Functiones et Approximatio
XXXIX.1 (2008), 145–148

REDUCTION IN K-THEORY OF SOME INFINITE EXTENSIONS OF NUMBER FIELDS

Wojciech Gajda

Dedicated to Władysław Narkiewicz
on the occasion of his 70th birthday

Abstract: For the cyclotomic extension $F(\mu_\infty)=\bigcup_{m\geq 1} F(\mu_m)$ of a number field F, we prove that the reduction map $K_{2n+1}(F(\mu_\infty)) \longrightarrow K_{2n+1}(\kappa_{\tilde{v}})$, when restricted to nontorsion elements, is surjective. Here $\kappa_{\tilde{v}}$ denotes the residue field at a prime $\tilde{v}$ of $F(\mu_\infty)$.
Keywords: number field, cyclotomic extension, K-groups

1. Statement of the result

Let F be a number field and let $\mathcal{O}_F$ denote its ring of algebraic integers. Fix an algebraic closure $\bar{F}$ of F. We will denote by $F(\mu_\infty)$ the subfield of $\bar{F}$, which is obtained by adding to F all roots of unity. Thus $F(\mu_\infty)=\bigcup_{m\geq 1} F(\mu_m)$, where $F(\mu_m)$ is the smallest extension of F ($\subset \bar{F}$) containing all m-th roots of unity. We put $\mathcal{O}_{F(\mu_\infty)}=\bigcup_{m\geq 1} \mathcal{O}_{F(\mu_m)}$. We fix a prime ideal $\tilde{v}$ of $\mathcal{O}_{F(\mu_\infty)}$ and denote by v the prime ideal of $\mathcal{O}_F$ lying below $\tilde{v}$. We denote by $\kappa_{\tilde{v}}=\mathcal{O}_{F(\mu_\infty)}/\tilde{v}$ and by $\kappa_v=\mathcal{O}_F/v$ the respective residue fields. It is clear that $\kappa_{\tilde{v}}$ is the algebraic closure of the finite field κ_v. In the paper n denotes a fixed natural number.

We investigate the map

$$r_{\tilde{v}} : K_{2n+1}(\mathcal{O}_{F(\mu_\infty)}) \longrightarrow K_{2n+1}(\kappa_{\tilde{v}})$$

induced on the Quillen's K-theory by the arithmetic reduction at the prime $\tilde{v}$. Since by the result of Quillen [5], the even dimensional K-groups of finite fields vanish, it follows by the localization sequence in K-theory that the groups $K_{2n+1}(L)$ and $K_{2n+1}(\mathcal{O}_L)$, for a number field L (the groups $K_{2n+1}(\mathcal{O}_{F(\mu_\infty)})$ and $K_{2n+1}(F(\mu_\infty))$, respectively) are isomorphic. In this note we will identify the odd K-group of the ring of integers with the odd K-group of the field of fractions by using these isomorphisms.

1991 Mathematics Subject Classification: primary 19D55, secondary 19D50, 19F27, 20G.

The author was partially supported by the grant NN201173933.

Theorem 1.1. *For any element t of the group $K_{2n+1}(\kappa_{\tilde{v}})$ there exists a nontorsion element x in the group $K_{2n+1}(F(\mu_\infty))$, such that $r_{\tilde{v}}(x)=t$.*

We also prove (see Remark below) that the reduction map $r_{\tilde{v}}$ is surjective when restricted to the torsion part of $K_{2n+1}(F(\mu_\infty))$. Together with the Theorem it shows that there are two disjoint subsets in the group $K_{2n+1}(F(\mu_\infty))$, which are mapped onto $K_{2n+1}(\kappa_{\tilde{v}})$ by the reduction at $\tilde{v}$.

In the paper [3] the author investigated the reduction $K_{2n+1}(\mathbb{Z}) \to K_{2n+1}(\mathbb{F}_p)$, on the odd dimensional K-theory of rational integers. One of the results of [3] gives the density of primes p for which the reduction is nontrivial. The density was computed by using special elements in K-groups of $\mathbb{Z}$ and the arithmetic of certain Kummer extensions of cyclotomic fields. The method applied in the present note is different. At a critical point in the proof of the Theorem we use Borel's computations of ranks of the K-groups of number rings and a result of Harris and Segal (cf. [4, Corollary 3.2]).

2. Proof of the Theorem

Observe that

$$K_{2n+1}(\kappa_{\tilde{v}}) = \lim_\kappa K_{2n+1}(\kappa)$$

where the direct limit is taken over all finite fields κ contained in $\kappa_{\tilde{v}}$. For each κ as above we choose a prime w of the number field L contained in $F(\mu_\infty)$ for which the residue field κ_w is κ. Note that maps of the direct system are inclusions of finite cyclic groups cf. [5]. In order to prove the Theorem it is enough to show that for any w and for every $t \in K_{2n+1}(\kappa_w)$ there exists a nontorsion $x \in K_{2n+1}(F(\mu_\infty))$ such that $r_{\tilde{v}}(x)$ equals the value at t of the inclusion $K_{2n+1}(\kappa_w) \to K_{2n+1}(\kappa_{\tilde{v}})$. Let us put $q_w := \#\kappa_w$. Consider the cyclic group $K_{2n+1}(\kappa_w) = \mathbb{Z}/m_w$, where $m_w = q_w^{n+1}-1$ by the theorem of Quillen [5]. Let

$$\mathbb{Z}/m_w = \bigoplus_{i=1}^{s} \mathbb{Z}/l_i^{k_i}$$

be the decomposition of the group into its Sylow primary subgroups. For every $1 \le i \le s$ we pick a generator t_i' of the l_i-primary part $\mathbb{Z}/l_i^{k_i}$ of $K_{2n+1}(\kappa_w)$. Next we choose the finite field extension L'/F with the following two properties:

1. $F(\mu_{m_w}) \subset L' \subset F(\mu_\infty)$
2. the rank of the group $K_{2n+1}(L')$ is bigger than s.

