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LOCAL VARIATION OF EULER PRODUCTS

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Dedicated to Władysław Narkiewicz
on the occasion of his 70th birthday

Abstract: We determine how big an Euler product can be at s_2, when its size at s_1 is known, and apply this via Halász's method to bound the mean value of a multiplicative function in terms of the size of the generating Dirichlet series.
Keywords: Euler product, multiplicative function

1. Statement of results

Throughout this paper we let $f(n)$ denote a totally multiplicative function such that $|f(n)| \le 1$ for all n, and for $\sigma > 1$ we set

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \Big(1 - \frac{f(p)}{p^s}\Big)^{-1}.$$

Our object is to determine what can be said about the sizes of $F(s_1)$ and $F(s_2)$ when s_1 and s_2 are nearby.

We first consider how an Euler product can vary for differing σ, with t fixed. Since

$$\sigma - 1 \asymp \frac{1}{\zeta(\sigma)} \le \frac{\zeta(2\sigma)}{\zeta(\sigma)} \le |F(s)| \le \zeta(\sigma) \asymp \frac{1}{\sigma-1} \tag{1.1}$$

uniformly in the strip $1 < \sigma \le 2$, the orders of magnitude arising all lie between $\sigma - 1$ and $1/(\sigma-1)$. Suppose that $1 < \sigma_1 \le \sigma_2$. Since

$$\frac{|F(\sigma_1)|}{|F(\sigma_2)|} = \exp\Big(\Re \sum_{n=2}^{\infty} \frac{\Lambda(n) f(n)}{\log n}\big(n^{-\sigma_1} - n^{-\sigma_2}\big)\Big),$$

it is immediate that

$$\frac{\zeta(\sigma_2)}{\zeta(\sigma_1)} \le \frac{|F(\sigma_1)|}{|F(\sigma_2)|} \le \frac{\zeta(\sigma_1)}{\zeta(\sigma_2)}.$$

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Thus if $1 < \sigma_1 \le \sigma_2 \le 2$, then

$$\frac{\sigma_1 - 1}{\sigma_2 - 1}|F(\sigma_2)| \ll |F(\sigma_1)| \ll \frac{\sigma_2 - 1}{\sigma_1 - 1}|F(\sigma_2)|\,. \tag{1.2}$$

Further insights on this topic are facilitated by

Theorem 1.1. *Let f and F be as above. For $\sigma > 1$, put $Q = Q(\sigma) = \exp(1/(\sigma - 1))$, and set*

$$T(s) = \Re \sum_{p \le Q} \frac{f(p)}{p^{1+it}}\,.$$

Then $|F(s)| \asymp \exp\big(T(s)\big)$ uniformly in the half-plane $\sigma > 1$.

Suppose that $1 < \sigma_1 \le \sigma_2 \le 2$. From Theorem 1.1 we see that any desired order of magnitude of $|F(\sigma_2)|$ between $\sigma_2 - 1$ and $1/(\sigma_2 - 1)$ can be attained by appropriately choosing $f(p)$ for $p \le Q(\sigma_2)$. Once this has been done, any desired size of $|F(\sigma_1)|$ in the interval (1.2) can be obtained by an appropriate choice of $f(p)$ for $Q(\sigma_2) < p \le Q(\sigma_1)$.

We now fix $\sigma > 1$, and consider what can be said concerning $|F(\sigma + it)|$ when $|F(\sigma)|$ is known. From (1.1) it follows that for each $\sigma > 1$ there is a number $\gamma \in [-1, 1]$ such that

$$|F(\sigma)| = \zeta(\sigma)^\gamma, \tag{1.3}$$

and a $\mu \in [-1, 1]$ depending on both σ and t such that

$$|F(s)| = \zeta(\sigma)^\mu\,. \tag{1.4}$$

Thus we want to know what pairs (γ, μ) can occur. This is a fairly straightforward issue when $|t| \ge 1$, but when $|t|$ is small, the reality is surprisingly intricate.

Theorem 1.2. *Let f and F be as above, and suppose that γ is defined by (1.3). Put*

$$X(u) = \int_0^1 \frac{u + \cos 2\pi\theta}{|u + e(\theta)|}\,d\theta, \qquad Y(u) = \int_0^1 \frac{1 + u\cos 2\pi\theta}{|1 + ue(\theta)|}\,d\theta\,. \tag{1.5}$$

Choose u so that $X(u) = \gamma$, and set $\mu = Y(u)$. Then

$$\zeta(\sigma)^{-\mu}(\log 4t)^{-A} \le |F(\sigma + it)| \le \zeta(\sigma)^{\mu}(\log 4t)^{A} \tag{1.6}$$

uniformly for $\sigma > 1$, $t \ge 1$. Here A is a suitable absolute constant. If

$$f(p) = \frac{u + p^i}{|u + p^i|} \tag{1.7}$$

for all p, then

$$|F(\sigma)| \asymp \zeta(\sigma)^{X(u)}, \qquad |F(\sigma + i)| \asymp \zeta(\sigma)^{Y(u)} \tag{1.8}$$

uniformly in u and $\sigma > 1$.

