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FORMAL SOLUTIONS OF BURGERS TYPE EQUATIONS

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To Bogdan Bojarski, teacher, mentor
and friend, with great respect

Abstract: We study formal power series solutions to the initial value problem for the Burgers type equation $\partial_t u - \Delta u = X\big(f(u)\big)$ with polynomial nonlinearity f and a vector field X, and prove that they belong to the formal Gevrey class G^2. Next we give counterexamples showing that the solution, in general, is not analytic in time at $t = 0$. We also prove the existence of non-constant globally analytic solutions.

Keywords: Burgers type equation, formal solutions, combinatorial estimates, Gevrey estimates, non-analyticity

1. Introduction

We consider the initial value problem for a Burgers type equation

$$\begin{cases} \partial_t u - \Delta u &= X\big(f(u)\big), \\ u|_{t=0} &= u_0, \end{cases} \tag{1}$$

where Δ is the Laplace operator on $\mathbb{R}^d$, $d \in \mathbb{N}$, X is a vector field on $\mathbb{R}^d$ and f is a polynomial of degree $r \geqslant 1$. A number of papers were written about this problem, mainly in the one dimensional case and $f(u) = u^{k+1}, k \in \mathbb{N}$. In particular, in that case when $k = 1$ D. B. Dix established in [2] local existence and uniqueness the initial value problem in the class $C([0, T), H^s)$ for $s > -1/2$, and in the general case $k \in \mathbb{N}$, D. Bekiranov in [1] done the work in the class of weighted L^p based Sobolev spaces.

Here we assume that the initial data are analytic on a domain $\Omega \subset \mathbb{R}^d$ and we are interested in formal power series solutions of (1),

$$\hat{u}(t, x) = \sum_{n=0}^{\infty} \varphi_n(x) t^n. \tag{2}$$

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Usually the growth properties of formal power series are controlled by Gevrey estimates.

Definition 1. *Let $\Omega \subseteq \mathbb{R}^d$ and $s \geqslant 1$. A formal power series (2) is said to belong to the* Gevrey class $G^s(\Omega)$ in time *if for any compact set $K \Subset \Omega$ one can find $L < \infty$ such that*

$$\sup_{n\in\mathbb{N}_0} \sup_{x\in K} \frac{|\varphi_n(x)|}{L^n (n!)^{s-1}} < \infty. \tag{3}$$

Remark that for $s = 1$ we get the convergence i.e. $G^1(\Omega) = \mathcal{A}(\Omega)$.

Let us mention here that formal power series solutions to nonlinear partial differential equations were studied by H. Chen and Z. Luo in [3], H. Chen, Z. Luo and H. Tahara in [4], H. Chen, Z. Luo and C. Zhang in [5], H. Chen and Z. Zhang in [6], R. Gérard and H. Tahara in [7], and by S. Ōuchi in [13], [14] and [15]. In particular, S. Ōuchi obtained in [13] Gevrey estimates of formal solutions for a quite general class of nonlinear PDEs. From his results one can infer the estimation for the equation (1).

We construct the power series solution (2) of (1) and prove that if u_0 is analytic on a domain $\Omega \subset \mathbb{R}^d$, then the solution (2) belongs to the formal Gevrey class $G^2(\Omega)$ in time. Our main result reads as follows.

Theorem 1. *Let Ω be a domain in $\mathbb{R}^d$, f a polynomial of degree $r \in \mathbb{N}$ vanishing at zero, X a constant vector field on Ω and let $u_0 \in \mathcal{A}(\Omega)$. Then the formal power series solution (2) of (1) belongs to $G^2(\Omega)$ in time.*

In fact, Theorem 1 follows from Theorem 1.8 in [13]. However its proof presented here in Section 3 is done by another and elementary method. It is based on some combinatorial identities and estimates obtained in Section 2, which may be of independent interest. Next, in Section 4 we prove that solutions to (1), in general, can not be analytic in time at $t = 0$. Finally, in Section 5 we show that (1) has always non-constant solutions which are analytic on $\mathbb{R}_t \times \mathbb{R}^d_x$.

2. Formal solutions and combinatorial estimates

In this section we shall construct the formal power series solutions of (1) and prove a combinatorial lemma useful in the proof of the Theorem 1.

