*Functiones et Approximatio*
*40.1 (2009), 69–74*

# A SECOND ORDER DIFFERENTIABILITY TECHNIQUE OF BOJARSKI-IWANIEC IN THE HEISENBERG GROUP

András Domokos, Juan J. Manfredi

Dedicated to Professor Bogdan Bojarski
on the occasion of his 75th birthday

**Abstract:** We adapt a technique developed by Bojarski and Iwaniec in their celebrated 1983 paper [2] to prove second order differentiability results for $p$-harmonic functions to the case of the Heisenberg group. We prove that for $2 \leqslant p < 4$ we have $X_i(|Xu|^{(p-2)/p} X_j u) \in L^2_{\text{loc}}(\Omega)$ and $X_i(|Xu|^p) \in L^2_{\text{loc}}(\Omega)$, where $u$ is a $p$-harmonic function in the Heisenberg group $\mathbb{H}^n$.

**Keywords:** Heisenberg group, p-Laplacian equation, p-harmonic function

## 1. Introduction

It is a great privilege to present this contribution in honor of Professor Bogdan Bojarski. As one of the creators of the modern theory of quasiregular mappings, we are all indebted to his dedication and vision. His seminal higher integrability result [3] opened a new era in quasiconformal analysis. He is not only a great scientist, but also a great teacher and expositor. A generation of students of quasiconformal and quasiregular mappings learned from the classical foundational article [4].

We are also personally indebted to Bojarski through two of this students Tadeusz Iwaniec and Piotr Hajłasz, whom we are lucky to count among our collaborators and colleagues.

There has been various recent advances in the study of the regularity of $p$-harmonic mappings in the Grušin plane and the Heisenberg group. Starting with the second differentiability results of [5], a mixed Moser iteration technique has been implemented in [6] and [7]. This approach gives the expected Lipschitz continuity of $p$-harmonic functions for the range $2 \leqslant p < 4$ as well as Hölder continuity of the horizontal derivatives for solutions of the non-degenerate $p$-Laplace equation. To the best of our knowledge the situation for $p \geqslant 4$ remains unsolved.

In [2] Bojarski and Iwaniec presented a proof of second differentiability of $p$-harmonic functions for $p \geqslant 2$ based on studying the difference quotients of the map

$$F(x) = |Du|^{\frac{p-2}{2}} Du.$$

**1991 Mathematics Subject Classification:** primary: 35H20, 35J70

They proved that $F \in W^{1,2}_{\text{loc}}$, thereby concluding that

$$D\left(|Du|^{\frac{p-2}{2}}\, Du\right) \in L^2_{\text{loc}} \qquad \text{and} \qquad D(|Du|^p) \in L^2_{\text{loc}}. \tag{1.1}$$

We caution the reader that, in general, we cannot expand the derivatives in (1.1) to conclude that $D^2u$ is in a weighted $L^p$-space.

In this note we have adapted this method to the Heisenberg group framework, where second derivatives do not commute in general so that every time we interchange the order of differentiation we pick up commutators. The new terms are handled with the help of [6, 7]. It is indeed quite plausible that our main result, Theorem 3.2 can be also obtained by the techniques of [6, 7], but the simplicity and the elegance of the Bojarski-Iwaniec approach might give new insights on the regularity of $p$-harmonic functions in the Heisenberg group.

## 2. Definitions and Preliminaries

The Heisenberg group $\mathbb{H}^n$ can be identified with $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ endowed with the group multiplication

$$\begin{aligned}(x_1, ..., x_{2n}, t) \cdot (y_1, ..., y_{2n}, u) &= \\ &= \left(x_1 + y_1, ..., x_{2n} + y_{2n}, t + u - \frac{1}{2}\sum_{i=1}^{n}(x_{n+i}y_i - x_i y_{n+i})\right).\end{aligned}$$

A basis of the Lie algebra is given by the horizontal left invariant vector fields

$$\begin{aligned} X_i &= \frac{\partial}{\partial x_i} - \frac{x_{n+i}}{2}\frac{\partial}{\partial t}, \\ X_{n+i} &= \frac{\partial}{\partial x_{n+i}} + \frac{x_i}{2}\frac{\partial}{\partial t}, \end{aligned}$$

and their only nonzero commutator

$$[X_i, X_{n+i}] = T = \frac{\partial}{\partial t}, \qquad 1 \leqslant i \leqslant n\,.$$

