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THE DILATATION FUNCTION OF A HOLOMORPHIC ISOTOPY

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To Bogdan Bojarski with admiration

Abstract: Every nonvanishing univalent function $f(z)$ in the disk $\Delta^* = \widehat{\mathbb{C}} \setminus \overline{\Delta}, \Delta = \{|z| < 1\}$, for example, with hydrodynamical normalization, generates a complex isotopy $f_t(z) = tf(t^{-1}z) : \Delta^* \times \Delta \to \widehat{\mathbb{C}}$, which is a special case of holomorphic motions and plays an important role in many topics. Let q_f denote the minimal dilatation among quasiconformal extensions of f to $\widehat{\mathbb{C}}$.

In 1995, R. Kühnau raised the questions whether the dilatation function $q_f(r) = q_{f_r}$ is real analytic and whether the function f can be reconstructed if $q_f(r)$ is given. The analyticity of q_f was known only for ellipses and for the Cassini ovals.

Our main theorem provides a wide class of maps with analytic dilatations and implies also a negative answer to the second question.

Keywords: Univalent function, quasiconformal map, dilatation, subharmonic function, universal Teichmüller space, hyperbolic metrics, pluricomplex Green function

1. Dilatation function generated by univalent function

1.1. We consider the nonvanishing univalent functions in the disk

$$\Delta^* = \{z \in \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} : |z| > 1\}$$

with hydrodynamical normalization, i.e., of the form

$$f(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots \quad (f(z) \neq 0).$$

The collection of such functions is denoted by Σ. Let $\Sigma(k)$ be its subset containing the function with k-quasiconformal extensions to the unit disk $\Delta = \{z : |z| < 1\}$, and $\Sigma^0 = \bigcup_k \Sigma(k)$. This subset is closed in Σ in the topology of locally uniform convergence in Δ^*.

For $f \in \Sigma^0$, we denote by $\mu_{\widehat{f}}$ the Beltrami coefficient (or the complex dilatation)

$$\mu_{\widehat{f}} = \partial_{\bar{z}} \widehat{f} / \partial_z \widehat{f}$$

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of a quasiconformal extension $\widehat{f}$ of f to $\widehat{\mathbb{C}}$ and consider the *minimal dilatation*

$$q_f = \inf\{\|\mu_{\widehat{f}}\|_\infty : \widehat{f}|\Delta^* = f\},$$

which is the Teichmüller norm of f.

Each function $f \in \Sigma$ admits a complex isotopy

$$f_t(z) = tf\left(\frac{z}{t}\right) = z + b_0 t + b_1 t^2 z^{-1} + b_2 t^3 z^{-2} + \dots : \Delta^* \times \Delta \to \widehat{\mathbb{C}} \qquad (1.1)$$

which connects f with the identity map id and is $\widehat{\mathbb{C}}$-holomorphic in $\Delta^* \times \Delta$. This isotopy is a special case of holomorphic motions and plays an important role in many subjects areas. It generates the *dilatation function*

$$q_f(t) = q_{f_t},$$

which is continuous and circularly symmetric, i.e., $q_f(t) = q_f(|t|)$. Other important properties of q_f will be given below.

Note that this function is closely connected, for example, with Fredholm eigenvalues of Jordan curves.

1.2. Let us first mention the quantitative properties of the dilatation function, which are given by

Theorem A.

(a) *If a function $f(z) = z + b_0 + b_1 z^{-1} + \dots$ belongs to $\Sigma(k)$, then for any $t \in \Delta$ the map $f_t(z) = tf(t^{-1}z)$ belongs to $\Sigma(k|t|^2)$. This bound $q_{f_t} \leqslant k|t|^2$ is sharp.*

(b) *If*

$$f(z) = z + \sum_{n=p}^{\infty} b_n z^{-n}, \quad b_p \neq 0, \qquad (1.2)$$

for some integer $p > 1$, then $q_{f_t} \leqslant k|t|^{p+1}$. This bound is also sharp.

(c) *If the equality $q_{f_t} = k|t|^2$ occurs for some $t_0 \neq 0$, then it holds for all $t \in \Delta$. This occurs only for the maps*

$$J(z) = z + b_0 + b_1 z^{-1} \qquad \text{with } |b_1| = k, \qquad (1.3)$$

for which $J_t(z) = z + b_0 t + b_1 t^2/z$ and the extremal extensions onto Δ are of the form $J_t(z) = z + b_0 t + kt^2\overline{z}$.

The equality $q_{f_t} = k|t|^{p+1}$ is attained on the functions

$$J_p(z) = \left[J(z^{(p+1)/2}) - b_0\right]^{2/(p+1)} + c = z + c + \frac{2b_1}{p+1}\frac{1}{z^p} + \dots,$$

where $|b_1| = k$, $c = \text{const}$.

The proof of this theorem for $k = 1$ is given in [Kr1] in the line of the Gardiner-Royden theorem on equality of the Kobayashi and Teichmüller metrics on Teichmüller spaces (see [GL], [Ro]). The case $k < 1$ requires different arguments and relies on the plurisubharmonicity of the Teichmüller metric of the universal Teichmüller space $\mathbf{T}$ (cf. [Kr2], [Kr3]).

This theorem is rich in applications. The related problems were considered, for example, in [KK], [Ku2].

For small $|t|$, there is the asymptotic estimate

$$q_{f_t} = |b_1||t|^2 + O(|t|^3), \quad t \to 0, \tag{1.4}$$

which is sharp when $b_1 \neq 0$; it was obtained by Kühnau (see [KK, p. 102]). The proof relies on the bound $|b_1| \leqslant k$ on $\Sigma(k)$ which holds for all $k \leqslant 1$. The arguments break down in getting a sharp estimate for the functions having in Δ^* expansions of the form (2).

