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ON THE EXISTENCE OF THE MINIMA OF DEGENERATE VARIATIONAL INTEGRALS

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Dedicated to Professor Bogdan Bojarski
on the occasion of his 75th birthday

Abstract: The central theme running through this article is the embedding of Sobolev weighted spaces into Sobolev-Orlicz spaces under minimal assumptions on the weight. As application we prove the existence and regularity of the minimum of functionals of the Calculus of Variations with growth governed by the weight we consider.

Keywords: Orlicz Spaces, Sobolev weighted spaces

1. Introduction

The main purpose of this note is to establish the existence and uniqueness of the minima of certain functionals of the Calculus of Variations. As prototype, we consider functionals of quadratic type

$$F(v,\Omega)=\int_\Omega \langle \mathcal{A}(x)\nabla v,\nabla v\rangle\,dx,\qquad \mathcal{A}^t(x)=\mathcal{A}(x), \tag{1.1}$$

where Ω is a bounded connected open set in $\mathbb{R}^n$. The coefficient matrix is measurable and satisfies the following conditions:

$$\begin{aligned}0\leqslant m(x)|\zeta|^2\leqslant \langle A(x)\zeta,\zeta\rangle \leqslant M(x)|\zeta|^2 \;\text{ a.e. } x\in\Omega,\,\forall\xi\in\mathbb{R}^n,\\ m^{-1},\,M\in\mathscr{L}^1(\Omega).\end{aligned} \tag{1.2}$$

Notice that the minimizer u of (1.1) subjected to the condition $u=u_0$ on $\partial\,\Omega$, if it exists, is also a solution to the Dirichlet problem

$$\begin{cases}\operatorname{div}\mathcal{A}\nabla u=0 & \text{in }\ \Omega\\ u=u_0 & \text{on }\ \partial\Omega\end{cases} \tag{1.3}$$

This type of equation is well-known in the literature. When $0<\lambda\leqslant m(x)\leqslant M(x)\leqslant\Lambda$, for some constant λ and Λ, the Hölder continuity of the weak solutions has been established by De Giorgi [D], Nash [N] and Moser [M]. Furthermore

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higher integrability for ∇u was obtained by Bojarski [B] and Meyers [Me]. Murthy and Stampacchia proved the existence, uniqueness and the boundedness of the weak solutions of (1.3) in the isotropic case; that is, assuming that $M(x) = c\, m(x)$; with suitable integrability assumptions on m and m^{-1} imposed as well, see also [T]. Under these assumptions there have also been established a lot of results concerning *gamma*-convergence, G-convergence or homogenization (see for example [MS], [FM]). These results were generalized in numerous ways. The most significant generalization of this classical results (see for example [FKS]) has been achieved when the weight $m(x)$ belong to the Muckhenhoupt class $\mathscr{A}_2$. Recall that a nonnegative function λ belongs to the Muckhenhoupt class $\mathscr{A}_2(\Omega)$ if and only if

$$\lambda \in \mathscr{A}_2 \Longleftrightarrow \sup_{B\subset\Omega} \fint_B \lambda \fint_B \lambda^{-1} < \infty$$

where B is a ball in $\mathbb{R}^n$. In the present paper we prove the existence and uniqueness of the minima of (1.1) in a suitable weighted Sobolev space $\mathscr{W}(\Omega)$ for anisotropic case as stated in (1.2). Then, we consider more general functionals and we obtain a similar existence and uniqueness result.

It is just routine to observe that if the weight is a constant function, then $u \in \mathscr{L}^{\frac{2n}{n-2}}(\Omega)$ by the classical Sobolev Imbedding Theorem. In our case, the only assumptions $m, \frac{1}{m} \in \mathscr{L}^1(\Omega)$ allow us to prove some additional regularity properties of a function u belonging to the weighted Sobolev spaces $\mathscr{W}(\Omega)$ (see (2.1) for the definition).

2. Preliminaries

In this section we describe the main properties of functional spaces to deal with.