Clearly one can easily construct a formal power series solution (2) of (1). Namely, if $f(u) = \sum_{l=1}^{r} a_l u^l$, the functions φ_n are given by the recurrence relations

$$\begin{cases} \varphi_0 &= u_0, \\ \varphi_{n+1} &= \frac{1}{n+1}\Big(\Delta\varphi_n + \psi_n\Big), \qquad n \in \mathbb{N}_0, \end{cases} \tag{4}$$

with

$$\psi_n = \sum_{l=1}^{r} a_l \sum_{\kappa\in\mathbb{N}_0^l,\, |\kappa|=n} X\big(\varphi_{\kappa_1}\cdots\varphi_{\kappa_l}\big), \qquad n \in \mathbb{N}_0. \tag{5}$$

Indeed

$$\partial_t \hat{u}(t,x) = \sum_{n=0}^{\infty}(n+1)\varphi_{n+1}(x)t^n, \qquad \Delta_x \hat{u}(t,x) = \sum_{n=0}^{\infty}\Delta\varphi_n(x)t^n$$

and

$$\begin{aligned} X\big(f(\hat{u})\big) &= \sum_{l=1}^{r} a_l X\Big(\big(\sum_{n=0}^{\infty}\varphi_n(x)t^n\big)^l\Big) \\ &= \sum_{n=0}^{\infty}\Big(\sum_{l=1}^{r} a_l \sum_{\kappa\in\mathbb{N}_0^l,\,|\kappa|=n} X\big(\varphi_{\kappa_1}...\varphi_{\kappa_l}\big)\Big)t^n. \end{aligned}$$

Hence we get (4). It is easy to note that (4) implies

$$\varphi_{n+1} = \frac{1}{(n+1)!}\Big(\Delta^{n+1}u_0 + \sum_{k=0}^{n} k!\Delta^{n-k}\psi_k\Big), \qquad n\in\mathbb{N}_0. \tag{6}$$

In the proof of Theorem 1 we use Lemma 1 stated below and the combinatorial identity

$$\begin{aligned} &\sum \frac{(|\beta^1|+\kappa_1)!}{\beta^1!\kappa_1!}\cdots\frac{(|\beta^j|+\kappa_j)!}{\beta^j!\kappa_j!} \\ &\qquad = \frac{(|\alpha|+|\kappa|+j-1)!}{\alpha!(|\kappa|+j-1)!}, \quad j\in\mathbb{N},\ \kappa\in\mathbb{N}_0^j,\ \alpha\in\mathbb{N}_0^d, \end{aligned} \tag{7}$$

where the sum is over $\beta^1,...,\beta^j\in\mathbb{N}_0^d$ with $\beta^1+\cdots+\beta^j=\alpha$. The formula (7) can be proved by the combinatorial interpretation of both sides (if $j=2$ see [16], Form. 4.2.5.36). To formulate Lemma 1 for $\ell=(l_1,...,l_j)\in\mathbb{N}_0^j$ with $j\geqslant 2$ and $|\ell|=l_1+\cdots+l_j=l\in\mathbb{N}_0$ define

$$\binom{l}{\ell} = \binom{l}{l_1,...,l_j} = \frac{l!}{l_1!\cdots l_j!}.$$

Lemma 1. *Let $\gamma,l\in\mathbb{N}$. For $n\in\mathbb{N}_0$ set*

$$A_l(n) = \sum_{\kappa\in\mathbb{N}_0^l,\,|\kappa|=n}\binom{n}{\kappa}^{\gamma-1}\Big/\binom{\gamma n}{\gamma\kappa}. \tag{8}$$

Then there exists a constant $L=L(l)<\infty$ (independent of γ) such that

$$A_l(n)\leqslant L \quad for \quad n\in\mathbb{N}_0. \tag{9}$$

Proof. Clearly $A_l(0)=1$ and $A_1(n)=1$ for $n\in\mathbb{N}_0$. Next for $n\geqslant 1$ note that

$$A_l(n) = \sum_{b=1}^{l}\binom{l}{b}A_l^b(n),$$

where

$$A_l^b(n) = \sum_{\kappa\in\mathbb{N}^b,\,|\kappa|=n} \binom{n}{\kappa}^{\gamma-1} \Big/ \binom{\gamma n}{\gamma\kappa, \mathbf{0}_{l-b}}. \tag{10}$$