1.3. The dilatation function is connected by

$$\tau_{\mathbf{T}}([f_t], \mathbf{0}) = \tanh^{-1} q_f(t) = \frac{1}{2} \log \frac{1 + q_f(t)}{1 - q_f(t)}$$

with the Teichmüller distance between the equivalence class $[f_t]$ of f_t (the collection of maps equal f on S^1) and the origin in the space $\mathbf{T}$.

Note also that every class $[f_t]$ is a *Strebel point* which means that it contains an extremal Teichmüller map whose Beltrami coefficient on the disk Δ is of the form

$$\mu_{f_t} = q_f(t)|\psi_t|/\psi_t, \tag{1.5}$$

where ψ_t is an integrable holomorphic function (or equivalently, a holomorphic quadratic differential $\psi_t dz^2$) on Δ. This is important in many applications.

Such differentials play a crucial role in the theory of extremal quasiconformal maps of the unit disk. We denote the space of holomorphic differentials in Δ with L_1 norm by $A_1(\Delta)$. We distinguish its subset

$$A_1^2 = \{\psi \in A_1(\Delta) : \psi = \omega^2, \omega \text{ holomorphic}\},$$

which consists of differentials having only zeros of even order in D. Such differentials naturally appear in many problems. We have also a natural pairing

$$\langle \mu, \psi \rangle_\Delta = \iint\limits_\Delta \mu(z)\psi(z)\, dxdy \quad (z = x + iy)$$

for every $\mu \in L_\infty(\Delta)$ and $\psi \in L_1(\Delta)$.

2. Two questions of Kühnau. Main results

2.1. In 1995, R. Kühnau raised the following questions which reveal rather surprising features of the dilatation function (see [KK, §4]).

Question 1. *Is the function $q_f(r)$ real analytic?*

Question 2. *Is it possible to reconstruct $f(z)$ if $q_f(r)$ is given?*

These intriguing problems still remain open. Both of them were arose from an important example constructed in [Ku2]. It is concerned with the Cassini ovals and shows that a conformal map f of the disk Δ^* onto the exterior of any loop of oval

$$L_c = \{w : |w^2 - c^2| = 1\} \quad \text{with} \quad 0 < c < 1$$

has the extremal quasiconformal dilatation

$$q_f = 1/\cosh\left[\frac{\pi}{2}\frac{K'(c^2)}{K(c^2)}\right],$$

where $K(k)$ is the complete elliptic integral of the first kind.

Nothing further related to solving these problems has been obtained. There is a conjecture that the analyticity must happen for most of the dilatation functions and depends on distribution of zeros of the corresponding quadratic differentials ψ_t.

Note that the general results on the smoothness of Teichmüller distance, established in [Ea], [Ga], [Re], [Ro], provide that this distance at generic points of the universal Teichmüller space is at most C^2 smooth.

2.2. Our goal is to prove the following

Theorem 2.1. *For every function*

$$f_*(z) = z + b_0^* + b_1^* z^1 + \cdots \in \Sigma^0,$$

whose extremal Beltrami coefficient μ_{f_*} on the disk Δ is defined (via (1.5)) by a holomorphic quadratic differential ψ_* having only zeros of even order, the dilatation function $q_{f_*}(r)$ is analytic on the interval $\{0 < r < 1\}$.

This theorem has two important consequences. First, it establishes the existence of a wide class of univalent functions (which contains the above conformal maps of the Cassini ovals), whose dilatation functions are analytic.

Second, the proof of the theorem, provides a representation of $q_f(r)$ as the Grunsky norm of f_r, interesting by itself, and implies a negative answer to the second question.

The proof involves the Grunsky coefficients of univalent functions and certain deep results of complex geometry of the universal Teichmüller space. It reduces to the construction and comparison of metrics with appropriate curvature properties. We precede the proof by a brief exposition of the needed auxiliary results.

3. The Grunsky inequalities

The classical Grunsky theorem states that a holomorphic function $f(z) = z + \mathrm{const} + O(z^{-1})$ in a neighborhood of the point at infinity $z = \infty$ is extended to a univalent holomorphic function on the disk

$$\Delta^* = \{z \in \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} : |z| > 1\}$$

if and only if its *Grunsky coefficients* α_{mn} satisfy the inequalities

$$\Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\,\alpha_{mn} x_m x_n \Big| \leqslant 1. \tag{3.1}$$

These coefficients are generated by

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = - \sum_{m,n=1}^{\infty} \alpha_{mn} z^{-m} \zeta^{-n}, \quad (z, \zeta) \in (\Delta^*)^2,$$

and $\mathbf{x} = (x_n)$ runs over the unit sphere $S(l^2)$ of the Hilbert space l^2 with $\|\mathbf{x}\| = \Big(\sum\limits_{1}^{\infty} |x_n|^2\Big)^{1/2}$, and the principal branch of logarithmic function is chosen (cf. [Gr]). In particular, this assumes that $f(z) \neq 0$ on Δ^*. The quantity

$$\varkappa_f := \sup \Big\{ \Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\,\alpha_{mn} x_m x_n \Big| : \mathbf{x} = (x_n) \in S(l^2) \Big\} \tag{3.2}$$

is called the *Grunsky norm* of f.

Grunsky's theorem has been essentially strengthened for the functions with quasiconformal extensions, for which we have instead of (3.1) a stronger inequality (see [Ku1])

$$\Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\,\alpha_{mn} x_m x_n \Big| \leqslant q_f, \tag{3.3}$$

which implies that the Grunsky and Teichmüller norms of $f \in \Sigma$ are related as follows:

$$\varkappa_f \leqslant q_f. \tag{3.4}$$

Moreover, it was established recently in [Kr6] that a stronger sharp relation

$$\varkappa_f \leqslant q_f \leqslant \frac{3}{2\sqrt{2}}\, \varkappa_f$$

holds, but we do not need the right inequality here.