As usual, $e(\theta) = e^{2\pi i\theta}$. The function $X(u)$ is continuous, odd, and strictly increasing since

$$\frac{\partial}{\partial u}\frac{u+\cos 2\pi\theta}{|u+e(\theta)|} = \frac{(\sin 2\pi\theta)^2}{|u+e(\theta)|^3} \geq 0$$

for all θ and u. Moreover, $\lim_{u\to\pm\infty} X(u) = \pm 1$, so for any $\gamma \in [-1,1]$ there is a unique u such that $X(u) = \gamma$. Apart from the log power in (1.6), the pairs (γ,μ) that can be achieved are those that lie in the body whose boundary is depicted in Figure 1. In Lemma 3.1 we show that this is a convex convex body that is symmetric about the x and y axes, and also about the lines $y = \pm x$. It has support lines $x = \pm 1$, $y = \pm 1$, $x + y = \pm 4/\pi$, and $x - y = \pm 4/\pi$.

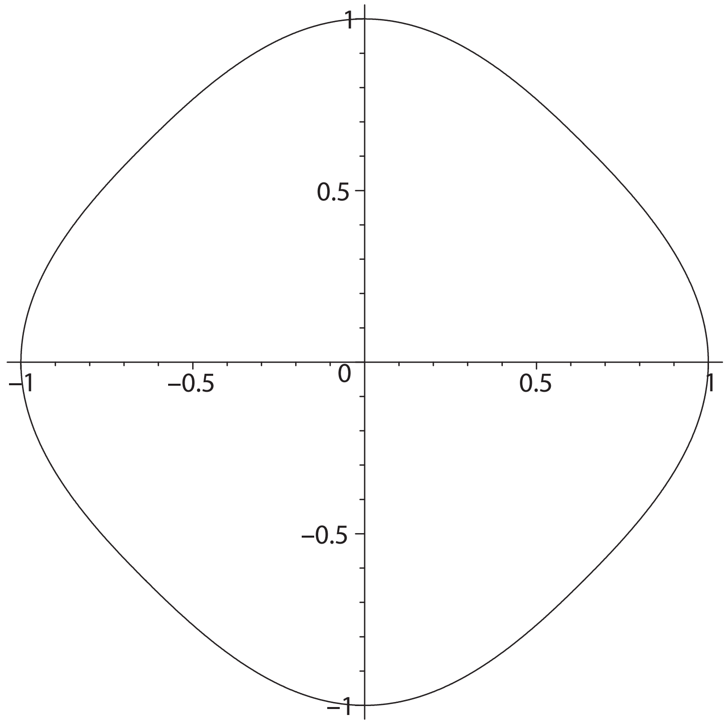

Figure 1: The parameterized curves $(X(u), \pm Y(u))$ for $-\infty \leq u \leq \infty$.

That one can simultaneously achieve $|F(\sigma)| \asymp (\sigma-1)^{2/\pi}$, $|F(\sigma+i)| \asymp (\sigma-1)^{-2/\pi}$ was used by the first author [5] to show that if $0 < c < 4/\pi - 1$ and $N > N_0(c)$, then the function

$$U_N(s) = \sum_{n=1}^{N} n^{-s} \tag{1.9}$$

has zeros in the half-plane

$$\sigma > 1 + \frac{c \log\log N}{\log N} . \tag{1.10}$$

Subsequently, the authors [6] showed that this is best possible to the extent that if $c > 4/\pi - 1$ and $N > N_1(c)$, then $U_N(s) \ne 0$ in the half-plane (1.10).

It is easy to show that $|F(s)| \asymp |F(\sigma)|$ when $0 \le t \le \sigma - 1$. Suppose that $\sigma - 1 \le t \le 1$, and let δ be defined by the relation

$$t = (\sigma - 1)^{1-\delta} . \tag{1.11}$$

Thus $0 \le \delta \le 1$, and larger values of δ correspond to larger values of t. The method used to prove Theorem 1.2 can be used to show that $F(s) \ll F(\sigma)\zeta(\sigma)^{4\delta/\pi}$. However, when $|F(\sigma)|$ is very small or very large, we can do better:

Theorem 1.3. *Let f and F be as described at the outset. If $0 \le t \le \sigma - 1$, then*

$$|F(s)| \asymp |F(\sigma)| . \tag{1.12}$$

Suppose that (1.3) *holds, that $\sigma - 1 \le t \le 1$, that δ is defined by* (1.11), *and that $X(u)$ and $Y(u)$ are defined as in Theorem* 1.2. *We have three cases:*

Case 1. $-1 \le \gamma \le -2/\pi$ and $(1+\gamma)/(1-2/\pi) \le \delta \le 1$. Then choose $u \le -1$ so that $X(u) = (\gamma + 1 - \delta)/\delta$, and set $\mu = \delta Y(u) + \delta - 1$.

Case 2. $0 \le \delta \le (1+\gamma)/(1-2/\pi)$ and $0 \le \delta \le (1-\gamma)/(1+2/\pi)$. Set $\mu = \gamma + 4\delta/\pi$.

Case 3. $-2/\pi \le \gamma \le 1$ and $(1-\gamma)/(1+2/\pi) \le \delta \le 1$. Choose $u \ge -1$ so that $X(u) = (\gamma - 1 + \delta)/\delta$, and set $\mu = \delta Y(u) + 1 - \delta$.

In all three cases,

$$F(s) \ll \zeta(\sigma)^{\mu} .$$

The three cases in Theorem 1.3 correspond to the three indicated regions in Figure 2.

