*Functiones et Approximatio*
*40.1 (2009), 91–103*

# REGULARITY OF MAPPINGS OF FINITE DISTORTION

Flavia Giannetti, Luigi Greco, Antonia Passarelli di Napoli

Dedicated to Professor Bogdan Bojarski
on the occasion of his 75th birthday

**Abstract:** We study the degree of regularity of the Jacobian determinant of a mapping of finite distortion $K$, under suitable integrability assumptions on $K$.

**Keywords:** Mappings of finite distortion, Jacobian determinant, higher integrability

## 1. Introduction

In this paper we consider mappings of finite distortion. Let $\Omega$ be a domain of $\mathbb{R}^n$. A mapping $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ is said to be of finite distortion if its Jacobian determinant $J_f = \det Df$ is locally integrable in $\Omega$ and there exists a measurable function $K \geqslant 1$ finite a.e. such that

$$|Df(x)|^n \leqslant K(x)\, J_f(x)\,, \tag{1.1}$$

for a.e. $x \in \Omega$. In the left hand side $|Df|$ is the operator norm of the differential matrix. Inequality (1.1) is termed distortion inequality and the smallest function $K \geqslant 1$ for which it holds is called the distortion function of $f$. If the distortion is bounded, $f$ is called a quasiregular mapping. Clearly, in this case $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. Quasiregular homeomorphisms are called quasiconformal mappings.

In his celebrated paper [11], Gehring proved that the derivatives of a quasiconformal mapping are actually in $L^p_{\mathrm{loc}}(\Omega)$ for some $p > n$. This result was already proved in the plane by Bojarski in [5]. This was subsequently extended to quasiregular mappings by Elcrat and Meyers [9], see also [27]. The above result is a cornerstone in the theory of multidimensional quasiregular mappings, but perhaps more importantly in [11] Gehring invented what is nowadays universally known as Gehring's lemma, showing that the reverse Hölder inequalities imply higher integrability. The extremely high number of generalizations and applications of Gehring's lemma testify for its character of fundamental tool in real analysis. We refer the reader to [12], [17], [18], [21] and the references therein.

**2000 Mathematics Subject Classification:** 26B10, 42B25

Partially supported by PRIN MIUR (2006): "Equazioni e sistemi ellittici e parabolici: stime a priori, esistenza e regolarità" and by GNAMPA

In many circumstances the assumption of boundedness of the distortion is restrictive and it is natural to ask how much this condition can be relaxed so as to still allow for a satisfactory theory. Starting with [8], the class of mappings with exponentially integrable distortion emerged. More recent developments may be found e.g. in the monograph [21] and references therein. Here, we are mainly interested in the integrability properties of the Jacobian determinant and the differential matrix. For this, we refer to [28], [3], [19], [20]. In [28] it was shown that, if $\exp(K^\gamma) \in L^1_{\rm loc}$ for some $\gamma > 1$, then $|Df| \in L^n \log^\alpha L_{\rm loc}$ for all $\alpha > 0$. In [19], [20], it was noticed that, if $\exp(\beta\, K) \in L^1_{\rm loc}$ for some $\beta > 0$, then $J_f \in L^1_{\rm loc}$ implies by the distortion inequality (1.1) that

$$|Df| \in L^n \log^{-1} L_{\rm loc}\,. \tag{1.2}$$

On the other hand, it is known [23] that, for a general map $f \in W^{1,1}_{\rm loc}$ with $J_f \geqslant 0$, the condition (1.2) implies local integrability of the Jacobian. Therefore, (1.2) is a natural condition for mappings with exponentially integrable distortion.

In [22], [3], [19], [20], it is shown that, for each $\alpha \geqslant 0$ there exists $\beta \geqslant 1$ such that, if $\exp(\beta\, K) \in L^1_{\rm loc}$, then $|Df| \in L^n \log^\alpha L_{\rm loc}$. In the recent paper [10] it is proved that, if $\exp(\beta\, K) \in L^1_{\rm loc}$ for some $\beta > 0$, then $J_f \in L \log^\alpha L_{\rm loc}$ with $\alpha = c(n)\beta > 0$. Very recently in [2] the authors obtained the optimal regularity in the plane showing that $J_f \in L \log^\alpha L_{\rm loc}$ for every $\alpha < \beta$.