Hence it is sufficient to show that there exists a constant $L = L(b,l) < \infty$ such that

$$A_l^b(n) \leqslant L \quad \text{for} \quad n \in \mathbb{N}_0. \tag{11}$$

To this end note that for $n < b$ the sum in (10) is empty and so $A_l^b(n) = 0$. Next setting $\kappa' = (\kappa_1, ..., \kappa_{b-1})$, we note that for $n \geqslant b$ a term in the sum (10) is equal to

$$\begin{aligned}\binom{n}{\kappa}^{\gamma-1} \Big/ \binom{\gamma n}{\gamma\kappa, \mathbf{0}_{l-b}} &= \left(\frac{n!}{\kappa!}\right)^{\gamma-1} \cdot \frac{(\gamma\kappa)!}{(\gamma n)!} \\ &= \frac{(\gamma\kappa')!}{(\kappa'!)^{\gamma-1}} \cdot \left(\frac{n!}{\kappa_b!}\right)^{\gamma-1} \cdot \frac{(\gamma\kappa_b)!}{(\gamma n)!} \\ &= \frac{(\gamma\kappa')!}{(\kappa'!)^{\gamma-1}} \cdot \frac{\big((\kappa_b+1)(\kappa_b+2)\cdots n\big)^{\gamma-1}}{(\gamma\kappa_b+1)(\gamma\kappa_b+2)\cdots(\gamma n)}.\end{aligned}$$

Clearly the numerator of the last factor is bounded by $(n+1)^{(\gamma-1)(n-\kappa_b)}$. Next assuming $1 \leqslant \kappa_1 \leqslant ... \leqslant \kappa_b$, we get $n - |\kappa'| = \kappa_b \geqslant \frac{n}{b}$. Hence $\gamma\kappa_b + k \geqslant \frac{\gamma n}{b} + k \geqslant \min\left(\frac{\gamma}{b}, k\right) \cdot (n+1)$ for $k = 1, 2, ..., \gamma n - \gamma\kappa_b$ and $n \geqslant 0$. So the nominator of the last factor is not less then

$$c_0 \big/ (n+1)^{\gamma n - \gamma\kappa_b} \quad \text{with some} \quad c_0 > 0.$$

Hence a term in the sum (10) is bounded by

$$L\big/(n+1)^{n-\kappa_b} \leqslant L\big/(n+1)^{b-1}$$

with some $L < \infty$ (since $n - \kappa_b = |\kappa'| \geqslant b-1$). Finally since the sum (10) contains no more then $(n+1)^{b-1}$ terms we get (11). ∎

3. Proof of the Main Theorem.

Let $f(u) = \sum_{l=1}^r a_l u^l$ with $r \geqslant 1$. Put $\gamma = \max\{2, r\}$. We shall prove inductively that for any compact set $K \Subset \Omega$ one can find $1 \leqslant C < \infty$ such that for any $n, m \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = m$,

$$\sup_{x\in K} |\partial^\alpha \varphi_n(x)| \leqslant b_n\, C^{m+n\gamma+1} \frac{(m+n\gamma)!}{(n!)^{\gamma-1}}, \tag{12}$$

$$\sup_{x\in K} |\partial^m \psi_n(x)| \leqslant c_n\, C^{m+(n+1)\gamma+1} \frac{(m+(n+1)\gamma)!}{(n!)^{\gamma-1}}, \tag{13}$$

where

$$b_n = b_0 d^n + \sum_{k=0}^{n-1} c_k (k+1)^{\gamma-2} d^{n-k-1}, \tag{14}$$

$$c_n = \sum_{l=1}^{r} |a_l| \frac{(m+n\gamma+l)!}{(n\gamma+l-1)!} \cdot \frac{(n\gamma)!}{(m+n\gamma+\gamma)!} \cdot \\ \cdot \sum_{\kappa\in\mathbb{N}_0^l,\, |\kappa|=n} b_{\kappa_1}\dots b_{\kappa_l} \binom{n}{\kappa}^{\gamma-1} \Big/ \binom{\gamma n}{\gamma\kappa}. \qquad (15)$$