We shall deal with $f \in \Sigma^0$ satisfying $\varkappa_f = q_f$. Such functions fill a rather sparse set in Σ^0, but they play a crucial role in many applications of the Grunsky inequalities. The following theorem proved in [Kr1], [Kr5] describes these functions completely and relates to our main Theorem 2.1.

Theorem B. *The equality $\varkappa_f = q_f$ holds if and only if the function f is the restriction to Δ^* of a quasiconformal self-map w^{μ_0} of $\widehat{\mathbb{C}}$ with Beltrami coefficient μ_0 satisfying the condition*

$$\sup |\langle \mu_0, \varphi \rangle_\Delta| = \|\mu_0\|_\infty, \tag{3.5}$$

where the supremum is taken over holomorphic functions $\varphi \in A_1^2(\Delta)$ such that $\|\varphi\|_{A_1(\Delta)} = 1$.

If, in addition, the equivalence class of f (the collection of maps equal f on S^1) is a Strebel point, then μ_0 is necessarily of the form

$$\mu_0(z) = \|\mu_0\|_\infty |\psi_0(z)|/\psi_0(z) \quad \textit{with} \quad \psi_0 \in A_1^2 \quad (z \in \Delta). \tag{3.6}$$

Using Parseval's equality, one obtains that the elements of A_1^2 are represented in the form

$$\psi(z) = \frac{1}{\pi} \sum_{m+n=2}^{\infty} \sqrt{mn} x_m x_n z^{m+n-2} \tag{3.7}$$

with $\mathbf{x} = (x_n) \in S(l^2)$ (see [Kr1]).

The condition (3.5) has a geometric nature. The equality (3.6) holds, for example, for all $f \in \Sigma^0$ which are asymptotically conformal on the unit circle S^1 (in particular, for f with C^{1+} smooth images $f(S^1)$).

For analytic curves $f(S^1)$ the equality (3.6) was obtained by a different method in [Ku3].

4. Universal Teichmüller space

4.1. The *universal Teichmüller space* $\mathbf{T}$ is the deformation space of conformal structures on the disk obtained by factorization of the space of quasisymmetric homeomorphisms of the unit circle $S^1 = \partial\Delta$ by Möbius maps. This space is a complex Banach manifold with rich complex geometry and pluripotential features.

The canonical complex Banach structure on $\mathbf{T}$ is defined by factorization of the ball of Beltrami coefficients

$$\mathbf{Belt}(\Delta)_1 = \{\mu \in L_\infty(\mathbb{C}) : \mu|\Delta^* = 0, \ \|\mu\| < 1\},$$

letting $\mu, \nu \in \mathbf{Belt}(\Delta)_1$ be equivalent if the corresponding maps $w^\mu, w^\nu \in \Sigma^0$ coincide on S^1 (hence, on $\overline{\Delta^*}$) and passing to the Schwarzian derivatives

$$S_f(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2, \quad f = w^\mu|\Delta^*.$$

These derivatives are the points of complex Banach space $\mathbf{B}$ of hyperbolically bounded holomorphic functions in Δ^* with the norm

$$\|\varphi\|_{\mathbf{B}} = \sup_{\Delta^*} (|z|^2 - 1)^2 |\varphi(z)|, \tag{4.1}$$

and the space $\mathbf{T}$ is biholomorphically equivalent to a bounded domain in $\mathbf{B}$. The points of this domain are those Schwarzian derivatives S_f which correspond to univalent functions in Δ^* with quasiconformal extensions. The defining projection $\phi_{\mathbf{T}} : \mu \to S_{w^\mu}$ is a holomorphic map from $L_\infty(\Delta)$ to $\mathbf{B}$.

The basic intrinsic complete metric on the space $\mathbf{T}$ is the *Teichmüller metric* defined by

$$\begin{aligned} \tau_{\mathbf{T}}(\phi_{\mathbf{T}}(\mu), \phi_{\mathbf{T}}(\nu)) = \frac{1}{2} \inf\bigl\{\log K\bigl(w^{\mu_*} \circ \bigl(w^{\nu_*}\bigr)^{-1}\bigr) :& \\ \phi_{\mathbf{T}}(\mu_*) = \phi_{\mathbf{T}}(\mu), \phi_{\mathbf{T}}(\nu_*) = \phi_{\mathbf{T}}(\nu)\bigr\}.& \end{aligned} \tag{4.2}$$

It is generated by the *Finsler structure* on the tangent bundle $\mathcal{T}(\mathbf{T}) = \mathbf{T} \times \mathbf{B}$ of $\mathbf{T}$ defined by

$$\begin{aligned} F_{\mathbf{T}}(\phi_{\mathbf{T}}(\mu), \phi'_{\mathbf{T}}(\mu)\nu) = \inf\bigl\{\bigl\|\nu_*(1-|\mu|^2)^{-1}\bigr\|_\infty :& \\ \phi'_{\mathbf{T}}(\mu)\nu_* = \phi'_{\mathbf{T}}(\mu)\nu;\ \ \mu \in \mathbf{Belt}(\Delta)_1;\ \ \nu, \nu_* \in L_\infty(\mathbb{C})\bigr\}.& \end{aligned} \tag{4.3}$$

The main invariant metric on the space $\mathbf{T}$ is the *Kobayashi metric* $d_{\mathbf{T}}$ which is defined as the largest pseudometric d on $\mathbf{T}$ contracted by holomorphic maps $h : \Delta \to \mathbf{T}$ so that for any two points $\psi_1, \psi_2 \in \mathbf{T}$, we have

$$d_{\mathbf{T}}(\psi_1, \psi_2) \leqslant \inf\{d_\Delta(0, t) : h(0) = \psi_1,\ h(t) = \psi_2\}.$$