There are also developments under the assumption of subexponential integrability for $K$, namely, $\exp(P(K)) \in L^1_{\rm loc}$, where $P$ is a nonnegative increasing function verifying the so-called divergence condition

$$\int_1^\infty \frac{P(t)}{t^2}\, dt = \infty\,, \tag{1.3}$$

see [4], [25]. Typical examples are

$$P(t) = \frac{t}{\log(\mathrm{e}+t)}\,, \qquad P(t) = \frac{t}{\log(\mathrm{e}+t)\log\log(9+t)}\,.$$

The divergence condition (1.3) turned out to be essential in many circumstances, see e.g. [21].

Our route to the study of mappings of finite distortion will take place through some precise estimates concerning the integrability of the Jacobian determinant, see Sections 2 and 3. At the present level, the study of integrability properties of the Jacobian originates with the works of Müller [30] and of Iwaniec-Sbordone [23]. The estimates we shall present have independent interest, not only in view of applications to the case of mappings of finite distortion, and extend and generalize many known results, besides [30] and [23], see [6], [15], [29], [16], [13].

Even though we consider the case of nonnegative Jacobian, it is appropriate to mention that it is of interest to know under which conditions one can conclude that the Jacobian belongs to a Hardy space. The systematic study of this problem was initiated in the paper [7].

## 2. The integrability of the Jacobian

For a map $h \in W^{1,1}(\mathbb{R}^n; \mathbb{R}^n)$, we shall denote by $Dh$ the differential matrix and by $J_h = \det Dh$ its Jacobian determinant. For the sake of simplicity, we shall consider mappings $h$ with compact support. Actually, we shall be mainly concerned with the case $h = \varphi f$, where $f$ is a Sobolev mapping on a domain $\Omega$ of $\mathbb{R}^n$, whose Jacobian $J_f$ is nonnegative, and $\varphi \in C_0^\infty(\Omega)$, $\varphi \geqslant 0$. Of course, we set $h(x) = 0$, for all $x \notin \Omega$. Clearly,

$$Dh = \varphi Df + f \otimes \nabla\varphi\,, \qquad J_h = \varphi^n J_f + F, \tag{2.1}$$

where $\varphi^n J_f \geqslant 0$ and $F$ has better integrability properties than $|Df|^n$. See Section 3 for more details.

Let $\mathscr{A} \in C^1([0, +\infty[)$ be a given increasing nonnegative function. We then define the function $\mathscr{B} \in C^1([0, +\infty[)$ by the rule

$$\mathscr{B}(t) = \frac{1}{t} \int_0^{2t} \tau\, \mathscr{A}'(\tau)\, d\tau. \tag{2.2}$$

For future reference, we note two inequalities.

**Lemma 2.1.** *For each $t \geqslant 0$, we have*

$$\mathscr{A}(2t) - \mathscr{A}(t) \leqslant \mathscr{B}(t) \leqslant 2\, \mathscr{A}(2t). \tag{2.3}$$

**Proof.** Both inequalities are straightforward consequences of (2.2). For the first, we have

$$\mathscr{A}(2t) - \mathscr{A}(t) = \int_t^{2t} \mathscr{A}'(\tau)\, d\tau \leqslant \frac{1}{t} \int_t^{2t} \tau\, \mathscr{A}'(\tau)\, d\tau \leqslant \mathscr{B}(t).$$

For the second,

$$\mathscr{B}(t) = 2\, \frac{1}{2t} \int_0^{2t} \tau\, \mathscr{A}'(\tau)\, d\tau \leqslant 2 \int_0^{2t} \mathscr{A}'(\tau)\, d\tau \leqslant 2\, \mathscr{A}(2t). \qquad \blacksquare$$

To present a statement in a simple form, we shall assume that $\mathscr{A}$ satisfies the condition

$$\lim_{t\to\infty} \mathscr{A}(t) = \infty. \tag{2.4}$$

However, we shall also consider the case $\mathscr{A}$ bounded, as the need will arise.

Our main result in this section is the following.

**Theorem 2.2.** *Let $\mathscr{A} \in C^1([0, +\infty[)$ be an increasing nonnegative function satisfying (2.4) and $\mathscr{B}$ be defined by (2.2).*
*There exists a constant $C = C(n) > 0$ such that*

$$\left| \int_{\mathbb{R}^n} J_h\, \mathscr{A}(|Dh|)\, dx \right| \leqslant C(n) \int_{\mathbb{R}^n} |Dh|^n\, \mathscr{B}(|Dh|)\, dx, \tag{2.5}$$

*provided the integral in the left hand side is absolutely converging.*

We stress that the constant $C$ in (2.5) will be found independent of the function $\mathscr{A}$.