Clearly, since $u_0 = \varphi_0 \in \mathcal{A}(\Omega)$, (12) holds for $n = 0$ with $b_0 = 1$ and some $1 \leqslant C < \infty$. Now assume that (12) holds for $n \in \mathbb{N}_0$. Since Δ is rotationally invariant we can assume that $X = \frac{\partial}{\partial x_1}$. Set $0' = (0, ..., 0) \in \mathbb{N}_0^{d-1}$ Then by (5), the Leibniz rule, the inductive assumption and (7) we estimate for $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = m$ and $x \in K$,

$$\begin{aligned}
|\partial^\alpha \psi_n(x)| &\leqslant \sum_{l=1}^{r} |a_l| \sum_{\kappa\in\mathbb{N}_0^l,\, |\kappa|=n} \left|\partial^{(\alpha+(1,0'))}\big(\varphi_{\kappa_1}(x)\cdots\varphi_{\kappa_l}(x)\big)\right| \\
&\leqslant \sum_{l=1}^{r} |a_l| \sum_{\substack{\kappa\in\mathbb{N}_0^l \\ |\kappa|=n}} \sum_{\substack{\beta^1,\dots,\beta^l\in\mathbb{N}_0^d \\ \beta^1+\dots+\beta^l=\alpha+(1,0')}} \frac{(\alpha+(1,0'))!}{\beta^1!\cdots\beta^l!} |\partial^{\beta_1}\varphi_{\kappa_1}(x)|\cdots|\partial^{\beta_l}\varphi_{\kappa_l}(x)| \\
&\leqslant \sum_{l=1}^{r} |a_l| \sum_{\substack{\kappa\in\mathbb{N}_0^l \\ |\kappa|=n}} \sum_{\substack{\beta^1,\dots,\beta^l\in\mathbb{N}_0^d \\ \beta^1+\dots+\beta^l=\alpha+(1,0')}} \frac{(\alpha+(1,0'))!}{\beta^1!\cdots\beta^l!} b_{\kappa_1} C^{|\beta^1|+\kappa_1\gamma+1} \\
&\quad \times \frac{(|\beta^1|+\kappa_1\gamma)!}{\kappa_1!^{\gamma-1}} \cdots b_{\kappa_l} C^{|\beta^l|+\kappa_l\gamma+1} \frac{(|\beta^l|+\kappa_l\gamma)!}{\kappa_l!^{\gamma-1}} \\
&\leqslant C^{m+\gamma n+\gamma+1} \sum_{l=1}^{r} |a_l| \sum_{\substack{\kappa\in\mathbb{N}_0^l \\ |\kappa|=n}} \frac{b_{\kappa_1}\cdots b_{\kappa_l}}{\kappa_1!^{\gamma-1}\cdots\kappa_l!^{\gamma-1}} (\alpha+(1,0'))! \\
&\quad \times \sum_{\substack{\beta^1,\dots,\beta^l\in\mathbb{N}_0^d \\ \beta^1+\dots+\beta^l=\alpha+(1,0')}} \frac{(|\beta^1|+\kappa_1\gamma)!}{\beta^1!} \cdots \frac{(|\beta^l|+\kappa_l\gamma)!}{\beta^l!} \\
&= C^{m+\gamma n+\gamma+1} \sum_{l=1}^{r} |a_l| \frac{(m+\gamma n+l)!}{(\gamma n+l-1)!} \cdot \sum_{\substack{\kappa\in\mathbb{N}_0^l \\ |\kappa|=n}} b_{\kappa_1}\cdots b_{\kappa_l} \frac{(\kappa_1\gamma)!\cdots(\kappa_l\gamma)!}{\kappa_1!^{\gamma-1}\cdots\kappa_l!^{\gamma-1}} \\
&= C^{m+\gamma n+\gamma+1} \sum_{l=1}^{r} |a_l| \frac{(m+\gamma n+l)!}{(\gamma n+l-1)!} \cdot \frac{(\gamma n)!}{n!^{\gamma-1}} \\
&\quad \times \sum_{\substack{\kappa\in\mathbb{N}_0^l \\ |\kappa|=n}} b_{\kappa_1}\cdots b_{\kappa_l} \binom{n}{\kappa}^{\gamma-1} \Big/ \binom{\gamma n}{\gamma\kappa} \\
&\leqslant c_n \cdot C^{m+\gamma(n+1)+1} \frac{(m+\gamma(n+1))!}{n!^{\gamma-1}},
\end{aligned}$$

where c_n is given by (15). Hence (13) and (15) hold for n.