Here d_Δ denotes the *hyperbolic Poincaré metric* on the unit disk Δ of Gaussian curvature -4, with the differential form

$$ds = \lambda_{\text{hyp}}(z)|dz| := \frac{|dz|}{1-|z|^2}. \tag{4.4}$$

Its differential (infinitesimal) form is defined for the points (ψ, v) of $\mathcal{T}(\mathbf{T})$ by

$$\mathcal{K}_{\mathbf{T}}(\psi, v) = \inf\{1/r : r > 0, h \in \text{Hol}(\Delta_r, \mathbf{T}),\ h(0) = \psi, dh(0) = v\},$$

where $\text{Hol}((\Delta_r, \mathbf{T})$ denotes the set of holomorphic maps of the disk $\Delta_r = \{|z| < r\}$ into $\mathbf{T}$ and $\psi = \phi_{\mathbf{T}}(\mu), v = \phi'_{\mathbf{T}}(\nu)$.

4.2. Due to the fundamental Gardiner-Royden theorem, the Kobayashi and Teichmüller metrics on Teichmüller spaces are equal (see [EKK, EM, GL, Ro]).

An essential strengthening of this theorem for the space $\mathbf{T}$ was established in [Kr3] by applying the technique of Grunsky coefficient inequalities (see also [Kr4]). It states that *the differential Kobayashi metric $\mathcal{K}_{\mathbf{T}}(\varphi, v)$ on the tangent bundle $\mathcal{T}(\mathbf{T})$ of $\mathbf{T}$, which coincides with the Finsler structure (4.3), is logarithmically plurisubharmonic in $\varphi \in \mathbf{T}$ and has constant holomorphic sectional curvature $\kappa_{\mathcal{K}}(\varphi, v) = -4$ on $\mathcal{T}(\mathbf{T})$.*

As an important corollary, one obtains that the Teichmüller distance $\tau_{\mathbf{T}}(\varphi, \psi)$ is logarithmically plurisubharmonic in each of its variables (moreover,

$$g_{\mathbf{T}}(\varphi, \psi) = \log\tanh \tau_{\mathbf{T}}(\varphi, \psi) = \log k(\varphi, \psi),$$

where $g_{\mathbf{T}}$ denotes the pluricomplex Green function of the space $\mathbf{T}$ and $k(\varphi, \psi)$ is the extremal dilatation of quasiconformal maps determining the distance between the points φ and ψ in $\mathbf{T}$). This implies that the dilatation function $q_f(t)$ of each $f \in \Sigma$ is a *circularly symmetric* (that means $q_f(t) = q_f(|t|)$) *logarithmically subharmonic* function on the disk Δ.

Recall that the (generalized) *Gaussian curvature* κ_λ of a upper semicontinuos Finsler metric $ds = \lambda(t)|dt|$ in a domain $\Omega \subset \mathbb{C}$ is defined by

$$\kappa_\lambda(t) = -\frac{\boldsymbol{\Delta} \log \lambda(t)}{\lambda(t)^2}, \tag{4.5}$$

where $\boldsymbol{\Delta}$ is the *generalized Laplacian*

$$\boldsymbol{\Delta}\lambda(t) = 4 \liminf_{r \to 0} \frac{1}{r^2} \Big\{ \frac{1}{2\pi} \int_0^{2\pi} \lambda(t + re^{i\theta})\, d\theta - \lambda(t) \Big\} \tag{4.6}$$

(provided that $-\infty \leqslant \lambda(t) < \infty$). For C^2 metrics we have the usual curvature.

The holomorphic curvature mentioned above is the supremum of curvatures (4.5) at $t = 0$ of metrics $\mathcal{K}_{\mathbf{T}}(g(t), g'(t))$ induced by appropriate holomorphic maps $g : \Delta \to \mathbf{T}$; it will not be used here.

Similar to C^2 functions, for which $\boldsymbol{\Delta}$ coincides with the usual Laplacian

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4 \frac{\partial^2}{\partial z \partial \overline{z}} \qquad (z = x + iy),$$

one obtains that λ is subharmonic on Ω if and only if $\boldsymbol{\Delta}\lambda(t) \geqslant 0$; hence, at the points t_0 of local maximuma of λ with $\lambda(t_0) > -\infty$, we have $\boldsymbol{\Delta}\lambda(t_0) \leqslant 0$.

4.3. It follows from Theorem B and part (b) of Theorem A that for $f \in \Sigma$ with $b_1 = \cdots = b_{p-1} = 0$, $b_p \neq 0$, the function

$$u(z) = \frac{q_f(r)}{r^{p+1}} \quad \text{with} \quad u(0) = \limsup_{r \to 0} v(r) =: a_p$$

is logarithmically subharmonic on the disk Δ. This implies a weaker extension of (1.4) to $p > 1$ in the form

$$q_f(t) = a_p |t|^{p+1} + o(|t|^{p+1}), \quad t \to 0. \tag{4.7}$$

The asymptotic equalities (1.4) and (4.7) estimate sharply the behavior of dilatation near the origin. Theorem 2.1 does not concern this point.

5. Proof of Theorem 2.1

The proof of this Theorem will be established in several stages. The underlying features are the same as in [Kr6].

Step 1: Metric generated by Grunsky coefficients. A fundamental property of the Grunsky coefficients $\alpha_{mn}(f) = \alpha_{mn}(S_f)$ is that these coefficients are holomorphic functions of the Schwarzians S_f on the universal Teichmüller space $\mathbf{T}$. Therefore, for every $f \in \Sigma^0$ and each $\mathbf{x} = (x_n) \in S(l^2)$, the series

$$h_{\mathbf{x}}(\varphi) = \sum_{m,n=1}^{\infty} \sqrt{mn}\alpha_{mn}(\varphi)x_m x_n \tag{5.1}$$

defines a holomorphic map of the space $\mathbf{T}$ into the unit disk Δ.