Halász [2] devised a method by which the summatory function $S_0(x) = \sum_{n \le x} f(n)$ of a multiplicative function could be estimated in terms of the size of the generating function $F(s)$ in the half-plane $\sigma > 1$. Later, Halász [3] gave a sharp quantitative form of this theorem. After further refinements by the first author [4] and Tenenbaum [8], we know that if f is multiplicative and $|f(n)| \le 1$ for all n, then

$$S_0(x) \ll \frac{x}{\log x} \int_{1/\log x}^{1} M_0(\alpha)\, \frac{d\alpha}{\alpha} \tag{1.13}$$

where for $\alpha > 0$ we set

$$M_0(\alpha) = \Big(\sum_{k=-\infty}^{\infty} \max_{\substack{|t-k| \le 1/2 \\ 1+\alpha \le \sigma \le 2}} \Big| \frac{F(\sigma + it)}{\sigma + it} \Big|^2 \Big)^{1/2} . \tag{1.14}$$

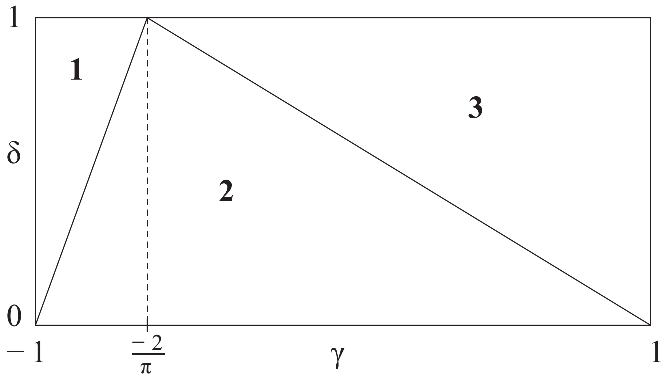

Figure 2: The three cases in Theorem 1.3.

If it is desired to bound $S_0(x)$ solely in terms of $F(\sigma)$, then one could use Theorem 1.2 to derive an estimate for M_0, and hence for S_0. Correspondingly, for the weighted summatory function $S_1(x) = \sum_{n\le x} f(n)/n$, the authors [6] showed that if f is totally multiplicative and $|f(n)| \le 1$ for all n, then

$$S_1(x) \ll \frac{1}{\log x}\int_{1/\log x}^{1} M_1(\alpha)\,\frac{d\alpha}{\alpha} \tag{1.15}$$

where for $\alpha > 0$ we put

$$M_1(\alpha) = \Big(\sum_{k=-\infty}^{\infty} \max_{\substack{|t-k|\le 1/2\\ 1+\alpha\le\sigma\le 2}} \Big|\frac{F(\sigma+it)}{\sigma-1+it}\Big|^2\Big)^{1/2}. \tag{1.16}$$

One can use Theorems 1.2 and 1.3 to derive a bound for M_1 from any given bound for $|F(\sigma)|$. In particular, we use Theorem 1.3 to establish

Theorem 1.4. *Let $X(u)$ and $Y(u)$ be as in Theorem 1.2, and let $u_0 = -0.822168\,39\ldots$ be the unique root of the equation $u + 2 = uX(u) + Y(u)$. For $-1 \le \gamma \le X(u_0) = -0.46019555\ldots$, put $\nu(\gamma) = Y(u_\gamma)$ where $X(u_\gamma) = \gamma$, and for $X(u_0) \le \gamma \le 1$ put $\nu(\gamma) = 2 - (1-\gamma)(Y(u_0)-2)/(X(u_0)-1)$. Let f and F be as at the outset, suppose that $1 < \sigma \le 2$, and that (1.3) holds. Then*

$$\frac{F(\sigma+it)}{\sigma-1+it} \ll \zeta(\sigma)^{\nu(\gamma)}$$

uniformly for $-1 \le t \le 1$.

2. Proof of Theorem 1.1

Since

$$F(s) = \exp\Big(\sum_{n=2}^{\infty}\frac{\Lambda(n)f(n)}{(\log n)n^s}\Big),$$

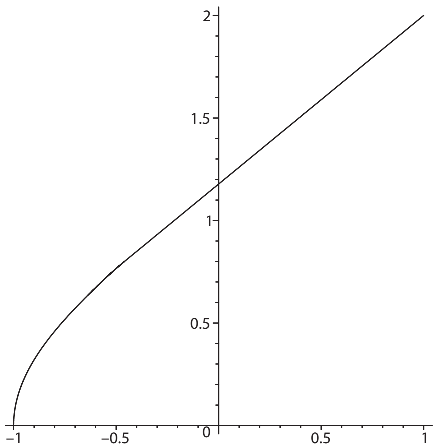

Figure 3: Plot of $\nu(\gamma)$ as defined in Theorem 1.4.

it is clear that

$$|F(s)| \asymp \exp\Big(\Re \sum_p \frac{f(p)}{p^s}\Big) .$$

But

$$\Re \sum_p \frac{f(p)}{p^s} = T(s) + \Re \sum_{p>Q} \frac{f(p)}{p^s} - \Re \sum_{p\le Q} \frac{f(p)}{p^{it}}\Big(\frac{1}{p} - \frac{1}{p^\sigma}\Big) ,$$

so to complete the proof it suffices to establish the two estimates

$$\sum_{p>Q} \frac{1}{p^\sigma} \ll 1 , \tag{2.1}$$

$$\sum_{p\le Q} \Big(\frac{1}{p} - \frac{1}{p^\sigma}\Big) \ll 1 . \tag{2.2}$$

By the Chebyshev upper bound for the number of primes in an interval, we see that

$$\sum_{2^kQ<p\le 2^{k+1}Q} \frac{1}{p^\sigma} \ll \frac{(2^kQ)^{1-\sigma}}{\log Q} \ll 2^{k(1-\sigma)}(\sigma-1) .$$

Since

$$\sum_{k=0}^{\infty} 2^{k(1-\sigma)} \ll \frac{1}{\sigma - 1} ,$$

we obtain (2.1).