Let $\Omega$ be a domain in $\mathbb{H}^n$. Consider the Sobolev space with respect to the horizontal vector fields $X_i$

$$HW^{1,p}(\Omega) = \left\{u \in L^p(\Omega) \;:\; X_i u \in L^p(\Omega), \text{ for all } \; i \in \{1, ..., 2n\}\right\}.$$

$HW^{1,p}(\Omega)$ is a Banach space with respect to the norm

$$||u||_{HW^{1,p}} = ||u||_{L^p} + \sum_{i=1}^{2n} ||X_i u||_{L^p}\,.$$

We denote by $HW_0^{1,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $HW^{1,p}(\Omega)$. The horizontal gradient of the function $u$ is given by

$$Xu = (X_1 u, \dots, X_n u, X_{n+1} u \dots, X_{2n} u).$$

For $p \geqslant 2$ consider the subelliptic p-Laplacian equation

$$\sum_{i=1}^{2n} X_i(|Xu|^{p-2} X_i u) = 0 \quad \text{in } \Omega. \tag{2.1}$$

A function $u \in HW_{\text{loc}}^{1,p}(\Omega)$ is called p-harmonic if it is a weak solution of (2.1); that is, we have

$$\sum_{i=1}^{2n} \int_\Omega |Xu|^{p-2} X_i u \; X_i \varphi \; dx = 0 \tag{2.2}$$

for all $\varphi \in HW_0^{1,p}(\Omega)$ with compact support in $\Omega$. The following regularity result is proved in [7]:

**Theorem 2.1.** *For the range $2 \leqslant p < 4$, p-harmonic functions $u \in HW_{\text{loc}}^{1,p}(\Omega)$ satisfy*

$$Xu \in L^\infty_{\text{loc}}(\Omega) \quad \textit{and} \quad Tu \in L^q_{\text{loc}}(\Omega) \quad \textit{for all } 1 < q < \infty \, .$$

## 3. Main result

Consider the differences

$$\Delta_{T,s} u(x) = u\big(xe^{sT}\big) - u(x) \quad \text{and} \quad \Delta_{T,-s} u(x) = u(x) - u\big(xe^{-sT}\big) \, .$$

Let $\eta \in C_0^\infty(\Omega)$ be a cut-off function and let us use the test function

$$\varphi = \Delta_{T,-s}(\eta^2 \Delta_{T,s} u)$$

in equation (2.2). Standard estimates based on the inequalities

$$\Big\langle |a|^{p-2}a - |b|^{p-2}b, a-b \Big\rangle \geqslant 2^{2-p}|a-b|^p \, , \tag{3.1}$$

$$\Big| \, |a|^{p-2}a - |b|^{p-2}b \, \Big| \leqslant C(p,n)\Big(|a|^p + |b|^p\Big)^{\frac{p-2}{p}} \; |a-b| \tag{3.2}$$

and on Theorem 2.1 lead to

$$\int |\Delta_{T,s} Xu(x)|^p \; \eta^p(x) \; dx \; \leqslant c \int |\Delta_{T,s} Xu(x)| \; |\Delta_{T,s} u(x)| \; |\eta(x)| \; |X\eta(x)| \; dx \, .$$

Young's inequality with the fact that $p \geqslant q$ if $\frac{1}{p} + \frac{1}{q} = 1$ gives that

$$\int |\Delta_{T,s} Xu|^p \; \eta^p \; dx \leqslant c \int |\Delta_{T,s} u|^p |X\eta|^p \; dx \, .$$

Dividing by $|s|^p$ and using the fact that $Tu \in L^p_{\text{loc}}(\Omega)$ we obtain the following result:

**Theorem 3.1.** *For $2 \leqslant p < 4$ and any $p$-harmonic function $u \in HW^{1,p}_{\mathrm{loc}}(\Omega)$ we have that*

$$XTu = TXu \in L^p_{\mathrm{loc}}(\Omega)\ .$$

Next, we implement the Bojarski-Iwaniec approach from [2]. Consider the mapping

$$F(x) = |Xu(x)|^{\frac{p-2}{2}} Xu(x)\ .$$

Clearly, $F \in L^2_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n})$. Consider $x_0 \in \Omega$ and $r > 0$ such that $B(x_0, 3r) \subset \Omega$. Also, let $\eta$ be a cut-off function between $B(x_0, r)$ and $B(x_0, 2r)$. Fix an arbitrary $k \in \{1, .., 2n\}$. We do the computations for $k \in \{1, .., n\}$ and leave the case $k \in \{n+1, .., 2n\}$ to the reader. In equation (2.2) use the test function