Next, in order to prove (12) and (14) for $n+1$, observe that Δ^k is a sum of d^k operators of the form $\partial^{2k}_{l_1 l_1 \ldots l_k l_k}$ with some $l_i \in \{1, ..., d\}$ for $i = 1, ..., k$, $k \in \mathbb{N}$. So by (6) we estimate for $x \in K$ and $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = m$,

$$\begin{aligned}
|\partial^\alpha \varphi_{n+1}(x)| \leqslant{} & \frac{1}{(n+1)!}\Big[b_0 C^{m+2n+3}(m+2n+2)!d^{n+1} \\
& + \sum_{k=0}^{n} k! c_k C^{m+2n-2k+(k+1)\gamma+1} \frac{(m+2n-2k+(k+1)\gamma)!}{k!^{\gamma-1}} d^{n-k}\Big] \\
={} & \frac{1}{(n+1)!}\Big[b_0 C^{m+2n+3}(m+2n+2)!d^{n+1} \\
& + \sum_{k=0}^{n} c_k C^{m+2n+k(\gamma-2)+\gamma+1} \frac{(m+2n+k(\gamma-2)+\gamma)!}{k!^{\gamma-2}} d^{n-k}\Big] \\
\leqslant{} & C^{m+\gamma n+\gamma+1} \frac{(m+(n+1)\gamma))!}{(n+1)!^{\gamma-1}} \cdot \Big[b_0 \frac{(m+2n+2)! \cdot (n+1)!^{\gamma-2}}{(m+(n+1)\gamma)!} d^{n+1} \\
& + \sum_{k=0}^{n} c_k \cdot \frac{(m+2n+k(\gamma-2)+\gamma)!}{k!^{\gamma-2}} \cdot \frac{(n+1)!^{\gamma-2}}{(m+(n+1)\gamma)!} d^{n-k}\Big] \\
\leqslant{} & C^{m+(n+1)\gamma+1} \frac{(m+(n+1)\gamma))!}{(n+1)!^{\gamma-1}} \\
& \times \Big(b_0 d^{n+1} + \sum_{k=0}^{n} c_k \cdot (k+1)^{\gamma-2} d^{n-k}\Big)
\end{aligned}$$

since $C \geqslant 1$, $\gamma \geqslant 2$,

$$\frac{(m+2n+2)! \cdot (n+1)!^{\gamma-2}}{(m+(n+1)\gamma)!} = 1 \Big/ \binom{m+(n+1)\gamma}{m+2n+2, n+1, \ldots, n+1} \leqslant 1$$

and

$$\begin{aligned}
& \frac{(m+2n+k(\gamma-2)+\gamma)!}{(k+1)!^{\gamma-2}} \cdot \frac{(n+1)!^{\gamma-2}}{(m+(n+1)\gamma)!} \\
& \quad = \binom{m+2n+2+(k+1)(\gamma-2)}{m+2n+2, k+1, \ldots, k+1} \Big/ \binom{m+\gamma(n+1)}{m+2n+2, n+1, \ldots, n+1} \leqslant 1
\end{aligned}$$

for $k = 0, 1, \ldots, n$. Hence (12) and (14) hold for $n+1$.

Now we shall prove that relations (14) and (15) imply for $n \in \mathbb{N}_0$,

$$b_n \leqslant (M+d)^n \quad \text{and} \quad c_k \leqslant M(M+d)^k/(k+1)^{\gamma-1} \quad \text{for} \quad k < n, \tag{16}$$

where $M = L \cdot \sum_{l=1}^{r} |a_l|$ and L is the constant in (13).

Clearly since $b_0 = 1$, (16) holds for $n = 0$. Next assuming (16) for $n \in \mathbb{N}_0$ we get by (15) and Lemma 1

$$\begin{aligned}
c_n &\leqslant \sum_{l=1}^{r} |a_l| \frac{(m+\gamma n+l)!}{(\gamma n+l-1)!} \frac{(\gamma n)!}{(m+\gamma n+\gamma)!} \cdot (M+d)^n \cdot A_l(n) \\
&\leqslant \sum_{l=1}^{r} |a_l| \frac{(\gamma n+l)!}{(\gamma n+l-1)!} \frac{(\gamma n)!}{(\gamma n+\gamma)!} \cdot (M+d)^n \cdot L \\
&\leqslant L \sum_{l=1}^{r} |a_l| \frac{(\gamma n+l)}{(\gamma n+1)\cdots(\gamma n+\gamma)} \cdot (M+d)^n \\
&\leqslant M \frac{(M+d)^n}{(n+1)^{\gamma-1}},
\end{aligned}$$

since for any $m \in \mathbb{N}_0$ and $l = 1, ..., r$

$$\frac{(m+\gamma n+l)!}{(m+\gamma n+\gamma)!} \leqslant \frac{(\gamma n+l)!}{(\gamma n+\gamma)!}.$$

