by a constant number of balls with radius $r/2$. Furthermore, a complete doubling space can be equipped with a doubling measure, but there exists noncomplete doubling spaces that do not carry doubling measures.

We also assume that the space supports a weak $(1,p)$–Poincaré inequality, that is there exist $c > 0$ and $\tau \geqslant 1$ such that

$$⨍_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leqslant cr \left(⨍_{B(x,\tau r)} g^p \, d\mu \right)^{1/p}$$

for all x in X, $r > 0$ and all pairs $\{u, g\}$ where u is a locally integrable function on X and g is a p–weak upper gradient of u. Here we used the convention

$$u_B = ⨍_B u \, d\mu = \frac{1}{\mu(B)} \int_B u \, d\mu.$$

A result of [10] shows that in a metric space with a doubling measure a weak $(1,p)$–Poincaré inequality implies a weak (t,p)–Poincaré inequality for some $t > p$ and possibly a new τ i.e. there exist $c' > 0$ and $\tau' \geqslant 1$ such that

$$\left(\fint_B |u - u_B|^t \, \mathrm{d}\mu\right)^{1/t} \leqslant c' r \left(\fint_{\tau' B} g^p \, \mathrm{d}\mu\right)^{1/p}, \tag{2.3}$$

where

$$\begin{cases} 1 \leqslant t \leqslant Qp/(Q-p) & \text{if} \quad p < Q, \\ 1 \leqslant t & \text{if} \quad p \geqslant Q, \end{cases}$$

for all balls B in X, and $Q = \log_2 c_d$. The exponent Q serves as a counterpart of dimension related to the measure.

We also recall the following properties of p–weak upper gradients. If $u \in N^{1,p}(X)$ and η is a bounded Lipschitz continuous function, then $uv \in N^{1,p}(X)$ and

$$g_{u\eta} \leqslant |u| g_\eta + |\eta| g_u$$

μ–a.e. For a nonnegative function $u \in N^{1,p}(\Omega)$ we have

$$g_{u^p} \leqslant p u^{p-1} g_u$$

μ–a.e. In addition, for every real number c the minimal p–weak upper gradient satisfies $g_u = 0$ μ–a.e. on the set $\{x \in X : u(x) = c\}$.

2.2. Local minimizers

Let Ω be an open subset of X, and consider the functional

$$\int_\Omega F(x, g_u(x)) d\mu(x), \tag{2.4}$$

where F satisfies (1.2) for $1 < p \leqslant q$, such that

$$\begin{cases} q \leqslant Qp/(Q-p) & \text{if} \quad p < Q, \\ q < \infty & \text{if} \quad p \geqslant Q, \end{cases}$$

and F is convex in the second variable.

The potential theory in the metric setting is usually studied in complete metric spaces that support a doubling measure and a weak Poincaré inequality. However, it is fairly easy to construct examples of noncomplete metric spaces that satisfy the other two standard assumptions. Additionally, the minimizition problem of (2.4) may not be equivalent in X and in its completion, even if the completion carries the same doubling measure and the weak Poincaré inequality. We invite the reader to see Björn and Marola [3] for more detailed discussion and motivation to work in noncomplete spaces.

In a possibly noncomplete space we have to precise what is meant by "local". We follow the lines in [3], and write $E \dot{\Subset} \Omega$ if E is bounded and

$$\operatorname{dist}(E, X \setminus \Omega) > 0.$$

We say that a function $u \in N^{1,p}_{loc}(\Omega)$ if u is in $N^{1,p}(\Omega')$ for every open (or, equivalently, measurable) $\Omega' \dot{\Subset} \Omega$. As $\Omega' \dot{\Subset} \Omega$ should be understood with respect to X, this means that the local Newtonian space depends on X. Notice, that if X is complete and equipped with a doubling measure, then it is proper (i.e. closed and bounded sets are compact). This implies that $E \dot{\Subset} \Omega$ if and only if $E \Subset \Omega$, and, that $N^{1,p}_{loc}(\Omega)$ is independent of the surrounding space.

In the same sense we say that u is locally bounded in Ω, if u is bounded in all balls $B \dot{\Subset} \Omega$.