Applying Theorem B, one concludes that the given function f_* must satisfy the equality $\varkappa_{f_*} = q_{f_*}$, and hence, by (3.7), its defining holomorphic quadratic differential ψ_* has the form

$$\psi_*(z) = \frac{1}{\pi}\sum_{m,n=1}^{\infty} \sqrt{mn}\alpha_{mn}(f_*)x_m^* x_n^*, \quad z \in \Delta, \tag{5.2}$$

where $\mathbf{x}^* = (x_n^*) \in S(l^2)$.

The isotopy (1.1) for f_1 determines in the space $\mathbf{T}$ the holomorphic disk

$$\Delta(S_{f_1}) = \gamma_{f_1}(\Delta) = \{S_{f_{1,t}} : t \in \Delta\}, \tag{5.3}$$

where γ_{f_1} denotes a holomorphic map $\Delta \to \mathbf{T}$ induced by holomorphic point-wise map

$$t \mapsto S_{f_{1,t}}(z) = \frac{1}{t^2}S_{f_1}\Big(\frac{z}{t}\Big) : \Delta \to \mathbb{C}$$

(see [Kr1]). This disk has only a singulary at the origin of $\mathbf{T}$.

The restrictions of the maps (5.1) to the disk (5.3) are, in terms of of parameter $t \in \Delta$, of the form

$$\widetilde{h}_{\mathbf{x}}(t) := h_{\mathbf{x}} \circ \gamma_{f_1}(t) = \sum_{m,n=1}^{\infty} \sqrt{mn}\alpha_{mn}(f_1)x_m x_n t^{m+n}. \tag{5.4}$$

Using this map, we pull back the hyperbolic metric (4.4) to the disk (5.3) and define on this disk (and on the unit disk $\{|t| < 1\}$) the conformal metric $ds = \lambda_{\widetilde{h}_{\mathbf{x}}}(t)|dt|$ with

$$\lambda_{\widetilde{h}_{\mathbf{x}}}(t) = \frac{|\widetilde{h}'_{\mathbf{x}}(t)||dt|}{1 - |\widetilde{h}_{\mathbf{x}}(t)|^2}. \tag{5.5}$$

Its Gaussian curvature equals -4 at noncritical points.

Take the upper envelope of these metrics

$$\lambda_{\varkappa}(t) = \sup\{\lambda_{\widetilde{h}_{\mathbf{x}}}(t) : \mathbf{x} \in S(l^2)\} \tag{5.6}$$

and its upper semicontinuous regularization

$$\lambda_{\varkappa}(t) = \limsup_{t' \to t} \lambda_{\varkappa}(t').$$

Then similar to [Kr6], we have

Lemma 5.1. *The metric $\lambda_\varkappa$ is logarithmically subharmonic on the punctured disk $\Delta_* = \Delta \setminus \{0\}$ and its generalized Gaussian curvature satisfies $k_{\lambda_\varkappa} \leqslant -4$.*

The last inequality is equivalent to the following one

$$\boldsymbol{\Delta} \log \lambda_\varkappa \geqslant 4\lambda_\varkappa^2,$$

or $\boldsymbol{\Delta} u_\varkappa \geqslant 4e^{2u_\varkappa}$, where $u_\varkappa = \log \lambda_\varkappa$. Here $\boldsymbol{\Delta}$ again means the generalized Laplacian.

Let us compare $\lambda_\varkappa$ with the infinitesimal Kobayashi metric $\lambda_\mathcal{K}$ of $\mathbf{T}$, restricted to the same disk $\Delta(S_{f_1})$. It is also logarithmically subharmonic and has generalized Gaussian curvature -4. This is done in the same way as in [Kr6], but we present the main details, because these will be used also for other metrics.

Using the Grunsky coefficients of the functions $f \in \Sigma^0$ and their generalization to arbitrary simply connected domains due to Milin, one can define on the tangent bundle $\mathcal{T}(\mathbf{T})$ a new Finsler structure $F_\varkappa(\varphi, v)$, which is dominated by the canonical Finsler structure (4.3) (for details see [Kr6]). This structure allows us to construct in a standard way on embedded holomorphic disks $\gamma(\Delta) \subset \mathbf{T}$ the Finsler metrics $\lambda_\gamma(t) = F_\varkappa(\gamma(t), \gamma'(t))$ and the corresponding distances

$$d_\gamma(\varphi_1, \varphi_2) = \inf \int_\gamma F_\varkappa(\gamma(t), \gamma'(t)) ds_t,$$

taking the infimum over C^1 smooth curves $\gamma : [0, 1] \to \mathbf{T}$ joining the points φ_1 and φ_2.

The following lemma on reconstruction of the Grunsky norm is crucial.

Lemma 5.2 ([Kr6]). *On any extremal Teichmüller disk $\Delta(\mu_0) = \{\phi_\mathbf{T}(t\mu_0) : t \in \Delta\}$ (and its isometric images in $\mathbf{T}$), we have the equality*

$$\tanh^{-1}[\varkappa(f^{r\mu_0})] = \int_0^r \lambda_\varkappa(t) dt. \tag{5.7}$$