2.1. Sobolev weighted spaces

Fix a non-negative integrable function $m = m(x)$ whose reciprocal is also integrable in Ω. We introduce the following weighted Sobolev spaces:

$$\mathscr{W}(\Omega) = \{u \in \mathscr{W}^{1,1}(\Omega) : \int_\Omega |u| + \int_\Omega m(x)|\nabla u|^2 < +\infty\}, \tag{2.1}$$

$$\mathscr{W}_0(\Omega) = \{u \in \mathscr{W}_0^{1,1}(\Omega) : \int_\Omega |u| + \int_\Omega m(x)|\nabla u|^2 < +\infty\}. \tag{2.2}$$

These spaces will be equipped with the norm (see [MS] for more details)

$$\|u\|^2_{\mathscr{W}(\Omega)} = \int_\Omega |u| + \int_\Omega m(x)|\nabla u|^2.$$

Observe that an equivalent norm on $\mathscr{W}_0(\Omega)$ is

$$\|u\|_{\mathscr{W}_0(\Omega)} = \sqrt{\int_\Omega m(x)|\nabla u|^2\, dx}.$$

2.2. Orlicz spaces

A function $A : [0, +\infty) \longrightarrow [0, +\infty)$ is called a Young function if it has the form

$$A(s) = \int_0^s a(r)\, dr \tag{2.3}$$

where $a : [0, +\infty) \longrightarrow [0, +\infty)$ is any increasing left-continuous function. We also assume that a is neither identically zero nor identically infinite on $(0, +\infty)$. The right-continuous inverse of A is defined on $[0, \infty]$ by the rule

$$A^{-1}(r) = \inf\{s : A(s) > r\} \qquad (\inf \emptyset = \infty),$$

thus,

$$A(A^{-1}(r)) \leqslant r \leqslant A^{-1}(A(r)) \qquad \text{for } r \geqslant 0.$$

The Young conjugate of A, denoted either by $\widetilde{A}$ or by $A^{\sim}$, is defined as

$$\widetilde{A}(s) = \sup\{sr - A(r) : r > 0\}.$$

Notice that when A is a Young function so is $\widetilde{A}$ and $\widetilde{\widetilde{A}} = A$.
The following relations hold for any Young function A:

$$r \leqslant A^{-1}(r)\widetilde{A}^{-1}(r) \leqslant 2r \qquad \text{for } r \geqslant 0.$$

Moreover,

$$A(s) \leqslant sa(s) \leqslant A(2s), \qquad \text{for } s \geqslant 0.$$

Given a Young function A the Orlicz space $\mathscr{L}^A(\Omega)$ is defined as

$$\mathscr{L}^A(\Omega) = \left\{ f : \int_\Omega A\left(\frac{|f(x)|}{\lambda}\right) dx < \infty, \text{ for some } \lambda > 0 \right\}.$$

We supply the Luxemburg norm $\|f\|_{\mathscr{L}^A(\Omega)}$ to this space,

$$\|f\|_{\mathscr{L}^A(\Omega)} = \inf\left\{ \lambda > 0 : \int_\Omega A\left(\frac{|f(x)|}{\lambda}\right)\, dx \leqslant 1 \right\}.$$

In this way $\mathscr{L}^A(\Omega)$ becomes a Banach space. Note that if $A(s) = s^p$ and $p \geqslant 1$, then $\mathscr{L}^A(\Omega)$ coincides with the usual Lebesgue space $\mathscr{L}^p(\Omega)$, and $\|\cdot\|_{\mathscr{L}^A(\Omega)} = \|\cdot\|_{\mathscr{L}^p(\Omega)}$.

The following generalized version of Hölder's inequality holds:

$$\int_\Omega f(x)g(x)\, dx \leqslant 2\|f\|_{\mathscr{L}^A(\Omega)}\|g\|_{\mathscr{L}^{\tilde{A}}(\Omega)}$$

Given a Young function A, we shall consider the (first order) Orlicz-Sobolev space

$$\mathscr{W}^{1,A}(\Omega) = \left\{ u \in \mathscr{L}^A(\Omega) : u,\ |Du| \in \mathscr{L}^A(\Omega) \right\}.$$

This space, equipped with the norm $\|u\|_{\mathscr{W}^{1,A}(\Omega)} = \|u\|_{\mathscr{L}^A(\Omega)} + \|Du\|_{\mathscr{L}^A(\Omega)}$, is a Banach space. Clearly if $A(s) = s^p$ with $p \geqslant 1$ then $\mathscr{W}^{1,A}(\Omega) = \mathscr{W}^{1,p}(\Omega)$, the standard Sobolev space.
As usually, $\mathscr{W}_0^{1,A}(\Omega)$ denotes the completion of $\mathscr{C}_0^\infty(\Omega)$ in $\mathscr{W}^{1,A}(\Omega)$.