We are now ready to define the local minimizers. We say that a function $u \in N^{1,p}_{loc}(\Omega)$ is a local *minimizer* of (2.4) if for all $\varphi \in N^{1,p}(\Omega)$ with $\operatorname{supp}\varphi \dot{\Subset} \Omega$, we have

$$\int_{\operatorname{supp}\varphi} F(x, g_u)\, \mathrm{d}\mu \leqslant \int_{\operatorname{supp}\varphi} F(x, g_{u+\varphi})\, \mathrm{d}\mu \tag{2.5}$$

If (2.5) holds true for all nonpositive $\varphi \in N^{1,p}(\Omega)$ with $\operatorname{supp}\varphi \dot{\Subset} \Omega$, u is called a local *subminimizer* of (2.4).

2.3. Preliminary results

The next lemma is a Sobolev type inequality for Newtonian functions with zero boundary values. For a proof; see [4] or [14].

Lemma 2.1. Let $1 < p < \infty$ and (X, d, μ) be a metric measure space, where μ is doubling and X supports a weak $(1,p)$–Poincaré inequality. Then there exists $c > 0$ such that for every ball $B(z,R)$ with $0 < R \leqslant \frac{1}{3}\operatorname{diam} X$ and every $u \in N^{1,p}_0(B(z,R))$ we have

$$\left(\fint_{B(z,R)} |u|^t \, d\mu\right)^{1/t} \leqslant cR\left(\fint_{B(z,R)} g_u^p \, d\mu\right)^{1/p},$$

where t is as in (2.3).

The proof of Lemma 2.2 can be found, for example, in [15].

Lemma 2.2. Let $(y_n)_{n=1}^{\infty}$ be a sequence of nonnegative real numbers satisfying

$$y_{n+1} \leqslant c b^n y_n^{1+\varepsilon}$$

for all $n = 0, 1, 2, \ldots$, where c, ε, b are positive constants and $b > 1$. Then

$$y_n \leqslant c^{\frac{(1+\varepsilon)^n - 1}{\varepsilon}} b^{\frac{(1+\varepsilon)^n - 1}{\varepsilon^2} - \frac{n}{\varepsilon}} y_0^{(1+\varepsilon)^n}.$$

In particular, if

$$y_0 \leqslant \theta = c^{-1/\varepsilon} b^{-1/\varepsilon^2},$$

then

$$y_n \leqslant \theta b^{-n/\varepsilon}.$$

Finally, we recall a well known iteration lemma; see, for example, [7].

Lemma 2.3. Let $f : [\rho, R] \subset \mathbb{R} \to [0, \infty)$ be a bounded nonnegative function. Suppose that for all ρ', R' such that $\rho \leqslant \rho' < R' \leqslant R$

$$f(\rho') \leqslant \theta f(R') + \big(A(R' - \rho')^{-\alpha} + C\big) \tag{2.6}$$

holds true for some constants $A, C \geqslant 0$, $\alpha > 0$ and $0 \leqslant \theta < 1$. Then

$$f(\rho) \leqslant c(\alpha, \theta)\big(A(R - \rho)^{-\alpha} + C\big). \tag{2.7}$$

3. The local boundedness result

Throughout this section we assume that (X, d, μ) is a metric measure space that supports a $(1, p)$–Poincaré inequality and μ is doubling. We suppose that Ω is an open subset of X.

If $p < Q$, we choose $t = Qp/(Q - p)$ in (2.3) and thus $t \geqslant q$. If $p \geqslant Q$, we can choose t large enough so that $t > q$. In addition, we point out that all constants will be denoted by c and they may not be the same everywhere.

We start by proving a Caccioppoli–type inequality for local minimizers. It remains true for any $q \geqslant p$.

Proposition 3.1. Let $u \in N^{1,p}_{loc}(\Omega)$ be a local minimizer of (2.4). Then for any ball $B(z, R) \Subset \Omega$ and $0 < \rho < R$ we have

$$\int_{A_{z,k,\rho}} g_u^p \, d\mu \leqslant \frac{c}{(R - \rho)^q} \int_{A_{z,k,R}} (u - k)^q \, d\mu + c\mu(A_{z,k,R}), \tag{3.1}$$

where $k > 0$ and

$$A_{z,k,r} = \{x \in B(z, r) \colon u(x) > k\}$$

for all $r > 0$. If $z \in \Omega$ is fixed, we drop the subscript z.

Obviously, we can equivalently write (3.1) in the form

$$\int_{B(z,\rho)} g^p_{(u-k)_+} \, d\mu \leqslant \frac{c}{(R - \rho)^q} \int_{B(z,R)} (u - k)_+^q \, d\mu + c\mu(A_{z,k,R}),$$

where $(u - k)_+ = \max\{u - k, 0\}$.