Step 2: Comparison with differential Kobayashi metric. Let us now assume (in this and in the next steps) that our function f_* is holomorphic on the closed disk $\overline{\Delta^*}$, i.e., that the curve $f_*(S^1)$ is analytic. Then $f_*|\Delta^*$ extends across the unit circle S^1 to a holomorphic univalent function on a larger disk $\Delta_a^* = \{z \in \widehat{\mathbb{C}} : |z| > a\}, (0 < a < 1)$. Let us take the minimal possible value of such a. Then

$$f_*(z) = af_1\Big(\frac{z}{a}\Big) =: f_a(z), \quad |z| > 1, \tag{5.8}$$

where

$$f_1(z) = a^{-1} f_*(az) = z + \sum_0^\infty b_n^1 z^{-n} \in \Sigma.$$

Note that

$$\alpha_{mn}(f_a) = \alpha_{mn}(f_1) a^{m+n}. \tag{5.9}$$

Taking into account that the disk $\Delta(S_{f_1})$ touches at the point $\varphi = S_{f_a}$ the Teichmüller disk centered at the origin of $\mathbf{T}$ and passing through this point and that the metric $\lambda_\varkappa$ does not depend on the tangent unit vectors whose initial points are the points of $\Delta(S_{f_1})$, one obtains from Lemma 5.2 and (3.3) that at the corresponding point $a \in \Delta$, we have the equality

$$\lambda_\varkappa(a) = \lambda_{\mathcal{K}}(a) \tag{5.10}$$

which means that $\lambda_\varkappa$ is a supporting metric for $\lambda_{\mathcal{K}}|\Delta(S_{f_1})$ at this point.

A more subtle comparison of these metrics is obtained by applying Minda's maximum principle:

Lemma 5.3 ([Mi]). *If a function $u : D \to [-\infty, +\infty)$ is upper semicontinuous in a domain $\Omega \subset \mathbb{C}$ and its generalized Laplacian satisfies the inequality $\boldsymbol{\Delta} u(z) \geqslant Ku(z)$ with some positive constant K at any point $z \in D$, where $u(z) > -\infty$, and if*

$$\limsup_{z\to\zeta} u(z) \leqslant 0 \quad \text{for all} \quad \zeta \in \partial D,$$

then either $u(z) < 0$ for all $z \in D$ or else $u(z) = 0$ for all $z \in \Omega$.

For a sufficiently small neighborhood U_0 of the point $t = a$, we put

$$M = \{\sup \lambda_{\mathcal{K}}(t) : t \in U_0\};$$

then in this neighborhood, $\lambda_{\mathcal{K}}(t) + \lambda_\varkappa(t) \leqslant 2M$. Consider the function

$$u = \log \frac{\lambda_\varkappa}{\lambda_{\mathcal{K}}}.$$

Then (cf. [Mi], [Di]) for $t \in U_0$,

$$\boldsymbol{\Delta} u(t) = \boldsymbol{\Delta} \log \lambda_\varkappa(t) - \boldsymbol{\Delta} \log \lambda_{\mathcal{K}}(t) = 4(\lambda_\varkappa^2 - \lambda_{\mathcal{K}}^2) \geqslant 8M(\lambda_\varkappa - \lambda_{\mathcal{K}}).$$

The elementary estimate

$$M \log(t/s) \geqslant t - s \quad \text{for} \quad 0 < s \leqslant t < M$$

(with equality only for $t = s$) implies that

$$M \log \frac{\lambda_{g_0}(t)}{\lambda_d(t)} \geqslant \lambda_{g_0}(t) - \lambda_d(t),$$

and hence,

$$\boldsymbol{\Delta} u(t) \geqslant 4M^2 u(t).$$

Applying Lemma 5.3, one obtains that, in view of (5.10), both metrics $\lambda_\varkappa$ and $\lambda_{\mathcal{K}}$ must be equal in the entire disk $\Delta(S_{f_1})$, which implies, by Lemma 5.2, the equality

$$\varkappa_{f_t} = q_f(t) \quad \text{for all} \quad t \in \Delta_*.$$

Consequently, for all $r \in (0,1)$, we have the equality

$$q_f(r) = \Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\alpha_{mn}(f_1) r^{m+n} x_m^r x_n^r \Big|. \tag{5.11}$$

The last equality implies, by Theorem B, that every map $f_{1,t}$ also has the extremal Beltrami coefficient in Δ of the form

$$\mu_{f_t} = q_f(t)|\psi_t|/\psi_t \quad \text{with} \quad \psi_t = \omega_t^2 \in A_1^2$$

and, hence, also satisfies the assumptions of Theorem 2.1.

Step 3. Analyticity of $q_f(r)$. Using the relations between the Grunsky coefficients of f_t and f_1, similar to (5.9), one can construct for each fixed $r = b \in (0,1)$ a holomorphic map

$$\widetilde{h}_b(t) := h_{\mathbf{x}^b}(t) = \sum_{m,n=1}^{\infty} \sqrt{mn}\alpha_{mn}(f_b) x_m^b x_n^b \Big(\frac{t}{b}\Big)^{m+n} : \Delta \to \Delta. \tag{5.12}$$

Here $\mathbf{x}^b = (x_n^b)$ is a point of $S(l^2)$ maximizing the right-hand side of (3.2) for $f = f_b$ on this sphere. Accordingly, the value of the metric $\lambda_{\varkappa}(r)$ via (5.6) is also attained on the map (5.12) with $r = b$.

On the other hand, this map generates, by pulling back, the conformal metric

$$\lambda_{\widetilde{h}_b}(t) = \frac{|\widetilde{h}_b'(t)||dt|}{1 - |\widetilde{h}_b(t)|^2} \tag{5.13}$$

whose Gaussian curvature is equal to -4 at noncritical points of $\widetilde{h}_b$ (and everywhere on Δ in the generalized sense). Equivalently, $\lambda_{\widetilde{h}_b}$ is a real analytic solution to the differential equation

$$\boldsymbol{\Delta} \log v = 4v^2,$$

on the domain

$$D_b := \Delta \setminus \mathrm{Crit}(\widetilde{h}_b),$$

where $\mathrm{Crit}(\widetilde{h}_b)$ denotes the set of critical points of the map $\widetilde{h}_b$ in the unit disk. This metric is a *supporting* metric to $\lambda_{\varkappa}$ at the point $t = b$ (which means that $\lambda_{\widetilde{h}_b}(b) = \lambda_{\varkappa}(b)$ and $\lambda_{\widetilde{h}_b}(t) \leqslant \lambda_{\varkappa}(t)$ in a neighborhood of b).