It is well-known that the validity of Poincaré-type inequalities and embeddings for space of functions defined in an open set Ω, which do not necessarily vanish on $\partial\Omega$, depends on the regularity of Ω. Let $E \subset \Omega$ we denote by $P(E,\Omega)$ the perimeter of E relative to Ω. For $n \geqslant 2$ and $\sigma \geqslant \frac{1}{n'}$ we set

$$\mathfrak{G}(\sigma) = \left\{ \Omega \subset \mathbb{R}^n : \begin{array}{l} \Omega \text{ is open and there exist positive} \\ \text{numbers } N \text{ and } Q \text{ such that} \\ |E|^\sigma \leqslant QP(E,\Omega) \text{ for all } E \subset \Omega : |E| \leqslant N \end{array} \right\}$$

We denote by Q_σ the number that makes the following inequality true

$$\min{}^\sigma\{|E|, |\Omega - E|\} \leqslant QP(E,\Omega)$$

for all $E \subset \Omega$.
For instance, any open set Ω having finite measure and satisfying the cone property belongs to the class $\mathfrak{G}(\frac{1}{n'})$. We end this section with the following Imbedding Theorem in Orlicz-Sobolev Space.

Theorem 2.1 ([C]). *Let $n \geqslant 2$ and let A be a Young function and let A_n be the Young function defined by*

$$A_n(s) = \int_0^s r^{n'-1}\left(B_n^{-1}\left(r^{n'}\right)\right)^{n'} dr$$

where B_n^{-1} is the (generalized right-continuous) inverse of

$$B_n(s) = \int_0^s \frac{\widetilde{A}(t)}{t^{n'+1}}\, dt.$$

- *If $\Omega \in \mathfrak{G}(\frac{1}{n'})$ is connected and has finite measure, then there exists a constant K depending on A, $|\Omega|$ and $Q_{\frac{1}{n'}}$ such that*

$$\|u - u_\Omega\|_{\mathscr{L}^{A_n}(\Omega)} \leqslant K\|\nabla u\|_{\mathscr{L}^A(\Omega)} \tag{2.4}$$

  where u_Ω is the mean value of u over Ω. If $\int_0^s \frac{\widetilde{A}(t)}{t^{n'+1}}\, dt < \infty$ then K depends only on $Q_{\frac{1}{n'}}$
- *For every $\Omega \in \mathfrak{G}\left(\frac{1}{n'}\right)$ the continuous imbedding holds*

$$\mathscr{W}^{1,A}(\Omega) \longrightarrow \mathscr{L}^{\overline{A}_n}(\Omega).$$

  Here $\overline{A}_n$ is the Young function defined by

$$\overline{A}_n = \begin{cases} A_n(s), & s \geqslant s_2 \\ A(s), & 0 \leqslant s \leqslant s_1 \end{cases}$$

  for suitable $0 < s_1 < s_2$.

3. Embedding of the Weighted Sobolev Spaces into Orlicz Sobolev Spaces

Let us note that given a weight $m \in \mathscr{L}^1(\Omega)$ such that $m^{-1} \in \mathscr{L}^1(\Omega)$, one can prove that m^{-1} belongs to a better space than $\mathscr{L}^1(\Omega)$. Using this property we can deduce also a degree of regularity for functions $u \in \mathscr{W}^{1,1}$ with finite energy, i.e. such that $F(u,\Omega) < \infty$.

In the following we will consider a symmetric $n \times n$ measurable matrix satisfying the bounds

$$0 \leqslant m(x)|\zeta|^2 \leqslant \langle \mathcal{A}(x)\zeta, \zeta\rangle \leqslant M(x)|\zeta|^2 \tag{3.1}$$

for a.e. $x \in \Omega$ and for any $\xi \in \mathbb{R}^n$ where

$$m^{-1} \in \mathscr{L}^1(\Omega), \qquad M \in \mathscr{L}^1(\Omega) \tag{3.2}$$

The classical embedding of $\mathscr{W}_0(\Omega)$ into $\mathscr{W}_0^{1,1}(\Omega)$ expressed by the inequality

$$\|\nabla u\|_{\mathscr{L}^1} \leqslant \|m^{-1}\|^{\frac{1}{2}}_{\mathscr{L}^1(\Omega)} \left(\int_\Omega \langle \mathcal{A}\nabla u, \nabla u\rangle\right)^{\frac{1}{2}}$$

which immediately follows by (3.1), (3.2) can be improved in the Orlicz-Sobolev setting as follows