**Remark 2.3.** Estimate (2.5) in some sense may be thought of as a higher integrability result for the Jacobian determinant. For the purpose of showing this more evidently, we consider the case of $\mathscr{A}$ satisfying the $\Delta_2$-condition, namely,

$$\mathscr{A}(2t) \leqslant C\, \mathscr{A}(t)\,, \qquad \forall t \geqslant 0,$$

for a suitable constant $C \geqslant 1$. Then, by the second inequality in (2.3), the integrand in the right hand side of (2.5) is controlled by

$$|Dh|^n\, \mathscr{A}(|Dh|)$$

and this dominates also the integrand $J_h\, \mathscr{A}(|Dh|)$ in the left hand side in view of Hadamard inequality. On the other hand, the function $\mathscr{B}$ can be essentially smaller than $\mathscr{A}$.

To prove Theorem 2.2, we shall follow the approach of [14], which is based on the well-known approximation result by Lipschitz function of Acerbi-Fusco [1] stated in the next lemma, and an argument due to Lewis [26].

**Lemma 2.4.** *There exists a constant $C = C(n)$ with the following property. For every $t > 0$, we can find a $C\,t$-Lipschitz map $g = g_t \colon \mathbb{R}^n \to \mathbb{R}^n$ which coincides with $h$ a.e. on the set*

$$\{x \in \mathbb{R}^n : \mathcal{M}\,|Dh|(x) \leqslant t\}.$$

Hereafter, $\mathcal{M}$ denotes the familiar Hardy-Littlewood maximal operator. We need to recall ([31]) the following well-known weak-type estimate:

$$|\{\mathcal{M}\, f(x) > t\}| \leqslant \frac{C(n)}{t} \int_{f(x)>t/2} f(x)\, dx, \tag{2.6}$$

for a nonnegative function $f$ on $\mathbb{R}^n$.

**Proof of Theorem 2.2.** Clearly, we can assume that the right hand side in (2.5) is finite. Moreover, as interchanging two components of $h$ turns the Jacobian determinant to the opposite, while $|Dh|$ is not affected, we can also assume without loss of generality that the integral in the left hand side is positive and drop the absolute value.

We first show that

$$\int_{\mathbb{R}^n} J_h\, dx = 0. \tag{2.7}$$

To this effect, we follow an idea of [14], see also [21]. To shorten notation, in the sequel we set

$$D = |Dh|\,, \qquad M = \mathcal{M}\, D = \mathcal{M}\,|Dh|.$$

For $t > 0$, let $g$ be a Lipschitz mapping given by Lemma 2.4. It is easily seen that also $g$ has compact support and thus

$$\int_{\mathbb{R}^n} J_g \, dx = 0, \tag{2.8}$$

that is, by the properties of $g$, using also (2.6)

$$\int_{M \leqslant t} J_h \, dx = - \int_{M > t} J_g \, dx \leqslant C\, t \int_{2D > t} D^{n-1} \, dx. \tag{2.9}$$

Now we want to pass to the limit in (2.9) as $t \to \infty$. To this aim, we notice that $J_h \, \mathscr{A}(D) \in L^1(\mathbb{R}^n)$ implies $J_h \in L^1(\mathbb{R}^n)$ and hence

$$\int_{\mathbb{R}^n} J_h \, dx = \lim_{t \to \infty} \int_{M \leqslant t} J_h \, dx. \tag{2.10}$$

On the other hand, if we multiply last term in (2.9) by $\mathscr{A}'(t) \geqslant 0$ and integrate over $(0, \infty)$ with respect to $t$, then by the Fubini theorem and recalling the definition (2.2) of $\mathscr{B}$, we see that the integral

$$\int_0^\infty t \, \mathscr{A}'(t) \, dt \int_{2D > t} D^{n-1} \, dx = \int_{\mathbb{R}^n} D^n \, \mathscr{B}(D) \, dx \tag{2.11}$$

is finite, that is, the function

$$t \mapsto t \, \mathscr{A}'(t) \int_{2D > t} D^{n-1} \, dx$$

is integrable. As by (2.4) clearly $\mathscr{A}'$ is not integrable, we have

$$\liminf_{t \to \infty} \; t \int_{2D > t} D^{n-1} \, dx = 0 \tag{2.12}$$

and (2.7) follows.