As for (2.2), we observe that

$$\sum_{p\le Q}\Big(\frac{1}{p}-\frac{1}{p^\sigma}\Big)=\sum_{p\le Q}\frac{\log p}{p}\int_0^{\sigma-1}p^{-u}\,du\le(\sigma-1)\sum_{p\le Q}\frac{\log p}{p}\ll 1\,.$$

The last estimate above is due to Mertens, and is found, for example, in Theorem 2.7(b) of Montgomery & Vaughan [7].

3. Lemmas

Lemma 3.1. *Let $X(u)$ and $Y(u)$ be defined as in Theorem 1.2. The region $\mathcal{C}$ bounded by the curves $(X(u),\pm Y(u))$ is convex, and the line $ux+y=\int_0^1|u+e(\theta)|\,d\theta$ is a support line of $\mathcal{C}$ passing through the point $(X(u),Y(u))$. The set $\mathcal{C}$ is symmetric about the x-axis, the y-axis, and about the lines $x=\pm y$. The function $X(u)$ is odd, while $Y(u)$ is even, and $Y(u)=\operatorname{sgn}(u)X(1/u)$.*

Proof. For $r\in L^1(\mathbb{T})$ we define Fourier coefficients $\widehat{r}(k)=\int_0^1 r(\theta)e(-k\theta)\,d\theta$. Let $\mathcal{C}_1$ consist of those points in the plane $\mathbb{R}^2$ that can be written in the form $(\Re\widehat{r}(0),\Re\widehat{r}(1))$ for some r such that $|r(\theta)|\le 1$ for all θ. Since the unit disk $|z|\le 1$ is convex, and the Fourier coefficient is linear, it follows that $\mathcal{C}_1$ is a convex set. Let $(a,b)\ne(0,0)$ define a direction in the plane. Since

$$a\Re\widehat{r}(0)+b\Re\widehat{r}(1)=\Re\int_0^1 r(\theta)(a+be(-\theta))\,d\theta\le\int_0^1|a+be(-\theta)|\,d\theta\,, \tag{3.1}$$

we see that $\mathcal{C}_1$ lies entirely in the closed half-plane $ax+by\le c$ where

$$c=c(a,b)=\int_0^1|a+be(-\theta)|\,d\theta\,.$$

Equality is achieved in (3.1) by taking

$$r(\theta)=\frac{a+be(\theta)}{|a+be(\theta)|}\,,$$

so the support line $ax+by=c$ contacts $\mathcal{C}_1$ at the point

$$\Big(\int_0^1\frac{a+b\cos 2\pi\theta}{|a+be(\theta)|}\,d\theta,\int_0^1\frac{b+a\cos 2\pi\theta}{|b+ae(\theta)|}\,d\theta\Big)\,.$$

Points of this form comprise the boundary of $\mathcal{C}_1$. On taking $(a,b)=(u,1)$ the above is $(X(u),Y(u))$, and we see that $\mathcal{C}=\mathcal{C}_1$.

We note that

$$\int_0^1\frac{a+b\cos 2\pi\theta}{|a+be(\theta)|}\,d\theta=\int_{1/2}^{3/2}\frac{a+b\cos 2\pi\theta}{|a+be(\theta)|}\,d\theta=\int_0^1\frac{a-b\cos 2\pi\theta}{|a-be(\theta)|}\,d\theta\,. \tag{3.2}$$

Thus if the pair (a,b) yields (X,Y), then $(a,-b)$ yields $(X,-Y)$, $(-a,b)$ yields $(-X,Y)$, $(-a,-b)$ yields $(-X,-Y)$, (b,a) yields (Y,X), and $(-b,-a)$ yields $(-Y,-X)$.

From (3.2) we see that

$$X(-u) = \int_0^1 \frac{-u+\cos 2\pi\theta}{|-u+e(\theta)|}\,d\theta = -\int_0^1 \frac{u-\cos 2\pi\theta}{|u-e(\theta)|}\,d\theta = -X(u)\,,$$

and that

$$Y(-u) = \int_0^1 \frac{1-u\cos 2\pi\theta}{|1-ue(\theta)|}\,d\theta = Y(u)\,.$$

■

By means of elementary calculations it is easy to show that

$$X(u) = 1 - \frac{1}{4u^2} + O(u^{-4}), \qquad Y(u) = \frac{1}{2u} + O(u^{-3}) \tag{3.3}$$

as $u \to +\infty$, and that

$$X(u) = \frac{u}{2} + O(|u|^3), \qquad Y(u) = 1 - \frac{u^2}{4} + O(u^4) \tag{3.4}$$

Further properties of X and Y may be elicited by observing that they can be expressed in terms of complete elliptic integrals. Let

$$K(k) = \int_0^1 \frac{1}{\sqrt{1-k^2t^2}\,\sqrt{1-t^2}}\,dt\,, \quad E(k) = \int_0^1 \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\,dt\,.$$

be the complete elliptic integrals of the first and second kind, in the Legendre normal form. This notation is employed both by Gradshteyn & Ryzhik [1], and by Maple. In the restricted range $0 \le u \le 1$ we find that

$$X(u) = \frac{2}{\pi u}(E(u) - (1-u^2)K(u)), \tag{3.5}$$

$$Y(u) = \frac{2}{\pi}E(u) \tag{3.6}$$

with the tangent line

$$ux + y = \frac{2}{\pi}(2E(u) - (1-u^2)K(u))\,. \tag{3.7}$$

Lemma 3.2. *Let $g(\theta) = |u + e(\theta)| = \sqrt{u^2 + 2u\cos 2\pi\theta + 1}$. Then $\widehat{g}(k) \ll k^{-2}$ for all $k \ne 0$, uniformly in u, and $g(\theta) = \sum_k \widehat{g}(k)e(k\theta)$ for all θ.*