Comparison of $\lambda_{\widetilde{h}_b}$ with $\lambda_{\varkappa}(t)$, similar to Step 1, implies

$$\lambda_{\widetilde{h}_b}(t) = \lambda_{\varkappa}(t) \quad \text{for all} \quad t \in D_b. \tag{5.14}$$

It follows from (5.14) that for every two points $b_1, b_2 \in (0,1)$, the metrics $\lambda_{\widetilde{h}_{b_1}}$ and $\lambda_{\widetilde{h}_{b_1}}$ are equal in the region $D_{12} = D_{b_1} \cap D_{b_2}$.

Equalizing the corresponding expressions (5.13), one derives from (5.12) that the coordinates $x_m^{b_2}$ are real analytic functions of $x_n^{b_1}$, and vice versa. In particular, for $0 < b < 1$, the coordinates x_m^b are analytic functions of r and x_n^a. Then the equality (5.11) implies the analyticity of the distortion function, and thereby, the theorem is proved for all $f \in \Sigma^0$ mapping the unit circle onto the analytic curves.

Step 4: Approximation. Let now f_* be an arbitrary function from Σ^0 whose extremal Beltrami coefficient $\mu_{f_*} = q_{f_*}|\psi_*|/\psi_*$ on Δ is determined by a quadratic differential

$$\psi_*(z) = \frac{1}{\pi} \sum_{m+n=2}^{\infty} \sqrt{mn} x_m^* x_n^* z^{m+n-2} \in A_1^2$$

with $\mathbf{x}^* = (x_n^*) \in S(l^2)$ having only zeros of even order.

Now select in $[0,1]$ a sequence $\{\tau_n\}$ approaching 1 and put

$$\psi_n(z) = c_n \psi_*(\tau_n z),$$

choosing $c_n > 0$ so that $\|\psi_n\|_{A_1} = 1$. The Beltrami coefficients $\mu_n = q_{f_*}|\psi_n|/\psi_n$, extended by zero to Δ^*, determine quasiconformal automorphisms $f_n := f^{\mu_n}$ of $\widehat{\mathbb{C}}$ with analytic images $f_n(S^1)$. Every ψ_n has in Δ also only zeros of even order. By the previous steps, we have for all $r \in [0,1]$ the equality

$$\varkappa_{f_n}(r) = q_{f_n}(r). \tag{5.15}$$

Since $\lim_{n\to\infty} \mu_n(z) = \mu_*(z)$ for all $z \in \widehat{\mathbb{C}}$, the maps f_n are convergent to f^{μ_0} in the spherical metric on $\widehat{\mathbb{C}}$. For a fixed $t \in \Delta$, the family $\{f_t : f \in \Sigma\}$ is compact in the spherical metric on $\widehat{\mathbb{C}}$. Applying this to the maps $f_{n,t}$ generated by $f_n, n = 0,1,2,\dots$, one concludes that for every $|t| < 1$ the homotopy maps $f_{n,t}$ are convergent to $f_{*,t}$ uniformly on the closed disk $\overline{D^*}$.

Now fix two values r_0 and r_1 close to 1 so that $r_0 < r_1$. Then, for any fixed integer $m > 1$, the derivatives of f_{n,r_1} of orders up to m are convergent to the corresponding derivatives of f_{0,r_1} uniformly on the closed disk $\overline{D^*}$, and therefore,

$$\lim_{n\to\infty} \|S_{f_{n,r_0}} - S_{f_{0,r_0}}\|_{\mathbf{B}} = 0, \tag{5.16}$$

and by (5.16),

$$\varkappa_{f_{n,r_0}}(r) = k_{f_{n,r_0}}(r). \tag{5.17}$$

Since the Teichmüller and Grunsky norms q_f and $\varkappa_f$ are both continuous with respect to convergence of $S_f \in \mathbf{T}$ in the norm (4.1) (see [Sh]), one obtains from (5.11), (5.16) and (5.17), letting $n \to \infty$, the limit equalities

$$q_{f_{*,r_0}}(r) = \varkappa_{f_{*,r_0}}(r) = \left| \sum_{m,n=1}^{\infty} \sqrt{mn}\, \alpha_{mn}(f_{*,r_0}) r^{m+n} x_m^r x_n^r \right|.$$

One can apply to f_{0,r_0} the arguments from the previous steps and derive, in view of the relation $[f_{*,r_0}]_r = [f_*]_{r_0 r}$, that both dilatations $q_{f_{*,r_0}}$ and $q_f(r)$ are analytic functions of $r \in (0,1)$. This completes the proof of the theorem.

6. On reconstruction of univalent function by dilatation

Consider again the functions $f_* \in \Sigma^0$ with analytic images $f_*(S^1)$. In the proof of Theorem 2.1, we have obtained the equalities

$$\begin{aligned} q_f(r) &= \Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn}(f_1) r^{m+n} x_m^r x_n^r \Big| \\ &= \Big| \sum_{m,n=1}^{\infty} \sqrt{mn} \alpha_{mn}(f_a) \Big(\frac{r}{a}\Big)^{m+n} x_m^r x_n^r \Big|, \end{aligned} \tag{6.1}$$

where $f_1 = f_*$ denotes the initial function and x_n^r are coordinates of a sequence from $S(l^2)$, which defines the quadratic differential ψ_r by (3.7) as well as the corresponding Beltrami coefficient $\mu_{f_r}|\Delta = q_f(r)|\psi_r|/\psi_r$. These x_m^r are analytic functions of r, provided that the point $a \in (0,1)$ is fixed.