Once we have established (2.7), using it in conjunction with (2.8), we find

$$\begin{aligned} \int_{D > t} J_h \, dx = - \int_{D \leqslant t} J_h \, dx &= - \int_{M \leqslant t} J_h \, dx - \int_{D \leqslant t < M} J_h \, dx \\ &= \int_{M > t} J_g \, dx - \int_{D \leqslant t < M} J_h \, dx \, . \end{aligned} \tag{2.13}$$

and hence

$$\int_{D > t} J_h \, dx \leqslant C\, t^n \, |\{M > t\}| \, . \tag{2.14}$$

Using also (2.6), therefore we find that

$$\int_{D > t} J_h \, dx \leqslant C\, t \int_{2D > t} D^{n-1} \, dx \, . \tag{2.15}$$

Now we argue as above: we multiply both sides of (2.15) by $\mathscr{A}'(t)$ and then integrate over $(0,\infty)$ with respect to $t$. Concerning the left hand side, by the Fubini theorem, we get

$$\int_0^\infty \mathscr{A}'(t)\,dt \int_{M>t} J_h\,dx = \int_{\mathbb{R}^n} J_h[\mathscr{A}(M) - \mathscr{A}(0)]\,dx$$

and using again (2.7)

$$\int_0^\infty \mathscr{A}'(t)\,dt \int_{M>t} J_h\,dx = \int_{\mathbb{R}^n} J_h\,\mathscr{A}(M)\,dx\,. \tag{2.16}$$

For the right hand side, we have (2.11). Therefore, we end up with estimate (2.5). ■

**Remark 2.5.** In Theorem 2.2 we assumed $J_h\,\mathscr{A}(|Dh|) \in L^1(\mathbb{R}^n)$, but as it is readily seen, the arguments proving (2.5) extend to handle the case where merely the negative part of $J_h\,\mathscr{A}(|Dh|)$ is assumed to be integrable. This seemingly trivial observation will be very useful when considering a map $h = \varphi f$, with $J_f \geqslant 0$, as mentioned at the beginning of this section.

**Remark 2.6.** We can dispense with condition (2.4), that is, consider $\mathscr{A}$ bounded, if we know that (2.7) holds. In fact, then the proof runs with no further modifications. Condition (2.7) in turn follows if $J_h \in L^1(\mathbb{R}^n)$ and (2.12) holds. Notice that, for $\mathscr{A}$ bounded (and $\mathscr{A} \not\equiv 0$), $J_h \in L^1(\mathbb{R}^n)$ is equivalent to $J_h\,\mathscr{A}(|Dh|) \in L^1(\mathbb{R}^n)$. A sufficient condition for (2.12) is $|Dh| \in L^n \log^{-1} L(\mathbb{R}^n)$, or more generally $P(|Df|^n) \in L^1(\mathbb{R}^n)$, for a function $P \in C^1([0,\infty[)$ verifying the divergence condition (1.3). Indeed, (1.3) implies (actually, it is equivalent to)

$$\int_1^\infty \left(\frac{P(2^{-n}t^n)}{t^{n-1}}\right)' \frac{1}{t}\,dt = \infty, \tag{2.17}$$

and then (2.12) can be proved as above, multiplying the last term in (2.9) by $\big(P(2^{-n}t^n)/t^{n-1}\big)'/t$ and integrating over $(1,\infty)$.

To illustrate better our estimate (2.5), we shall make now several applications of it choosing the function $\mathscr{A}$.

**Example 2.7.** Here we consider the case $\mathscr{A}(t) = t^{p-n}$, for $p > n$. Then (2.2) gives

$$\mathscr{B}(t) = \frac{p-n}{t}\int_0^{2t} \tau^{1+p-n-1}\,d\tau \leqslant n\,2^{p-n+1}\left(1 - \frac{n}{p}\right) t^{p-n}.$$

Hence, assuming also $p \leqslant 2n$, (2.5) yields

$$\left|\int_{\mathbb{R}^n} |Dh|^{p-n}\,J_h\,dx\right| \leqslant C(n)\left(1 - \frac{n}{p}\right)\int_{\mathbb{R}^n} |Dh|^p\,dx, \tag{2.18}$$

compare with inequality (7.98) of [21], pg. 164. Notice that (2.18) trivially holds with constant 1 for each $p$, by the Hadamard inequality. Therefore, the essence of this estimate is in the constant appearing in front of the integral in the right hand side, for $p$ close to $n$; this in turn depends on identity (2.7). Also, we can easily dispense with the condition that $h$ has compact support.