$$\varphi = \Delta_{X_k,-s}(\eta^2 \Delta_{X_k,s} u)$$

to get

$$\sum_{i=1}^{2n} \int_\Omega |Xu|^{p-2} X_i u\, X_i\Big(\Delta_{X_k,-s}(\eta^2 \Delta_{X_k,s} u)\Big)\, dx = 0.$$

By the identities

$$X_{k+n}\Big(v\big(x \cdot e^{sX_k}\big)\Big) = X_{k+n} v\big(x \cdot e^{sX_k}\big) - sTv\big(x \cdot e^{sX_k}\big)$$

and

$$X_{k+n}\Big(v\big(x \cdot e^{-sX_k}\big)\Big) = X_{k+n} v\big(x \cdot e^{-sX_k}\big) + sTv\big(x \cdot e^{-sX_k}\big)\,,$$

we get that

$$\sum_{i=1}^{2n} \int_\Omega |Xu|^{p-2} X_i u\, \Delta_{X_k,-s}\Big(X_i(\eta^2 \Delta_{X_k,s} u)\Big)\, dx$$
$$= -s \int_\Omega |Xu|^{p-2} X_{k+n} u\, T\Big(\eta^2 \Delta_{X_k,s} u\Big)\big(xe^{-sX_k}\big)\, dx\,.$$

From here, by the fact that the adjoint of $\Delta_{X_k,-s}$ is $-\Delta_{X_k,s}$, it follows that

$$\sum_{i=1}^{2n} \int_\Omega \Delta_{X_k,s}\Big(|Xu|^{p-2} X_i u\Big)\, \Big(X_i(\eta^2 \Delta_{X_k,s} u)\Big)\, dx$$
$$= s \int_\Omega |Xu|^{p-2} X_{k+n} u\, T\Big(\eta^2 \Delta_{X_k,s} u\Big)\big(xe^{-sX_k}\big)\, dx\,,$$

and hence

$$\sum_{i=1}^{2n} \int_\Omega \Delta_{X_k,s}\Big(|Xu|^{p-2} X_i u\Big)\, \eta^2\, X_i \Delta_{X_k,s} u\, dx$$
$$= -\sum_{i=1}^{2n} \int_\Omega \Delta_{X_k,s}\Big(|Xu|^{p-2} X_i u\Big)\, 2\eta\, X_i \eta\, \Delta_{X_k,s} u\, dx$$
$$+ s \int_\Omega |Xu|^{p-2} X_{k+n} u\, T\Big(\eta^2 \Delta_{X_k,s} u\Big)\big(xe^{-sX_k}\big)\, dx\,.$$

One more switch between $X_i$ and $\Delta_{X_k,s}$ gives

$$\begin{aligned}\sum_{i=1}^{2n}\int_\Omega \Delta_{X_k,s}\Big(|Xu|^{p-2}X_iu\Big)\,\eta^2\,\Delta_{X_k,s}X_iu\,dx\qquad\qquad&\\ = s\int_\Omega \Delta_{X_k,s}\Big(|Xu|^{p-2}X_{k+n}u\Big)\,\eta^2\,Tu\big(xe^{sX_k}\big)\,dx&\\ -\sum_{i=1}^{2n}\int_\Omega \Delta_{X_k,s}\Big(|Xu|^{p-2}X_iu\Big)\,2\eta\,X_i\eta\,\Delta_{X_k,s}u\,dx&\\ + s\int_\Omega |Xu|^{p-2}X_{k+n}u\,T\Big(\eta^2\Delta_{X_k,s}u\Big)\big(xe^{-sX_k}\big)\,dx\,.&\end{aligned}\tag{3.3}$$