Proof. Choose ρ' and R' such that $\rho \leqslant \rho' < R' \leqslant R$. Let η be a Lipschitz cut–off function such that $0 \leqslant \eta \leqslant 1$, $\eta = 1$ on $B(z, \rho')$, the support of η is contained in $B(z, R')$ and $g_\eta \leqslant c/(R' - \rho')$. Then set

$$\varphi = -\eta^q (u - k)_+.$$

Now $A_{k,\rho'} \subset \operatorname{supp}\varphi \subset A_{k,R'}$, and we have $\operatorname{supp}\varphi \Subset \Omega$ (note, that $\operatorname{supp}\varphi$ may not be compact). Furthermore, on $A_{k,R}$

$$u + \varphi = u - \eta^q(u-k) = (1-\eta^q)(u-k) + k$$

and hence μ–a.e. on this set

$$\begin{aligned} g_{u+\varphi} &\leqslant (u-k)g_{(1-\eta^q)} + (1-\eta^q)g_{(u-k)} \\ &\leqslant q(u-k)(g_\eta/\eta)\eta^q + (1-\eta^q)g_u. \end{aligned}$$

Since u is a local minimizer of (2.4), we have

$$\begin{aligned} \int_{A_{k,\rho'}} F(x, g_u)\, d\mu &\leqslant \int_{\operatorname{supp}\varphi} F(x, g_u)\, d\mu \\ &\leqslant \int_{\operatorname{supp}\varphi} F(x, g_{u+\varphi})\, d\mu \leqslant \int_{A_{k,R'}} F(x, g_{u+\varphi})\, d\mu \\ &\leqslant \int_{A_{k,R'}} (1-\eta^q)F(x, g_u)\, d\mu + \int_{A_{k,R'}} \eta^q F(x, q(u-k)g_\eta\eta^{-1})\, d\mu, \end{aligned}$$

where we used the convexity of F. Furthermore, using the properties of η and the right–hand side of (1.2) we get

$$\begin{aligned} \int_{A_{k,\rho'}} F(x, g_u)\, d\mu &\leqslant \int_{A_{k,R'}\setminus A_{k,\rho'}} F(x, g_u)\, d\mu \\ &\quad + \int_{A_{k,R'}} \eta^q \Big(1 + (q(u-k)g_\eta\eta^{-1})^q\Big)\, d\mu \\ &\leqslant \int_{A_{k,R'}\setminus A_{k,\rho'}} F(x, g_u)\, d\mu + \mu(A_{k,R}) \\ &\quad + \frac{c}{(R'-\rho')^q} \int_{A_{k,R'}} (u-k)^q\, d\mu. \end{aligned}$$

We use the hole–filling technique of K.–O. Widman, adding $c\int_{A_{k,\rho'}} F(x, g_u)\, d\mu$ on both sides of the above inequality and dividing by $(1+c)$ to get

$$\begin{aligned} \int_{A_{k,\rho'}} F(x, g_u)\, d\mu \leqslant \theta \int_{A_{k,R'}} F(x, g_u)\, d\mu + c\mu(A_{k,R}) \\ + \frac{c}{(R'-\rho')^q} \int_{A_{k,R'}} (u-k)^q\, d\mu, \end{aligned}$$

where $0 < \theta < 1$. Now the assertion follows from Lemma 2.3 and the left–hand side of (1.2). ■

We point out, that the test function constructed in the proof satisfies $v \leqslant u$, which implies that Proposition 3.1 remains true for local subminimizers of (2.4).

Theorem 3.2. Let $u \in N^{1,p}(\Omega)$ be a local minimizer of (2.4). Then u is locally bounded in Ω.

Proof. Fix a ball $B(z,R) \dot{\Subset} \Omega$ and suppose first that $0 < R \leqslant \frac{1}{3} \operatorname{diam} X$ (which is obvious if X is unbounded). Define

$$\rho_n = \frac{R}{2} + \frac{R}{2^{n+1}}, \qquad \bar\rho_n = \frac{\rho_n + \rho_{n+1}}{2},$$
$$k_n = k\big(1 - \frac{1}{2^{n+1}}\big), \qquad n = 0,1,2,\dots,$$

where $k > 0$ is to be chosen later. Observe, that with these definitions we have $\rho_n > \bar\rho_n > \rho_{n+1}$ and $k_{n+1} > k_n$ for all $n = 0,1,2,\dots$.