**Example 2.8.** Now, for $\alpha > 0$, we take

$$\mathscr{A}(t) = \log^{\alpha}(\mathrm{e} + t).$$

We then immediately estimate

$$\mathscr{B}(t) \leqslant \frac{\alpha}{t} \log^{\alpha}(\mathrm{e} + 2t) \int_0^{2t} \log^{-1}(\mathrm{e} + \tau)\, d\tau.$$

Moreover, it can be shown that, $\forall T > 0$,

$$\int_0^T \log^{-1}(\mathrm{e} + \tau)\, d\tau \leqslant C\, T \log^{-1}(\mathrm{e} + T), \tag{2.19}$$

with $C > 0$. Indeed, the continuous function

$$T \mapsto \frac{1}{T} \int_0^T \frac{\log(\mathrm{e} + T)}{\log(\mathrm{e} + \tau)}\, d\tau$$

converges both as $T \to 0$ and as $T \to \infty$ and so it is bounded. Hence we have

$$\mathscr{B}(t) \leqslant C\, \alpha \log^{\alpha - 1}(\mathrm{e} + 2t)$$

and (2.5) gives

$$\int_{\mathbb{R}^n} J_h \log^{\alpha}(\mathrm{e} + |Dh|)\, dx \leqslant C(n)\, \alpha \int_{\mathbb{R}^n} |Dh|^n \log^{\alpha - 1}(\mathrm{e} + 2|Dh|)\, dx. \tag{2.20}$$

Estimate (2.20) is qualitatively well-known; local variants of it for a mapping $f$ with $J_f \geqslant 0$ may be used to prove the Müller theorem [30] ($\alpha = 1$), the Iwaniec and Sbordone theorem [23] ($\alpha = 0$), a result by Brezis, Fusco and Sbordone [6] ($0 < \alpha < 1$), a result by Greco and Iwaniec [15] ($\alpha = 2$) and by Greco, Iwaniec and Moscariello [16] ($\alpha > 0$).

The novelty in (2.20) is the presence of the factor $\alpha$ in front of the integral in the right hand side, which again is a consequence of (2.7). In this form, (2.20) may be suggested by the considerations of [10].

**Example 2.9.** This example will correspond to the case considered in [28]. We set

$$\mathscr{A}(t) = \exp(\alpha \log^{\vartheta}(\mathrm{e} + t)),$$

with $\alpha > 0$ and $0 < \vartheta < 1$. By the definition (2.2) we find

$$\mathscr{B}(t) \leqslant \alpha\vartheta \exp\left(\alpha \log^{\vartheta}(\mathrm{e} + 2t)\right) \log^{\vartheta}(\mathrm{e} + 2t) \frac{1}{t} \int_0^{2t} \log^{-1}(\mathrm{e} + \tau)\, d\tau$$

and hence, using also (2.19), we see that

$$\mathscr{B}(t) \leqslant C\alpha\vartheta \exp\left(\alpha \log^{\vartheta}(\mathrm{e}+2t)\right) \log^{\vartheta-1}(\mathrm{e}+2t),$$

where $C$ is a universal constant (independent of $\alpha$ and $\vartheta$). Therefore,

$$\begin{aligned} &\int_{\mathbb{R}^n} J_h \exp(\alpha \log^{\vartheta}(\mathrm{e}+|Dh|))\, dx \\ &\qquad \leqslant C(n)\alpha\vartheta \int_{\mathbb{R}^n} |Dh|^n \exp(\alpha \log^{\vartheta}(\mathrm{e}+2|Dh|)) \log^{\vartheta-1}(\mathrm{e}+2|Dh|)\, dx. \end{aligned} \tag{2.21}$$

**Remark 2.10.** It is interesting to examine (2.5) when $\mathscr{A}$ is of the form

$$\mathscr{A}_\alpha(t) = (\mathscr{A}_1(t))^\alpha, \tag{2.22}$$

where $\mathscr{A}_1$ is a fixed nonnegative increasing function, and $\alpha > 0$. This was the case in all the examples we considered. Then (2.2) gives

$$\mathscr{B}(t) = \mathscr{B}_\alpha(t) = \frac{\alpha}{t}\int_0^{2t} \tau(\mathscr{A}_1(\tau))^{\alpha-1}\, \mathscr{A}_1'(\tau)\, d\tau. \tag{2.23}$$