Proof. By direct computation we see that

$$g'(\theta) = \frac{-2\pi \sin 2\pi\theta}{\sqrt{u^2 + 2u\cos 2\pi\theta + 1}}\,.$$

We observe not only that g' has bounded variation but also that this variation is uniformly bounded as a function of u. Hence $\widehat{g}(k) \ll 1/k^2$ for all $k \neq 0$, uniformly in u. This estimate is best possible when $u = \pm 1$. Since g is continuous, and its Fourier series is absolutely convergent, it follows that its Fourier series converges to $g(\theta)$ for all θ. ■

With more work it can be shown that if $|u|$ is large, then

$$\widehat{g}(k) \ll \Big(\frac{2}{|u|}\Big)^{|k|-1}\,.$$

Lemma 3.3. *Let u be a given real number, and put $h(\theta) = (u+\cos 2\pi\theta)/|u+e(\theta)|$. Then $\widehat{h}(k) \ll k^{-2}$ for $k \neq 0$, uniformly in u, and $h(\theta) = \sum_k \widehat{h}(k)e(k\theta)$ for all θ.*

Proof. From the formula

$$h'(\theta) = \frac{-u\sin 2\pi\theta\cos 2\pi\theta - \sin 2\pi\theta}{(u^2 + 2u\cos 2\pi\theta + 1)^{3/2}}$$

we see not only that h' has bounded variation, but that its variation is uniformly bounded in u. Thus $\widehat{h}(k) \ll k^{-2}$ for $k \neq 0$. Since h is continuous, and its Fourier series is absolutely convergent, it follows that its Fourier series converges to $h(\theta)$ for all θ. ■

4. Proof of Theorem 1.2

We note that

$$\log|F(s)| = \Re \sum_{n=2}^{\infty} \frac{\Lambda(n)f(n)}{(\log n)n^s}\,.$$

Let u be a real number. Then

$$u\log|F(\sigma)| + \log|F(s)| = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{(\log n)n^{\sigma}} \Re f(n)(u + n^{-it}) \le \sum_{n=2}^{\infty} \frac{\Lambda(n)}{(\log n)n^{\sigma}} |u + n^{-it}|\,.$$

Let $g(\theta)$ be defined as in Lemma 3.2. Then the right hand side above is

$$\begin{aligned} &= \sum_{n=2}^{\infty} \frac{\Lambda(n)}{(\log n)n^{\sigma}} g\Big(\frac{-t\log n}{2\pi}\Big) \\ &= \sum_{k=-\infty}^{\infty} \widehat{g}(k) \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} n^{-\sigma - ikt} = \sum_{k=-\infty}^{\infty} \widehat{g}(k) \log\zeta(\sigma + ikt)\,. \end{aligned}$$

But (as found in Theorem 6.7 of Montgomery & Vaughan [7]) $|\log\zeta(\sigma+it)| \le \log\log 4t + O(1)$ uniformly for $\sigma \ge 1$, $t \ge 1$, so the above is

$$= \widehat{g}(0)\log\zeta(\sigma) + O(\log\log 4t)\,.$$

Thus far we have shown that

$$|F(\sigma)|^u |F(s)| \le \zeta(\sigma)^{\widehat{g}(0)}(\log 4t)^A\,.$$

We recall that $|F(\sigma)| = \zeta(\sigma)^\gamma$, and note that $uX(u) + Y(u) = \widehat{g}(0)$. Thus if we choose u so that $X(u) = \gamma$, then the above gives

$$|F(s)| \le \zeta(\sigma)^{Y(u)}(\log 4t)^A\,.$$

To obtain the corresponding lower bound we argue similarly:

$$\begin{aligned} uT(\sigma) - T(s) &= \sum_{n=2}^\infty \frac{\Lambda(n)}{(\log n)n^\sigma}\Re f(n)(u - n^{-it}) \le \sum_{n=2}^\infty \frac{\Lambda(n)}{(\log n)n^\sigma}|u - n^{-it}| \\ &= \sum_{n=2}^\infty \frac{\Lambda(n)}{(\log n)n^\sigma} g\Big(\frac{1}{2} - \frac{t\log n}{2\pi}\Big) = \sum_{k=-\infty}^\infty (-1)^k \widehat{g}(k) \sum_{n=2}^\infty \frac{\Lambda(n)}{\log n} n^{-\sigma-ikt} \\ &= \sum_{k=-\infty}^\infty (-1)^k \widehat{g}(k)\log\zeta(\sigma+ikt) = \widehat{g}(0)\log\zeta(\sigma) + O(\log\log 4t), \end{aligned}$$

so that

$$\frac{|F(\sigma)|^u}{|F(s)|} \le \zeta(\sigma)^{\widehat{g}(0)}(\log 4t)^A\,.$$

But $|F(\sigma)| = \zeta(\sigma)^\gamma$, $uX(u) + Y(u) = \widehat{g}(0)$, and $X(u) = \gamma$, so

$$\frac{1}{|F(s)|} \le \zeta(\sigma)^{Y(u)}(\log 4t)^A\,,$$

which gives the desired lower bound.