For each n we choose a Lipschitz function η_n such that $0 \leqslant \eta_n \leqslant 1$, $\eta_n = 1$ on $B(z,\rho_{n+1})$, the support of η_n is contained in $B(z,\bar\rho_n)$ and $g_\eta \leqslant c/(\rho_n - \rho_{n+1}) \leqslant c2^{n+1}/R$. Moreover, let

$$J_n = \int_{A_{k_n,\rho_n}} (u - k_n)^q \, d\mu.$$

Since $\eta_n = 1$ on $B(z,\rho_{n+1})$, by the Hölder inequality and Lemma 2.1 (here we use the fact that $\rho_n \leqslant R \leqslant \frac{1}{3} \operatorname{diam} X$ for all n) we get

$$\begin{aligned} J_{n+1} &= \int_{A_{k_{n+1},\rho_{n+1}}} (u - k_{n+1})^q \eta_n^q \, d\mu \\ &\leqslant \mu(A_{k_{n+1},\rho_{n+1}})^{1-q/t} \Big(\int_{B(z,\bar\rho_n)} (u - k_{n+1})_+^t \eta_n^t \, d\mu \Big)^{q/t} \\ &\leqslant \mu(A_{k_{n+1},\rho_n})^{1-q/t} \Big[c\bar\rho_n \mu(B(z,\bar\rho_n))^{1/t} \Big(\fint_{B(z,\bar\rho_n)} g^p_{(u-k_{n+1})_+\eta_n} \, d\mu \Big)^{1/p} \Big]^q \\ &= c^q \bar\rho_n^q \mu(B(z,\bar\rho_n))^{q/t - q/p} \mu(A_{k_{n+1},\rho_n})^{1-q/t} \Big(\int_{B(z,\bar\rho_n)} g^p_{(u-k_{n+1})_+\eta_n} \, d\mu \Big)^{q/p}. \end{aligned}$$

Since $R/2 < \bar\rho_n \leqslant R$,

$$c^q \bar\rho_n^q \mu(B(z,\bar\rho_n))^{q/t - q/p} \leqslant c$$

for some c depending on q, t. We also observe that

$$\begin{aligned} J_n &\geqslant \int_{A_{k_{n+1},\rho_n}} (u - k_n)^q \, d\mu \geqslant \int_{A_{k_{n+1},\rho_n}} (k_{n+1} - k_n)^q \, d\mu \\ &= k^q 2^{-q(n+2)} \mu(A_{k_{n+1},\rho_n}) \end{aligned}$$

and hence

$$\mu(A_{k_{n+1},\rho_n}) \leqslant k^{-q} 2^{q(n+2)} J_n \leqslant c 2^{qn} J_n \qquad \text{for all } k \geqslant 1. \tag{3.2}$$

Furthermore, $(u-k_{n+1})_+=0$ on $B(z,\bar\rho_n)\setminus A_{k_{n+1},\bar\rho_n}$ so that $g_{(u-k_{n+1})_+\eta_n}=0$ μ–a.e. on the same set. This implies

$$J_{n+1}\leqslant c2^{qn(1-q/t)}J_n^{1-q/t}\bigg(\int_{A_{k_{n+1},\bar\rho_n}} g^p_{(u-k_{n+1})\eta_n}\,d\mu\bigg)^{q/p}. \tag{3.3}$$

On the other hand,

$$\begin{aligned} g_{(u-k_{n+1})\eta_n} &\leqslant \eta_n g_{(u-k_{n+1})}+(u-k_{n+1})g_{\eta_n}\leqslant g_u+(u-k_{n+1})(c2^{n+1}/R)\\ &\leqslant g_u+c2^n(u-k_{n+1})\end{aligned}$$

μ–a.e. on $A_{k_{n+1},\bar\rho_n}$, so that

$$\int_{A_{k_{n+1},\bar\rho_n}} g^p_{(u-k_{n+1})\eta_n}\,d\mu\leqslant c\int_{A_{k_{n+1},\bar\rho_n}} g_u^p\,d\mu + c2^{np}\int_{A_{k_{n+1},\bar\rho_n}}(u-k_{n+1})^p\,d\mu. \tag{3.4}$$