Theorem 3.1. *Let $\mathcal{A}(x)$ satisfy conditions* (3.1) *and* (3.2). *Then for any $q > 1$ there exists a convex increasing function $H : [0,+\infty] \to [0,+\infty]$ satisfying*

$$\lim_{t \longrightarrow +\infty} \frac{H(t)}{t} = +\infty \tag{3.3}$$

such that for $u \in \mathscr{W}_0(\Omega)$

$$\|\nabla u\|_{\mathscr{L}^H(\Omega)} \leqslant 2q^{\frac{q}{2}} \|m^{-1}\|^{\frac{1}{2}}_{\mathscr{L}^1(\Omega)} \left(\int_\Omega \langle \mathcal{A}\nabla u, \nabla u\rangle\right)^{\frac{1}{2}}. \tag{3.4}$$

Proof. If we define

$$G(t) = \int_0^t \frac{1}{(\Phi(\tau))^{\frac{1}{q}}}\ d\tau$$

where

$$\Phi(\tau) = \left|\left\{x \in \Omega : m(x) < \frac{1}{\tau}\right\}\right|,$$

$G(t)$ is a convex increasing function such that

$$\lim_{t \longrightarrow +\infty} \frac{G(t)}{t} = +\infty. \tag{3.5}$$

Moreover

$$\left\|m^{-1}\right\|_{\mathscr{L}^G} \leqslant q^q \left\|m^{-1}\right\|_{\mathscr{L}^1}. \tag{3.6}$$

In fact, by the inequality

$$\int_\Omega G(m^{-1}) \leqslant q\left(\int_\Omega m^{-1}\,dx\right)^{\frac{1}{q}}$$

we obtain

$$\begin{aligned}\|m^{-1}\|_{\mathscr{L}^G} &= \inf\left\{\lambda>0 : \int_\Omega G\left(\frac{m^{-1}(x)}{\lambda}\right) \leqslant 1\right\} \leqslant \inf\left\{\lambda>0 : \int_\Omega \frac{m^{-1}(x)}{\lambda} \leqslant \frac{1}{q^q}\right\}\\ &= \inf\left\{\lambda>0 : q^q\int_\Omega m^{-1}(x) \leqslant \lambda\right\} = q^q\,\|m^{-1}\|_{\mathscr{L}^1}\,.\end{aligned}$$

Now we define the function H as

$$H(t) = \min_{s>0}\left\{G(s) + \frac{t^2}{s}\right\}. \tag{3.7}$$

This definition is well-posed; in fact, if we consider fixed $t>0$

$$g(s) = G(s) + \frac{t^2}{s},$$

$g'(s)=0$ if and only if $s=s_0$ with s_0 such that

$$s_0^2 G'(s_0) = t^2.$$

As g is strictly convex, the minimum is unique. From the above considerations the definition of $H(t)$ can be also reformulated as follows

$$H(t) = \min_{s>0}\left\{G(s) + \frac{t^2}{s}\right\} = G(s_0) + \frac{t^2}{s_0}.$$

From this remark it is easy to prove that the function H is convex increasing and satisfies

$$\lim_{t\longrightarrow+\infty} \frac{H(t)}{t} = +\infty.$$

Moreover for any $\lambda>0$ and $\mu>0$ taking $s=\frac{m^{-1}}{\mu}$ $t=\frac{\nabla u}{\mu}$ by (3.7) we have

$$H\left(\frac{|\nabla u|}{\mu}\right) \leqslant G\left(\frac{m^{-1}}{\lambda}\right) + \lambda m\left(\frac{|\nabla u|}{\mu}\right)^2.$$

Next, integrating both sides on Ω we obtain

$$\int_\Omega H\left(\frac{|\nabla u|}{\mu}\right) \leqslant \int_\Omega G\left(\frac{m^{-1}}{\lambda}\right) + \frac{\lambda}{\mu^2}\int_\Omega m|\nabla u|^2.$$