It remains to prove (1.8). The function f is the one defined in (1.7). If $u = 0$, then $|F(\sigma)| = |\zeta(\sigma - i)| \asymp 1$ and $F(\sigma + i) = \zeta(\sigma)$. If $u = +\infty$, then $F(\sigma) = \zeta(\sigma)$ and $|F(\sigma+i)| = |\zeta(\sigma+i)| \asymp 1$. If $u = -\infty$, then $F(\sigma) = \zeta(2\sigma)/\zeta(\sigma) \asymp \zeta(\sigma)^{-1}$ and $|F(\sigma+i)| = |\zeta(2\sigma+2i)/\zeta(\sigma+i)| \asymp 1$. Thus (1.8) holds in these three cases, so it remains to treat finite, non-zero u. Let $h(\theta)$ be as in Lemma 3.3. In $\log|F(\sigma)|$ we wish to replace $\Re f(n)$ by the more convenient quantity $h((\log n)/(2\pi))$. When n is prime, this involves no change at all, and when n is a higher power of a prime, we are replacing one bounded coefficient by another, so

$$\begin{aligned}\log|F(\sigma)| &= \sum_{n=2}^{\infty} \frac{\Lambda(n)}{(\log n)n^\sigma} h\Big(\frac{\log n}{2\pi}\Big) + O(1)\\ &= \sum_{k=-\infty}^{\infty} \widehat{h}(k) \sum_{n=2}^{\infty} \frac{\Lambda(n)}{(\log n)n^{\sigma-ik}} + O(1)\\ &= \sum_{k=-\infty}^{\infty} \widehat{h}(k)\log(\sigma - ik) + O(1)\,.\end{aligned}$$

But $|\log\zeta(\sigma - it)| \le \log\log 4|k| + O(1)$ for all $k \ne 0$, so the above is

$$= \widehat{h}(0)\log\zeta(\sigma) + O(1)\,.$$

Since $\widehat{h}(0) = X(u)$, we have the first relation in (1.8). We note that

$$\Re\frac{f(p)}{p^i} = \frac{u\cos\log p + 1}{|u + p^i|} = \operatorname{sgn}(u)\frac{1/u + \cos\log p}{|1/u + p^i|}\,.$$

This is in the form $\operatorname{sgn}(u)h((\log p)/(2\pi))$ but with the parameter u replaced by $1/u$. Thus we repeat the above argument with u replaced by $1/u$. With this change of parameter, $\widehat{h}(0) = X(1/u) = \operatorname{sgn}(u)Y(u)$, so we obtain the second part of (1.8), and the proof is complete.

5. Proof of Theorem 1.3

We may assume throughout that $1 < \sigma \le 2$, as all quantities under discussion are uniformly $\asymp 1$ when $\sigma > 2$. Since

$$\frac{F'}{F}(s) = -\sum_{n=2}^{\infty} \frac{\Lambda(n)f(n)}{n^s} \ll \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^\sigma} = -\frac{\zeta'}{\zeta}(\sigma) \ll \frac{1}{\sigma - 1}\,,$$

it follows that $\log F(s) - \log F(\sigma) \ll t/(\sigma-1)$. Thus we have (1.12) if $0 \le t \le \sigma - 1$.

We observe that if $1 < \sigma_1 \le \sigma_2$, then

$$\log F(\sigma_1 + iv) - \log F(\sigma_2 + iv) = \sum_{n=2}^{\infty} \frac{\Lambda(n)f(n)}{\log n}\Big(\frac{1}{n^{\sigma_1}} - \frac{1}{n^{\sigma_2}}\Big)n^{-iv}\,.$$

We set $\sigma_1 = \sigma$, $\sigma_2 = 1 + t$, apply the above with $v = 0$ and $v = t$, and take real parts to find that

$$\begin{aligned}&u\log\Big|\frac{F(\sigma)}{F(1+t)}\Big| + \log\Big|\frac{F(\sigma+it)}{F(1+t+it)}\Big|\\ &\qquad = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n}\Big(\frac{1}{n^\sigma} - \frac{1}{n^{1+t}}\Big)\Re f(n)(u + n^{-it})\\ &\qquad \le \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n}\Big(\frac{1}{n^\sigma} - \frac{1}{n^{1+t}}\Big)|u + n^{-it}|\,.\end{aligned}$$

Let $g(\theta)$ be defined as in Lemma 3.2. Then the above is

$$= \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n}\Big(\frac{1}{n^\sigma} - \frac{1}{n^{1+t}}\Big) g(-t(\log n)/(2\pi))$$
$$= \sum_{k=-\infty}^{\infty} \widehat{g}(k) \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n}\Big(\frac{1}{n^\sigma} - \frac{1}{n^{1+t}}\Big) n^{-ikt}\,.$$

The function g is even and real-valued, so the sequence $\widehat{g}(k)$ is even and real-valued. Hence the above is

$$= \widehat{g}(0) \log \frac{\zeta(\sigma)}{\zeta(1+t)} + 2\sum_{k=1}^{\infty} \widehat{g}(k) \log \Big| \frac{\zeta(\sigma+ikt)}{\zeta(1+t+ikt)} \Big|\,.$$

If $1 \le k \le 1/t$, then $|\zeta(\sigma+ikt)| \asymp 1/|\sigma-1+ikt| \asymp 1/|t+ikt| \asymp |\zeta(1+t+ikt)|$. Thus the logarithm above is $O(1)$ for these k. If $k > 1/t$, then $\log \zeta(\sigma+ikt) \ll \log\log 4k$ and $\log \zeta(1+t+ikt) \ll \log\log 4k$, and $\widehat{g}(k) \ll k^{-2}$ for all $k \ne 0$, so the entire sum is $O(1)$. We note that $\widehat{g}(0) = uX(u) + Y(u)$. Thus

$$u \log \Big| \frac{F(\sigma)}{F(1+t)} \Big| + \log \Big| \frac{F(\sigma+it)}{F(1+t+it)} \Big| \le (uX(u)+Y(u)) \log \frac{\zeta(\sigma)}{\zeta(1+t)} + O(1)\,. \tag{5.1}$$

Our use of this depends on the various cases.