Hence by (14)

$$\begin{aligned}
b_{n+1} &\leqslant d^{n+1} + \sum_{k=0}^{n} c_k (k+1)^{\gamma-2} d^{n-k} \\
&\leqslant d^{n+1} + \sum_{k=0}^{n} M \frac{(M+d)^k}{(k+1)^{\gamma-1}} (k+1)^{\gamma-2} d^{n-k} \\
&\leqslant d^{n+1} + M \sum_{k=0}^{n} (M+d)^k d^{n-k} = (M+d)^{n+1}.
\end{aligned}$$

So (16) holds for $n+1$.

Now to end the proof of Theorem 1 note that (12) and (16) imply

$$\sup_{x \in K} |\varphi_n(x)| \leqslant (M+d)^n C^{n\gamma+1} \frac{(n\gamma)!}{(n!)^{\gamma-1}}, \qquad n \in \mathbb{N}_0.$$

Hence we get (3) with $s = 2$ and $L = (M+d)(C\gamma)^{\gamma}$. ■

4. Nonanalytic solutions

In this section we shall give a few examples of initial value problems (1), which do not admit solutions analytic in time at $t = 0$.

Firstly, we shall apply the method of J. Gorsky and A. Himonas [8] to the Burgers type equation

$$\partial_t u - \partial_x^2 u = \partial_x (u^r), \qquad r \geqslant 2. \tag{17}$$

Following [8], for a solution u of (17) we define the *homogeneity degree* of the term

$$(\partial_x^{\alpha_1}u)\cdots(\partial_x^{\alpha_m}u), \alpha\in\mathbb{N}_0^m, m\in\mathbb{N}$$

to be

$$\Big(\alpha_1+\frac{1}{r-1}\Big)+\cdots+\Big(\alpha_m+\frac{1}{r-1}\Big)=|\alpha|+\frac{m}{r-1}.$$

Lemma 2. *Let u satisfy (17). Then for every $k\in\mathbb{N}$*

$$\partial_t^k u=\partial_x^{2k}u+\sum_{l=1}^{k}\sum_{\substack{\alpha\in\mathbb{N}_0^m,\\ |\alpha|=2k-l\\ m=1+l(r-1)}} C_\alpha^l(\partial_x^{\alpha_1}u)\dots(\partial_x^{\alpha_m}u) \quad \textit{with some} \quad C_\alpha^l\geqslant 0. \tag{18}$$

Observe that the homogeneity degree of every summand on the right-hand side of (18) is equal to $2k+\frac{1}{r-1}$. A similar lemma for the case of generalized KdV equation was obtained by H. Hannah, A. Himonas and G. Petronilho [9], Lemma 2.2, see also [12], Lemma 2. So we omit its proof.

Repeating the computations of Examples 4 and 6 from [12] we get

Example 1. Let $r\geqslant 2, b\in\mathbb{C}\setminus\mathbb{R}$ and $\delta=1/(r-1)$. Then the formal solution of (17) with the Cauchy data $u(0,x)=(b-x)^{-\delta}$ does not belong to the Gevrey class $G^s(\Omega)$ in time for any $s<2$ and for any neighborhood Ω of the origin. In particular, it is not analytic in time at $t=0$.

For the periodic Cauchy data we have

Example 2. Let $r\geqslant 2$, $\delta=1/(r-1)$ and $M>1$. Then the formal solution of (17) with the Cauchy data $u(0,x)=i^\delta\frac{e^{ix}}{M-e^{ix}}$ does not belong to the Gevrey class $G^s(\Omega)$ in time for any $s<2$ and for any neighborhood Ω of the origin. Thus, it is not analytic in time at $t=0$.

It appears that the above method can not be easily adapted to the real-valued data. So to give examples of non-analytic solutions with real-valued Cauchy data we follow the method presented in [10]. Namely, put

$$u_0(x)=\frac{cx}{1+x^2}. \tag{19}$$

Then, if r is even the coefficients φ_n of the formal power solutions to (2) are given by

$$\varphi_n(x)=\frac{1}{n!}\sum_{k=0}^{\infty}(-1)^{n+k}A(n,2k+1)x^{2k+1}, \qquad n\in\mathbb{N}_0, \tag{20}$$