Using Proposition 3.1 (note, that $B(z,\rho_n)\Subset\Omega$ for all n) and (3.2) to the first integral on the right–hand side we get

$$\begin{aligned}\int_{A_{k_{n+1},\bar\rho_n}} g_u^p\,d\mu &\leqslant \frac{c}{(\rho_n-\bar\rho_n)^q}\int_{A_{k_{n+1},\rho_n}}(u-k_{n+1})^q\,d\mu+c\mu(A_{k_{n+1},\rho_n})\\ &\leqslant c\bigg(\frac{2^{n+1}}{R}\bigg)^q\int_{A_{k_n,\rho_n}}(u-k_n)^q\,d\mu+c\mu(A_{k_{n+1},\rho_n})\\ &\leqslant c2^{nq}J_n.\end{aligned} \tag{3.5}$$

To the second integral on the right–hand side of (3.4) we use the Hölder inequality and (3.2), which imply

$$\begin{aligned}\int_{A_{k_{n+1},\bar\rho_n}}(u-k_{n+1})^p\,d\mu &\leqslant \mu(A_{k_{n+1},\rho_n})^{1-p/q}\bigg(\int_{A_{k_{n+1},\rho_n}}(u-k_{n+1})^q\,d\mu\bigg)^{p/q}\\ &\leqslant (c2^{nq}J_n)^{1-p/q}J_n^{p/q}\leqslant c2^{nq(1-p/q)}J_n.\end{aligned} \tag{3.6}$$

Finally, by (3.3), (3.4), (3.5) and (3.6) we get

$$\begin{aligned} J_{n+1}&\leqslant c2^{nq(1-q/t)}J_n^{1-q/t}\Big[\big(c2^{np}J_n\big)^{q/p}+\big(c2^{np}2^{nq(1-p/q)}J_n\big)^{q/p}\Big]\\ &\leqslant c2^{2nq^2/p}J_n^{1-q/t+q/p}.\end{aligned} \tag{3.7}$$

Set $\varepsilon=q/p-q/t$ and $b=2^{2q^2/p}$. By the choice of q,p and t we have $\varepsilon>0$ and $b>1$. Moreover, we can write (3.7) in the form

$$J_{n+1}\leqslant cb^nJ_n^{1+\varepsilon},$$

and Lemma 2.2 implies

$$J_n \leqslant c^{-1/\varepsilon} b^{-1/\varepsilon^2} b^{-n/\varepsilon}$$

provided that we choose $k \geqslant 1$ such that

$$J_0 = \int_{A_{k/2,R}} (u - k/2)^q \, d\mu \leqslant c^{1/\varepsilon} b^{-1/\varepsilon^2}.$$

Then $J_n \to 0$ as $n \to \infty$, so that

$$\int_{A_{k,R/2}} (u-k)^q \, d\mu = 0$$

or equivalently

$$\operatorname*{ess\,sup}_{B(z,R/2)} u \leqslant k.$$

The proof is complete for the case $R \leqslant \frac{1}{3} \operatorname{diam} X$.

Assume then that $R > \frac{1}{3} \operatorname{diam} X$ and let $R' = \frac{1}{12} \operatorname{diam} X$. Then we can find $z' \in B(z, R/2)$ such that

$$\operatorname*{ess\,sup}_{B(z,R/2)} u \leqslant \operatorname*{ess\,sup}_{B(z',R')} u$$

Moreover, $B(z', 2R') \subset B(z, R) \Subset \Omega$, so that by the previous case we get

$$\operatorname*{ess\,sup}_{B(z',R')} u \leqslant k'$$

for some $k' \geqslant 1$.

Using the same argument for $-u$ we can prove that u is also locally bounded from below and the theorem follows. ■

In the proof of Theorem 3.2 the (sub)minimizing property of u is needed only for Proposition 3.1. Hence, it follows from the proof that the subminimizers of (2.4) are locally bounded from above.

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Addresses: Outi Elina Maasalo: Helsinki University of Technology, Institute of Mathematics, P.O. Box 1100, FI-02015 TKK, Finland;
Bianca Stroffolini: Dipartimento di Matematica e Applicazioni, Università degli Studi di Napoli Federico II, Monte S. Angelo, Via Cintia, 80126 Napoli, Italy;
Anna Verde: Dipartimento di Matematica e Applicazioni, Università degli Studi di Napoli Federico II, Monte S. Angelo, Via Cintia, 80126 Napoli, Italy.

E-mail: outi.elina.maasalo@hut.fi, bstroffo@unina.it, anverde@unina.it

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