We need the following inequalities that are similar but not identical to (3.1) and (3.2):

$$\Big\langle |a|^{p-2}a-|b|^{p-2}b\,,\,a-b\Big\rangle \geqslant \frac{4}{p^2}\,\Big||a|^{\frac{p-2}{2}}a-|b|^{\frac{p-2}{2}}b\Big|^2\,,\tag{3.4}$$

and

$$\Big||a|^{p-2}a-|b|^{p-2}b\Big| \leqslant \frac{2(p-1)}{p}\Big(|a|^p+|b|^p\Big)^{\frac{p-2}{2p}}\,\Big||a|^{\frac{p-2}{2}}a-|b|^{\frac{p-2}{2}}b\Big|\,.\tag{3.5}$$

Applying (3.4) we get

$$\text{left side of (3.3)} \;\geqslant \frac{4}{p^2}\,\int_\Omega \eta^2(x)\,\Big|\Delta_{X_k,s}F(x)\Big|^2\,dx\,.$$

For the first two lines of the right hand side we have to use (3.5) and Theorem 2.1 and absorb $\Delta_{X_k,s}F(x)$ into the left side, while for the third line we reuse Theorem 2.1 and get

$$\int_\Omega \eta^2(x)\,\Big|\Delta_{X_k,s}F(x)\Big|^2\,dx \leqslant c\int_{B(x_0,2r)} s^2|Tu|^2+|\Delta_{X_k,s}u|^2+|\Delta_{X_k,s}Tu|^2\,dx.$$

Dividing this inequality by $s^2$ and using Theorems 2.1 and 3.1 gives the following result:

**Theorem 3.2.** *For $2\leqslant p<4$ and any $p$-harmonic function $u\in HW^{1,p}_{\mathrm{loc}}(\Omega)$ we have that*

$$X_i(|Xu|^{\frac{p-2}{2}}X_ju)\in L^2_{\mathrm{loc}}(\Omega)$$

*for every $i,j\in\{1,..,2n\}$.*

**Corollary 3.1.** *For $2\leqslant p<4$ and any $p$-harmonic function $u\in HW^{1,p}_{\mathrm{loc}}(\Omega)$ we have that*

$$X_i(|Xu|^p)\in L^2_{\mathrm{loc}}(\Omega)\,.$$

**Proof.** The proof follows from Theorems 2.1 and 3.2 via the equation:

$$X_i\Big(|Xu|^p\Big) = \sum_{j=1}^{2n} X_i\Big(|Xu|^{\frac{p-2}{2}} X_j u \cdot |Xu|^{\frac{p-2}{2}} X_j u\Big)$$
$$= 2\sum_{j=1}^{2n} X_i\Big(|Xu|^{\frac{p-2}{2}} X_j u\Big) \cdot |Xu|^{\frac{p-2}{2}} X_j u\ . \qquad \blacksquare$$

To finish the paper we note that our results apply also to viscosity solutions in the given range of $p$'s since Bieske [1] has shown that weak solutions and viscosity solutions of the $p$-Laplace equation in the Heisenberg group coincide.

## References

[1] T. Bieske, *Equivalence of Weak and Viscosity Solutions to the p-Laplace Equation in the Heisenberg group*, Ann. Acad. Sci. Fenn. Math. **31** (2006), 363–379.

[2] B. Bojarski, T. Iwaniec, *p-Harmonic equation and quasiregular mappings*, University of Bonn, preprint no. 617, Sonderforschungsbereich **72** (1983) and Banach Center Publications, vol. 19, PWN Warsaw (1987), 25–38.

[3] B. Bojarski, *Homeomorphic solutions of Beltrami systems*, (Russian) Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 661–664.

[4] B. Bojarski, T. Iwaniec, *Analytical foundations of the theory of quasiconformal mappings in $R^n$*, Ann. Acad. Sci. Fenn. Ser. A I Math. **8** (1983), no. 2, 257–324.

[5] A. Domokos, *Differentiability of solutions for the non-degenerate p-Laplacian in the Heisenberg group*, J. Differential Equations **204** (2004), 439–470.

[6] J. Manfredi, G. Mingione, *Regularity results for quasilinear elliptic equations in the Heisenberg group*, Math. Ann. **339** (2007), 485–544.

[7] G. Mingione, A. Zatorska-Goldstein, X. Zhong, *Gradient regularity for elliptic equations in the Heisenberg group*, preprint.

**Addresses:** András Domokos: Department of Mathematics and Statistics, California State University Sacramento, Sacramento, CA, 95819, USA;
Juan J. Manfredi: Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA.

**E-mail:** domokos@csus.edu, manfredi@pitt.edu

**Received:** 3 April 2008; **revised:** 20 July 2008