where the coefficients $A(n, 2k+1)$ satisfy

$$A(0, 2k+1) = c,$$
$$A(n+1, 2k+1) = \begin{cases} (2k+2)(2k+3)A(n, 2k+3), & \text{if } k \leqslant \frac{r}{2} - 2, \\ (2k+2)(2k+3)A(n, 2k+3) + (-1)^{r/2}\,(2k+2) \\ \cdot \sum\limits_{\substack{\eta \in \mathbb{N}_0^r \\ |\eta| = n}} \binom{n}{\eta} \sum\limits_{\substack{\kappa \in \mathbb{N}_0^r \\ |\kappa| = k+1-r/2}} A(\eta_1, 2\kappa_1 + 1) \cdots A(\eta_r, 2\kappa_r + 1), \\ & \text{if } k \geqslant \frac{r}{2} - 1. \end{cases}$$

Hence

$$A(n, 2k+1) \leqslant (2k+2)\cdots(2k+2n+1)c \quad \text{if } r \bmod 4 = 2 \text{ and } c < 0,$$
$$A(n, 2k+1) \geqslant (2k+2)\cdots(2k+2n+1)c \quad \text{if } r \bmod 4 = 0 \text{ and } c > 0.$$

In both cases $\frac{|\varphi_n(x)|}{x}\Big|_{x=0} \geqslant \frac{(2n+1)!}{n!}|c|$ which implies that the formal solution does not belong to G^s in time for any $1 \leqslant s < 2$.

If r is odd the power series expansions of φ_n contains both odd and even powers of x. In that case the analysis is more laborious, but still can be done.

5. Analytic solutions

In this section we shall show that one can always find a non-constant initial data for which (1) has a solution analytic in time and space variables. Let us start with the Burgers equation

$$\partial_t u = \partial_x^2 u + a\partial_x(u^2), \qquad a \neq 0. \tag{21}$$

Looking for a solution to (21) in the form $u(t,x) = v(x - bt)$ we obtain the ODE for the function $v(y), y = x - bt$,

$$-bv' = v'' + (av^2)'.$$

Hence

$$v' = -av^2 - bv - c \quad \text{with some} \quad c \in \mathbb{R}. \tag{22}$$

Put $\Delta = b^2 - 4ac$. If $\Delta < 0$ the solution of (22) is

$$v(y) = \frac{-b}{2a} + \frac{\sqrt{-\Delta}}{2a}\tan\Big(\frac{\sqrt{-\Delta}}{2}(C - y)\Big), \qquad C \in \mathbb{R}.$$

If $\Delta = 0$ we have a stationary solution $v(y) = \frac{-b}{2a}$ and

$$v(y) = \frac{-b}{2a} + \frac{1}{ay - C}, \qquad C \in \mathbb{R}.$$

Finally, if $\Delta > 0$ we have two stationary solutions $v(y) = \frac{-b\pm\sqrt{\Delta}}{2a}$ and

$$v(y) = \frac{-b-\sqrt{\Delta}}{2a} + \frac{\sqrt{\Delta}}{a\big(1 + C\exp\{-\sqrt{\Delta}y\}\big)}, \qquad C \in \mathbb{R}.$$

Observe that in the case $\Delta > 0$ any solution of (22) extends to a holomorphic function in a strip along $\mathbb{R}$ if $C > 0$. Hence in that case we obtain solutions of (21) which are analytic on $\mathbb{R}_t \times \mathbb{R}_x$. Note also that in all cases the function v has at most simple poles with residue equal to $1/a$. Let us mention here that a complete characterization of convergent solutions of the Burgers equation (21) will be done in a forthcoming paper [11].

Now let us return to the general equation (1) and look for its solution in the form $u(t,x) = v(x_1 - bt), b \in \mathbb{R}$. Then for the function $v(y), y = x_1 - bt$, we get an analog of (22),

$$v' = -f(v) - bv - c \tag{23}$$

with some $c \in \mathbb{R}$. Clearly, in general, this equation can not be solved explicitly. However one can always find $b \neq 0$ and $c \in \mathbb{R}$ such that the equation $f(v) + bv + c = 0$ has two real roots $v_1 < v_2$. Then if v_0 is not a root of $f(v) + bv + c = 0$ and $v_1 < v_0 < v_2$, the solution to (23) with $v(0) = v_0$ is a non-constant analytic function on $\mathbb{R}$. Hence we get a non-constant solution to (1) which is analytic on $\mathbb{R}_t \times \mathbb{R}^d_x$.

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