Case 1. First we note that

$$\log \Big| \frac{F(\sigma)}{F(1+t)} \Big| = \gamma \log \zeta(\sigma) - \Re \sum_{n=2}^{\infty} \frac{\Lambda(n) f(n)}{(\log n) n^{1+t}} \tag{5.2}$$
$$\le \gamma \log \zeta(\sigma) + \sum_{n=2}^{\infty} \frac{\Lambda(n)}{(\log n) n^{1+t}} = \gamma \log \zeta(\sigma) + \log \zeta(1+t)\,.$$

Clearly

$$\log \Big| \frac{F(s)}{F(1+t+it)} \Big| - \log \Big| \frac{F(\sigma)}{F(1+t)} \Big|$$
$$= u \log \Big| \frac{F(\sigma)}{F(1+t)} \Big| + \log \Big| \frac{F(s)}{F(1+t+it)} \Big| - (u+1) \log \Big| \frac{F(\sigma)}{F(1+t)} \Big|\,. \tag{5.3}$$

If $u \le -1$, then by (5.1) and (5.2) the above is

$$\le (uX(u)+Y(u)) \log \frac{\zeta(\sigma)}{\zeta(1+t)} - (u+1)(\gamma \log \zeta(\sigma) + \log \zeta(1+t)) + O(1)$$
$$= u\big((X(u)-\gamma) \log \zeta(\sigma) - (X(u)+1) \log \zeta(1+t)\big)$$
$$+ (Y(u)-\gamma) \log \zeta(\sigma) - (Y(u)+1) \log \zeta(1+t) + O(1)\,. \tag{5.4}$$

Since $\zeta(1+t) \asymp 1/t = (\sigma-1)^{\delta-1} \asymp \zeta(\sigma)^{1-\delta}$, we have

$$\log\zeta(1+t) = (1-\delta)\log\zeta(\sigma) + O(1)\,. \tag{5.5}$$

We insert this in (5.4), and in doing so note that the $O(1)$ error term is multiplied by $Y(u)+1$, which is bounded, and also by $u(X(u)+1)$. Since $X(u)$ is an odd function, it follows from (3.3) that $X(u) = -1 + 1/(4u^2) + O(u^{-4})$ as $u \to -\infty$. Thus $u(X(u)+1)$ is uniformly bounded for $u \le -1$, and we find that the expression (5.4) is

$$= \big(u(\delta X(u) - \gamma - 1 + \delta) + \delta Y(u) - \gamma - 1 + \delta\big)\log\zeta(\sigma) + O(1)\,.$$

Our choice of u ensures that $\delta X(u) - \gamma - 1 + \delta = 0$. By (1.12) it follows that

$$\log|\zeta(1+t)| = \log|\zeta(1+t+it)| + O(1)\,. \tag{5.6}$$

Thus

$$\log|F(\sigma+it)| \le (\delta Y(u) + \delta - 1)\log\zeta(\sigma) + O(1)\ ,$$

which gives the stated bound in this case.

Case 2. We take $u = -1$ in (5.1). Since $X(-1) = -2/\pi$ and $Y(-1) = 2/\pi$, by (5.5) it follows that the right hand side is $4\delta/\pi\log\zeta(\sigma) + O(1)$. By (5.6) it follows that

$$\log|F(\sigma+it)| \le (\gamma + 4\delta/\pi)\zeta(\sigma) + O(1)\ ,$$

which is the desired bound in this case.

It is noteworthy that in Case 2 we do not use the hypotheses that define this case. Thus the bound in this case holds universally. The point is that in the other cases one can do better.

Case 3. We first note that

$$\begin{aligned}\log\Big|\frac{F(\sigma)}{F(1+t)}\Big| &= \gamma\log\zeta(\sigma) - \Re\sum_{n=2}^{\infty}\frac{\Lambda(n)f(n)}{(\log n)n^{1+t}}\\ &\ge \gamma\log\zeta(\sigma) - \sum_{n=2}^{\infty}\frac{\Lambda(n)}{(\log n)n^{1+t}} = \gamma\log\zeta(\sigma) - \log\zeta(1+t)\,.\end{aligned}$$

By (5.1) and (5.3) it follows that if $u \ge -1$, then

$$\begin{aligned}&\log\Big|\frac{F(s)}{F(1+t+it)}\Big| - \log\Big|\frac{F(\sigma)}{F(1+t)}\Big|\\ &\quad\le (uX(u)+Y(u))\log\frac{\zeta(\sigma)}{\zeta(1+t)} - (u+1)(\gamma\log\zeta(\sigma) - \log\zeta(1+t))\\ &\quad= u\big((X(u)-\gamma)\log\zeta(\sigma) + (1-X(u))\log\zeta(1+t)\big)\\ &\qquad + (Y(u)-\gamma)\log\zeta(\sigma) + (1-Y(u))\log\zeta(1+t)\,.\end{aligned}$$

We appeal to (5.5), and note—as we did in Case 1—that the multipliers of the $O(1)$ error term are uniformly bounded. Thus the above is

$$= \big(u(\delta X(u) - \gamma - \delta + 1) + \delta Y(u) - \gamma - \delta + 1\big) \log \zeta(\sigma) + O(1) \,.$$

We choose u so that $\delta X(u) - \gamma - \delta + 1 = 0$, and appeal to (5.6) to see that

$$\log |F(\sigma + it)| \le (\delta Y(u) - \delta + 1) \log \zeta(\sigma) + O(1) \,.$$

This gives the stated bound, so our proof is complete.