Each of the representations (3.2), (3.7) and (6.1) contains only the products $X_{mn}^r = x_m^r x_n^r$, and the coordinates x_m^r are not separated there. Substituting the series

$$X_{mn}^r = X_{mn}^a + \big(X_{mn}^a\big)'(a)(r-a) + \dots \tag{6.2}$$

into (6.1) and recollecting the terms, one obtains

$$q_f(r) = \sum_{m,n \geqslant 1} A_{mn}\big(a, \alpha_{mn}^a X_{mn}^a\big) r^{m+n}, \tag{6.3}$$

where A_{mn} depend only on a and on products $\alpha_{mn}^a X_{mn}^a$.

The radii of convergence of the series (6.2) can approach zero; in this case, (6.3) must be regarded as a formal power series. However, if ψ_a is, for example, a polynomial, then the series (6.3) is convergent absolutely and uniformly in a neighborhood of $r = a$.

Now, given a dilatation function $q_{f_1}(r)$, we expand it near the point $r = a$ and, comparing the series

$$q_f(r) = q_f(a) + q_f'(a)(z-a) + \frac{q_f''(a)}{2}(r-a)^2 + \dots$$

with (6.3), obtain an infinite system of nonlinear equations for defining the products

$$\alpha_{mn}(f_a) X_{mn}^a, \quad m,n = 1,2,\dots.$$

Any function f_a is uniquely restored by each of collections $\{\alpha_{mn}(f_a)\}$ or $\{X_{mn}^a\}$, but the function $q_f(r) = q_{f_a}(r/a)$ itself, in the generic case, does not separate these factors. This yields a negative answer to Question 2.

References

[Di] S. Dineen, *The Schwarz Lemma*, Clarendon Press, Oxford, 1989.

[Ea] C. J. Earle, *The Teichmüller distance is differentiable*, Duke Math. J. **4**4 (1977), 389–397.

[EKK] C. J. Earle, I. Kra, S. L. Krushkal, *Holomorphic motions and Teichmüller spaces*, Trans. Amer. Math. Soc. **944** (1994), 927–948.

[EM] C. J. Earle, S. Mitra, *Variation of moduli under holomorphic motions*, In the tradition of Ahlfors and Bers (Stony Brook, NY, 1998), Contemp. Math. **256**, Amer. Math. Soc., Providence, RI, 2000, 39–67.

[Ga] F. P. Gardiner, *On the variation of Teichmüller's metric*, Proc. Roy. Soc. Edinburgh Sect. A **85** (1980), 143–152.

[GL] F. P. Gardiner, N. Lakic, *Quasiconformal Teichmüller Theory*, Amer. Math. Soc., 2000.

[Gr] H. Grunsky, *Koeffizientenbedingungen für schlicht abbildende meromorphe Funktionen*, Math. Z. **45** (1939), 29–61.

[Kr1] S. L. Krushkal, *Grunsky coefficient inequalities, Carathéodory metric and extremal quasiconformal mappings*, Comment. Math. Helv. **64** (1989), 650–660.

[Kr2] S. L. Krushkal, *Extension of conformal mappings and hyperbolic metrics*, Siberian Math. J. **30** (1989), 730–744.

[Kr3] S. L. Krushkal, *Plurisubharmonic features of the Teichmüller metric*, Publications de L'Institut Mathématique-Beograd, Nouvelle série **75(89)** (2004), 119–138.

[Kr4] S. L. Krushkal, *Complex geometry of the universal Teichmüller space*, Siberian Math. J. **45** (2004), 646–668.

[Kr5] S. L. Krushkal, *Quasiconformal extensions and reflections*, Ch. 11 in: Handbook of Complex Analysis: Geometric Function Theory, Vol. II (R. Kühnau, ed.), Elsevier Science, Amsterdam, 2005, 507–553.

[Kr6] S. L. Krushkal, *Strengthened Moser's conjecture, geometry of Grunsky coefficients and Fredholm eigenvalues*, Central European J. Math **5(3)** (2007), 551–580.

[KK] S. L. Krushkal, R. Kühnau, *A quasiconformal dynamic property of the disk*, J. Anal. Math. **72** (1997), 93–103.

[Ku1] R. Kühnau, *Verzerrungssätze und Koeffizientenbedingungen vom Grunskyschen Typ für quasikonforme Abbildungen*, Math. Nachr. **48** (1971), 77–105.

[Ku2] R. Kühnau, *Möglichst konforme Spiegelung an einer Jordankurve*, Jber. Deutsch. Math. Verein. **90** (1988), 90–109.

[Ku3] R. Kühnau, *Wann sind die Grunskyschen Koeffizientenbedingungen hinreichend für Q-quasikonforme Fortsetzbarkeit*? Comment. Math. Helv. **61** (1986), 290–307.

[Mi] D. Minda, *The strong form of Ahlfors' lemma*, Rocky Mountain J. Math., **17** (1987), 457–461.

[Re] M. Rees, *Teichmüller distance is not* $C^{2+\varepsilon}$, Proc. London Math. Soc. (3) **88** (2004), 114–134.

[Ro] H. L. Royden, *Automorphisms and isometries of Teichmüller space*, Advances in the Theory of Riemann Surfaces (Ann. of Math. Stud. vol. 66), Princeton Univ. Press, Princeton, 1971, pp. 369–383.

[Sh] Y. L. Shen, *Pull-back operators by quasisymmetric functions and invariant metrics on Teichmüller spaces*, Complex Variables **42** (2000), 289–307.

[Ve] I. N. Vekua *Remarks on properties of the equation* $\Delta u = -2Ke^u$, Siberian Math. J. **1** (1960), no. 3, 331–342 (Russian).

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