By definition of the Luxemburg norm taking $\lambda = 2\|m^{-1}\|_{\mathscr{L}^G(\Omega)}$ we get

$$\int_\Omega G\left(\frac{m^{-1}}{\lambda}\right) \leqslant \frac{1}{2}.$$

At the same time, with the following choice of $\mu = 2\|\sqrt{m}\nabla u\|_{\mathscr{L}^2}\sqrt{\|\frac{1}{m}\|_{\mathscr{L}^G}}$ we arrive to

$$\frac{\lambda}{\mu^2}\int_\Omega m|\nabla u|^2 = \frac{1}{2}.$$

Collecting the previous inequalities we can conclude with the desire estimate

$$\int_\Omega H\left(\frac{|\nabla u|}{\mu}\right) \leqslant \frac{1}{2} + \frac{1}{2} = 1.$$

This, in particular, means that $\mu \geqslant \|\nabla u\|_{\mathscr{L}^H}$; that is

$$\|\nabla u\|_{\mathscr{L}^H} \leqslant 2\left(\int_\Omega m(x)|\nabla u|^2\right)^{\frac{1}{2}}\sqrt{\|\frac{1}{m}\|_{\mathscr{L}^G}} \tag{3.8}$$

Therefore using (3.6) we get the desired estimate

$$\|\nabla u\|_{\mathscr{L}^H} \leqslant 2q^{\frac{q}{2}}\|\sqrt{m}\nabla u\|_{\mathscr{L}^2}\sqrt{\|m^{-1}\|_{\mathscr{L}^1}}.$$ ■

The following example gives an idea of the relations between the functions H and G.

Example 3.2. Let $\alpha > 1$; if we take $G(s) = s^\alpha$ then $H(t) = t^{\frac{2\alpha}{\alpha+1}}$. In fact, using the same notation of Theorem 3.1,

$$H(t) = \min_s\left(s^\alpha + \frac{t^2}{s}\right) = \min g(s) \tag{3.9}$$

The minimum of $g(s)$ is obtained for $s = \alpha^{-(\frac{1}{\alpha+1})}t^{(\frac{2}{\alpha+1})}$, that means

$$H(t) = C(\alpha)t^{(\frac{2\alpha}{\alpha+1})}.$$

Example 3.3. Let $\alpha > 0$, if $G(s) = s\log^\alpha(s)$ then $H(t) \approx t\log^{\frac{\alpha}{2}}(t)$. In fact,

$$H(t) = \min_s\left(s\lg^\alpha s + \frac{t^2}{s}\right) \tag{3.10}$$

we get

$$g'(s) = \lg^\alpha s + \alpha\lg^{\alpha-1} s - \frac{t^2}{s^2} = \frac{1}{s^2}\left(s^2\lg^\alpha s + \alpha s^2\lg^{\alpha-1} s - t^2\right).$$

For s big enough up to lower infinite order terms, we get the minimum for $s^2\lg^\alpha s = t^2$, that means $s = t\lg^{-\frac{\alpha}{2}} t$, that is

$$H(t) \approx 2t\lg^{\frac{\alpha}{2}} t.$$

Similarly, let $\alpha > 0$ if $G(s) = s(\lg\lg s)^\alpha$ then $H(t) = t(\lg\lg t)^{\frac{\alpha}{2}}$.

3.1. Higher integrability

Let us observe that the better integrability argument applied to the reciprocal of the weight is similar to that already employed in [T]. However, here we are able to give an explicit relation between the higher integrability of m^{-1} with respect the higher integrability of a function $u \in \mathscr{W}(\Omega)$. The novelty of our approach is that the gradient of a function u belongs to an Orlicz space $\mathscr{L}^H$ where H does not depend on u but just on the reciprocal of the weight. In fact:

Remark 3.4. By (3.8) if $u \in \mathscr{W}(\Omega)$ then $\nabla u \in \mathscr{L}^H$; now we can apply the imbedding Theorem 2.1 to get:

$$u \in \mathscr{W}(\Omega) \Longrightarrow u \in \mathscr{L}^{\overline{H}_n}(\Omega).$$

If $m^{-1} \in \mathscr{L} \lg \mathscr{L}(\Omega)$ by (3.8) and Example 3.3 with $\alpha = 1$ then $\nabla u \in \mathscr{L} \lg^{\frac{1}{2}} \mathscr{L}(\Omega)$, so using imbedding Theorem 2.1 in the planar case we have:

Corollary 3.5. *Let us assume that* $\frac{1}{m} \in \mathscr{L} \lg \mathscr{L}(\Omega)$*, if* $u \in \mathscr{W}(\Omega)$ *then* $u \in \mathscr{L}^2 \lg \mathscr{L}(\Omega)$.