6. Proof of Theorem 1.4

For $0 \le t \le \sigma - 1$ we have $F(\sigma) \asymp F(\sigma + it)$, so the order of magnitude in this interval is the same as at its upper endpoint, $|F(\sigma + i(\sigma - 1))|$. This corresponds to $\delta = 0$ in (1.11). Thus in the sequel we may suppose that (1.11) holds with $0 \le \delta \le 1$. By Theorem 1.3 we may take

$$\nu(\gamma) = \max_{0 \le \delta \le 1} \mu(\gamma, \delta) + 1 - \delta \,. \tag{6.1}$$

In Case 2 of Theorem 1.3 we have $\nu(\gamma, \delta) \ge \gamma + 1 + (4/\pi - 1)\delta$, which is an increasing function of δ. Thus in Case 2 we may assume that δ is as large as possible. It is easy to check that if $-1 \le \gamma \le -2/\pi$ and $\delta = (1 + \gamma)/(1 - 2/\pi)$, then the bound given in Case 1 is the same as that given in Case 2. Similarly, if $-2/\pi \le \gamma \le 1$ and $\delta = (1 - \gamma)/(1 + 2/\pi)$, then the bound given in Case 2 is the same as that given in Case 3. Hence if $-1 \le \gamma \le -2/\pi$, we may assume that we are in Case 1, and if $-2/\pi \le \gamma \le 1$, we may assume that we are in Case 3.

Suppose that we are in Case 1. Then we choose u so that

$$X(u) = (\gamma + 1)/\delta - 1, \tag{6.2}$$

and the expression (6.1) is $\max_\delta \delta Y(u)$. As δ increases, $X(u)$ decreases, and hence u decreases. Since the values of δ and of u are in one-to-one correspondence, we may take u to be the independent variable. As δ runs from $(1 + \gamma)/(1 - 2/\pi)$ up to 1, $X(u)$ runs from $-2/\pi$ down to γ, and hence u runs from -1 down to the u_γ for which $X(u_\gamma) = \gamma$. From (6.2) we find that $\delta = (\gamma + 1)/(X(u) + 1)$. Hence

$$\delta Y(u) = (\gamma + 1) \frac{Y(u)}{X(u) + 1} \,, \tag{6.3}$$

and our object is to determine the largest size of this quantity. The quantity $Y/(X + 1)$ is the slope of the chord from $(-1, 0)$ to (X, Y), and the curve $(X(u), Y(u))$ is concave downwards, so the slope is a decreasing function of u. Hence its largest value occurs when u is smallest, which is to say when $X(u) = \gamma$. But then the quantity is $Y(u)$, so we have shown that we may take $\nu(\gamma) = Y(u)$ when $-1 \le \gamma \le -2/\pi$.

Suppose we are in Case 3. Then we choose u so that

$$X(u) = (\gamma - 1 + \delta)/\delta\,, \tag{6.4}$$

and the quantity (6.1) is $\max_\delta \delta Y(u)+2-2\delta$. As δ increases from $(1-\gamma)/(1+2/\pi)$ to 1, $X(u)$ increases from $-2/\pi$ to γ, and u increases from -1 to the u_γ for which $X(u_\gamma) = \gamma$. Since the values of δ and u are in one-to-one correspondence, we may take u to be the independent variable. From (6.4) we see that $\delta = (1-\gamma)/(1-X(u))$. Hence

$$\delta Y(u) + 2 - 2\delta = 2 - (1-\gamma)\frac{Y(u)-2}{X(u)-1}\,, \tag{6.5}$$

and we need to determine the maximum of this quantity for u in the interval $[-1, u_\gamma]$. The quotient $(Y(u)-2)/(X(u)-1)$ is the slope of the line passing through the two points $(X(u), Y(u))$ and $(1, 2)$. This is at first decreasing (which is to say that the expression (6.5) is increasing), until u reaches the point u_0 at which the tangent line $ux + y = uX(u) + Y(u)$ passes through the point $(1, 2)$. For $u > u_0$, the slope $(Y(u)-2)/(X(u)-1)$ is increasing, which is to say that the expression is decreasing. Thus if $-1 \le u_\gamma \le u_0$, then the expression (6.5) is maximised by taking $u = u_\gamma$, in which case the quantity (6.5) is $Y(u_\gamma)$. However, if $u_0 \le u_\gamma \le 1$, then the quantity (6.5) is largest in the interval $[-1, u_\gamma]$ when $u = u_0$, and then its value is

$$2 - (1-\gamma)\frac{Y(u_0)-2}{X(u_0)-1}\,.$$

To summarize, we find that if $-1 \le \gamma \le X(u_0)$, then the point (γ, ν) lies on the curve $(X(u), Y(u))$, and the worst δ is $\delta = 1$. On the other hand, if $X(u_0) \le \gamma \le 1$, then the point (γ, ν) lies on the tangent line $u_0x + y = u_0X(u_0) + Y(u_0)$, and the worst δ decreases linearly from 1 to 0.

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