Moreover, we have

$$\mathscr{B}(t) \leqslant \alpha\, \mathscr{A}_\alpha(2t)\left(\frac{1}{t}\int_0^{2t} \tau\frac{\mathscr{A}_1'(\tau)}{\mathscr{A}_1(\tau)}\, d\tau\right). \tag{2.24}$$

## 3. The case of nonnegative Jacobian

In this section we derive some consequences of (2.5) for the case alluded to at the beginning of Section 2, concerning a mapping of the form

$$h = \varphi f,$$

with $f$ a Sobolev mapping on a domain $\Omega$ of $\mathbb{R}^n$, having nonnegative Jacobian $J_f$, and $\varphi \in C_0^\infty(\Omega)$, $\varphi \geqslant 0$. All the expressions containing $\varphi$ or its derivatives as a factor are extended to the entire space $\mathbb{R}^n$ to be 0, $\forall x \notin \Omega$. We have formulas (2.1) relating the differential matrices and the Jacobian determinants of the two maps $f$ and $h$, where we recall that $\varphi^n J_f \geqslant 0$ and $F$ has better integrability properties than $|Df|^n$. Indeed, we have

$$|F| \leqslant C(n)\, |\varphi Df|^{n-1}\, |f \otimes \nabla\varphi|. \tag{3.1}$$

**Remark 3.1.** As an illustration, if we can prove (2.7), then essentially taking $\varphi$ a cut-off function, we get

$$\fint_Q J\, dx \leqslant C(n)\left(\fint_{2Q} |Df|^{\frac{n^2}{n+1}} dx\right)^{\frac{n+1}{n}} \tag{3.2}$$

for every cube $Q$ such that $2Q \subset \Omega$. More details are in [21]. Inequality (3.2) was crucial in many results on the Jacobian. It is standard for example under the natural assumption $f \in W^{1,n}_{\mathrm{loc}}$. It seems to be proven under some conditions below the natural one for the first time in [13].

In order to estimate $\varphi^n J_f\,\mathscr{A}(|\varphi Df|)$ in terms of $J_h\,\mathscr{A}(|Dh|)$, we shall note some point-wise inequalities relating these quantities. We start with the trivial estimate

$$\begin{aligned}\varphi^n J_f\,\mathscr{A}(|\varphi Df|) \leqslant J_h\,\mathscr{A}(|Dh|) + |F|\,\mathscr{A}(|Dh|) \\ + \varphi^n J_f\big[\,\mathscr{A}(|\varphi Df|) - \mathscr{A}(|Dh|)\big].\end{aligned} \tag{3.3}$$

Using (3.1) and the monotonicity of $\mathscr{A}$, we have easily

$$|F|\,\mathscr{A}(|Dh|) \leqslant C(n)\,|\varphi Df|^{n-1}\,|f\otimes\nabla\varphi|\;\mathscr{A}(|\varphi Df| + |f\otimes\nabla\varphi|). \tag{3.4}$$

To estimate the last term in the right hand side of (3.3) we only need to consider the case $|\varphi Df| \geqslant |Dh|$. If $|Dh| \leqslant |\varphi Df|/2$, then

$$|\varphi Df| \leqslant 2|f\otimes\nabla\varphi|$$

and by the Hadamard inequality we can estimate again by the right hand side in (3.4):

$$\begin{aligned}\varphi^n J_f\big[\,\mathscr{A}(|\varphi Df|) - \mathscr{A}(|Dh|)\big] \\ \leqslant C(n)\,|\varphi Df|^{n-1}\,|f\otimes\nabla\varphi|\;\mathscr{A}(|\varphi Df| + |f\otimes\nabla\varphi|).\end{aligned} \tag{3.5}$$

For the case $|\varphi Df| \geqslant |Dh| \geqslant |\varphi Df|/2$, we use (2.3) with $t = |\varphi Df|/2$ and obtain

$$\varphi^n J_f\big[\,\mathscr{A}(|\varphi Df|) - \mathscr{A}(|Dh|)\big] \leqslant |\varphi Df|^n\,\mathscr{B}(|\varphi Df|/2). \tag{3.6}$$

In the right hand side of (2.5) we use the monotonicity of the function $t \mapsto t^n\,\mathscr{B}(t)$:

$$\begin{aligned}|Dh|^n\,\mathscr{B}(|Dh|) &\leqslant (|\varphi Df| + |f\otimes\nabla\varphi|)^n\,\mathscr{B}(|\varphi Df| + |f\otimes\nabla\varphi|) \\ &\leqslant 2^n\big[|\varphi Df|^n\,\mathscr{B}(2|\varphi Df|) + |f\otimes\nabla\varphi|^n\,\mathscr{B}(2|f\otimes\nabla\varphi|)\big].\end{aligned} \tag{3.7}$$

Moreover, using the second inequality in (2.3) we find

$$\mathscr{B}(2|f\otimes\nabla\varphi|) \leqslant 2\,\mathscr{A}(4|f\otimes\nabla\varphi|).$$

We are now in a position to state a Corollary of Theorem 2.2.