4. Existence of the minima

Now we are in a position to state the following result:

Theorem 4.1. *Under the assumption in* (1.2) *if* $u_0 \in \mathscr{W}(\Omega)$*, then there exists a unique solution to the following variational problem*

$$\min\left\{\int_\Omega \langle \mathcal{A}(x)\nabla u, \nabla u\rangle : u \in u_0 + \mathscr{W}_0(\Omega)\right\}. \tag{4.1}$$

Proof. We appeal to the direct method of Calculus of Variations. So let u_j be a minimizing sequence; that is, $u_j \in u_0 + \mathscr{W}_0(\Omega)$ with

$$F(u_j, \Omega) \longrightarrow \inf_{v \in u_0 + \mathscr{W}_0(\Omega)} F(v, \Omega). \tag{4.2}$$

For any measurable set $E \subset \Omega$, using Hölder's inequality, we obtain:

$$\int_E |\nabla u_j| \leqslant \left(\int_E \frac{1}{m(x)}\right)^{\frac{1}{2}} \left(\int_\Omega m(x)\,|\nabla u_j|^2\right)^{\frac{1}{2}} \leqslant \left\|\frac{1}{m}\right\|^{\frac{1}{2}}_{\mathscr{L}^1(E)} F(u_j, \Omega)^{\frac{1}{2}}. \tag{4.3}$$

Since (4.2), we get from (4.3) that the sequence $\{|\nabla u_j|\}$ is equi-integrable. Hence, by the De La Vallèe-Poussin theorem, there exists a subsequence of $\{|\nabla u_j|\}$, not relabeled for convenience, weakly converging in $\sigma(\mathscr{L}^1, \mathscr{L}^\infty)$.

On the other hand, by the classical Sobolev Imbedding Theorem, we gain a strong convergence in $\mathscr{L}^1$ of the sequence u_j and so, up to a subsequence, we get that $u_j \rightharpoonup u$ weakly in $\mathscr{W}^{1,1}(\Omega)$.

Then, thanks to the lower semicontinuity of the integral functional

$$G(u,\Omega) = \int_\Omega |u| + \int_\Omega m(x)|\nabla u|^2$$

with respect to the weak convergence in $\mathscr{W}^{1,1}(\Omega)$ we infer that $u \in u_0 + \mathscr{W}_0(\Omega)$.

Finally, the weak convergence in $\mathscr{W}^{1,1}$ guarantees the lower semicontinuity of the functional F that implies that u is a minimum. Moreover since the strict convexity of the functional F we have also the uniqueness. ■

Nevertheless previous result can hold true in a more general setting.
Let $f : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be a Carthèodory integrand; that is,

(i) $f(\cdot, z)$ measurable for every $z \in \mathbb{R}^n$ and
(ii) $f(x, \cdot)$ continuous for a.e. $x \in \Omega$

Let us also assume that

(iii) $f(x, \cdot)$ is convex for a.e. $x \in \Omega$

We now consider functionals of the type

$$\mathcal{F}(u,\Omega) = \int f(x, \nabla u)\, dx \tag{4.4}$$

where we also assume the following p-growth conditions:

$$0 \leqslant m(x)|\zeta|^p \leqslant f(x,\zeta) \leqslant M(x)|\zeta|^p \qquad m, M \in \mathscr{L}^1(\Omega), \left(\frac{1}{m}\right)^{\frac{1}{p-1}} \in \mathscr{L}^1(\Omega). \tag{4.5}$$

On the analogy to the previous case we consider the following weighted Sobolev Spaces

$$\mathscr{W}^p(\Omega) = \{u \in \mathscr{W}^{1,1}(\Omega) : \int_\Omega |u| + \int_\Omega m(x)|\nabla u|^p < +\infty\} \tag{4.6}$$

$$\mathscr{W}_0^p(\Omega) = \{u \in \mathscr{W}_0^{1,1}(\Omega) : \int_\Omega |u| + \int_\Omega m(x)|\nabla u|^p < +\infty\} \tag{4.7}$$

Following the same ideas as in the proof of Theorem 4.1 it is just routine to show that:

Theorem 4.2. *Under the assumption in* (i), (ii), (iii) *and* (4.5) *there exists a unique solution of the following variational problem*

$$\min_{u\in W} \left\{ \int_\Omega f(x, \nabla u) : u \in u_0 + \mathscr{W}_0^p(\Omega) \right\} \tag{4.8}$$

for any $u_0 \in \mathscr{W}^p(\Omega)$.

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