**Corollary 3.2.** *Under the above assumptions, we have*

$$\begin{aligned}\int_{\mathbb{R}^n} \varphi^n J_f\,\mathscr{A}(|\varphi Df|)\,dx \leqslant C\int_{\mathbb{R}^n} |\varphi Df|^n\,\mathscr{B}(2|\varphi Df|)\,dx \\ + C\int_{\mathbb{R}^n} |\varphi Df|^{n-1}\,|f\otimes\nabla\varphi|\;\mathscr{A}(|\varphi Df| + |f\otimes\nabla\varphi|)\,dx \\ + C\int_{\mathbb{R}^n} |f\otimes\nabla\varphi|^n\,\mathscr{A}(4|f\otimes\nabla\varphi|)\,dx.\end{aligned} \tag{3.8}$$

## 4. Mappings of finite distortion

The aim of this section is to show that the estimate (3.8) implies regularity results for mappings of finite distortion, when coupled with a suitable Young type inequality. The following theorem may be seen as an interpolation between Gehring's result and the result of Theorem 1.1 of [10]. It improves the conclusion of [28].

**Theorem 4.1.** *Let $f$ be a mapping of finite distortion $K$ such that $\exp(\beta K^\gamma) \in L^1_{\mathrm{loc}}(\Omega)$, for some $\beta > 0$ and $\gamma > 1$. Then there exists $\alpha > 0$ such that*

$$\begin{gathered} J_f \exp\big(\alpha \log^{1-1/\gamma}(\mathrm{e} + J_f)\big) \in L^1_{\mathrm{loc}}(\Omega), \\ |Df|^n \exp\big(\alpha \log^{1-1/\gamma}(\mathrm{e} + |Df|)\big) \log^{-1/\gamma}(\mathrm{e} + |Df|) \in L^1_{\mathrm{loc}}(\Omega). \end{gathered} \tag{4.1}$$

We shall need the following

**Lemma 4.2 (Young inequality).** *Let $0 < \alpha \leqslant 1$, $\beta > 0$, $\gamma > 1$ and set $\vartheta = 1 - 1/\gamma$. Then, for any $K \geqslant 0$, $J \geqslant 0$, $D \geqslant 0$ with $J \leqslant D^n$, we have*

$$\begin{aligned} &KJ \exp\big(\alpha \log^\vartheta(\mathrm{e} + D)\big) \log^{\vartheta - 1}(\mathrm{e} + D) \\ &\qquad \leqslant 2\frac{n+1}{\beta^{1/\gamma}} \Big\{ \exp(\beta K^\gamma) + J \exp\big(\alpha \log^\vartheta(\mathrm{e} + D)\big)\Big\} \end{aligned} \tag{4.2}$$

**Proof.** Actually, we shall prove the following

$$\begin{aligned} &KD^n \exp\big(\alpha \log^\vartheta(\mathrm{e} + D)\big) \log^{\vartheta - 1}(\mathrm{e} + D) \\ &\qquad \leqslant 2\frac{n+1}{\beta^{1/\gamma}} \Big\{ \exp(\beta K^\gamma) + D^n \exp\big(\alpha \log^\vartheta(\mathrm{e} + D)\big)\Big\} \end{aligned} \tag{4.3}$$

which implies (4.2) as $J/D^n \leqslant 1$. We use the elementary inequality

$$ab \leqslant \exp(a^\gamma) + 2b \log^{1/\gamma}(\mathrm{e} + b), \quad a \geqslant 0,\ b \geqslant 0,$$

with $a = \beta^{1/\gamma} K$ and $b = D^n \exp\big(\alpha \log^\vartheta(\mathrm{e} + D)\big) \log^{\vartheta-1}(\mathrm{e} + D)$. Inequality (4.3) follows, as then $b \log^{1/\gamma}(\mathrm{e} + b)$ is easily bounded by

$$\begin{aligned} &D^n \exp\big(\alpha \log^\vartheta(\mathrm{e} + D)\big) \log^{\vartheta-1}(\mathrm{e} + D) \log^{1/\gamma}\big(\mathrm{e} + D^n \exp\big(\log^\vartheta(\mathrm{e} + D)\big)\big) \\ &\qquad \leqslant D^n \exp\big(\alpha \log^\vartheta(\mathrm{e} + D)\big) \log^{\vartheta-1}(\mathrm{e} + D) \log^{1/\gamma}\big((\mathrm{e} + D)^{n+1}\big) \\ &\qquad \leqslant (n+1) D^n \exp\big(\alpha \log^\vartheta(\mathrm{e} + D)\big). \end{aligned}$$

■

**Proof of Theorem 4.1.** We choose $\varphi \in C_0^\infty(\Omega)$, $\varphi \geqslant 0$, and apply Corollary 3.2 with the function

$$\mathscr{A}(t) = \exp\big(\alpha \log^\vartheta(\mathrm{e} + t)\big)$$

considered in Example 2.9. Accordingly,

$$\begin{aligned} &\int_{\mathbb{R}^n} J \exp\big(\alpha \log^\vartheta(\mathrm{e} + D)\big)\, dx \\ &\qquad \leqslant \int_{\mathbb{R}^n} H\, dx + C(n)\alpha \int_{\mathbb{R}^n} D^n \exp\big(\alpha \log^\vartheta(\mathrm{e} + D)\big) \log^{\vartheta-1}(\mathrm{e} + D)\, dx, \end{aligned} \tag{4.4}$$

where for simplicity we wrote $J = \varphi^n J_f = \det(\varphi Df)$, $D = |\varphi Df|$, and $H \in L^1(\mathbb{R}^n)$ is supported in $\operatorname{supp}\varphi$. We also used that $\mathscr{A}(4t) \leqslant 4\mathscr{A}(t)$, $\forall t \geqslant 0$. In the last integral of (4.4) we use the distortion inequality $D^n \leqslant KJ$ and then Young inequality (4.2):

$$\begin{aligned}\int_{\mathbb{R}^n} J\exp\big(\alpha\log^\vartheta(\mathrm{e}+D)\big)\,dx \leqslant \int_{\mathbb{R}^n} H\,dx + C\int_{\operatorname{supp}\varphi}\exp(\beta K^\gamma)\,dx \\ + \frac{C(n)\alpha}{\beta^{1/\gamma}}\int_{\mathbb{R}^n} J\exp\big(\alpha\log^\vartheta(\mathrm{e}+D)\big)\,dx.\end{aligned} \tag{4.5}$$

Choosing $\alpha$ small enough so that $C(n)\alpha\beta^{-1/\gamma} \leqslant 1/2$, in (4.5) last integral can be absorbed in the left hand side to give

$$\int_{\mathbb{R}^n} J\exp\big(\alpha\log^\vartheta(\mathrm{e}+D)\big)\,dx \leqslant 2\int_{\mathbb{R}^n} H\,dx + C\int_{\operatorname{supp}\varphi}\exp(\beta K^\gamma)\,dx, \tag{4.6}$$

provided we known a priori that $J\exp\big(\alpha\log^\vartheta(\mathrm{e}+D)\big)$ is integrable. To get rid of this condition, we apply the above argument with a truncated version of the function $\mathscr{A}$:

$$\mathscr{A}_T(t) = \min\{\mathscr{A}(t),\ \mathscr{A}(T)\} = \begin{cases}\mathscr{A}(t), & \text{for } 0 \leqslant t \leqslant T\\ \mathscr{A}(T), & \text{for } t \geqslant T\end{cases}$$

where $T > 0$; that is, $\mathscr{A}_T'(t) = 0$, $\forall t > T$. We can use Remark 2.6, as $f$ is a map of exponentially integrable distortion. All the estimates hold uniformly with respect to $T$ and we get again (4.6) letting $T \to \infty$. ■

**Remark 4.3.** Examinating how the above estimates depend on $\gamma$ and $\beta$ will reveal that the case of bounded distortion follows from Theorem 4.1 letting $\gamma \to \infty$. Indeed, it suffices to take $\beta = \|K\|_\infty^{-\gamma}$ for fixed $\gamma > 1$, and then pass to the limit.

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**Address:** Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università degli Studi di Napoli "Federico II", Via Cintia – 80126 Napoli

**E-mail:** giannett@unina.it, luigreco@unina.it, antonia.passarelli@unina.it

**Received:** 31 May 2008; **revised:** 11 September 2008