By the well-known results of Borel [2] and Quillen [6], for any number field L, the rank of $K_{2n+1}(L)$ is not smaller than the number of complex places of L. The number of complex places of $F(\mu_m)$ is an increasing and unbounded function of m. This shows that the field L' with properties (1) and (2) exists, since clearly the natural map:

$$K_{2n+1}(F(\mu_{m_w}))\otimes\mathbb{Q} \longrightarrow K_{2n+1}(L')\otimes\mathbb{Q}$$

is injective (cf. [1, Theorem 2, p.68]). Let w' be the prime of $\mathcal{O}_{L'}$ which lies below $\tilde{v}$. Consider the commutative diagram:

$$\begin{array}{ccc} K_{2n+1}(F(\mu_\infty)) & \xrightarrow{r_{\tilde{v}}} & K_{2n+1}(\kappa_{\tilde{v}}) \\ {\scriptstyle\alpha}\uparrow & & \uparrow \\ K_{2n+1}(L') & \xrightarrow{r_{w'}} & K_{2n+1}(\kappa_{w'}) \\ \uparrow & & {\scriptstyle\beta}\uparrow \\ K_{2n+1}(L) & \xrightarrow{r_w} & K_{2n+1}(\kappa_w) \end{array} \tag{2.1}$$

with the reduction maps as the horizontal arrows. The vertical arrows on the right hand side of the diagram (2.1) are embeddings of the direct system of K-groups of finite fields. Since $F(\mu_{m_w}) \subset L'$, it follows by [4], Corollary 3.2, p.27, that for every $1 \leq i \leq s$ there exists an element $t_i \in K_{2n+1}(L')$ of order $l_i^{k_i}$ such that $r'_w(t_i) = \beta(t'_i)$. Let us choose elements $y_1, y_2, \dots, y_s$ of $K_{2n+1}(L')$ in such a way that their images in the rational K-group $K_{2n+1}(L')$ are linearly independent. It is possible because of the property (2) of the field L'. Clearly, $y_1, y_2, \dots, y_s$ are nontorsion. Put $y'_i := t_i + y_i$, for every $1 \leq i \leq s$. Let π_i denote the projection of the cyclic group $K_{2n+1}(\kappa_{w'})$ onto its l_i-primary summand. It is clear that either $\pi_i(r_{w'}(y_i))$ or $\pi_i(r_{w'}(y'_i))$ is of order $l_i^{k_i}$. Without loss of generality, we can assume that $\pi_i(r_{w'}(y'_i))$ is of order $l_i^{k_i}$, for every $1 \leq i \leq s$. We choose integers $a_1, a_2, \dots, a_s$ such that, for every $1 \leq i \leq s$, the element

$$y''_i = a_i \frac{m_w}{l_i^{k_i}} y'_i$$

meets the condition:

$$r_{w'}(y''_i) = \beta(t_i).$$

It follows that the element $r_{w'}(\sum_{i=1}^s y''_i)$ generates the cyclic group $\beta(K_{2n+1}(\kappa_w))$. By the choice of the elements $y_1, y_2, \dots, y_s$, we know that $\sum_{i=1}^s y''_i$ is nontorsion. Since the map $\alpha \otimes \mathbb{Q} : K_{2n+1}(L') \otimes \mathbb{Q} \longrightarrow K_{2n+1}(F(\mu_\infty)) \otimes \mathbb{Q}$ is injective (cf. [1, Corollary 1, p.70]), it follows that the element $\alpha(\sum_{i=1}^s y''_i)$ is nontorsion in the group $K_{2n+1}(F(\mu_\infty))$. To finish the proof it is enough to put

$$x := \alpha(\sum_{i=1}^{s} y''_i)$$

and apply the commutativity of the diagram (2.1). ■

Corollary 2.1. *Let L be an algebraic extension of F which contains $F(\mu_\infty)$. For any prime $\bar{v}$ of L the reduction map $r_{\bar{v}} : K_{2n+1}(L) \longrightarrow K_{2n+1}(\kappa_{\bar{v}})$ is surjective, when restricted to the subset of nontorsion elements.*

Proof. The claim of the corollary follows by the Theorem, because $K_{2n+1}(\kappa_{\bar{v}}) = K_{2n+1}(\kappa_{\tilde{v}})$ and the map $K_{2n+1}(F(\mu_\infty)) \longrightarrow K_{2n+1}(L)$ is injective on nontorsion elements by [1] *loc. cit.* ■

Remark. Applying [4, Corollary 3.2], to the torsion part of $K_{2n+1}(F(\mu_\infty))$ in the same way as it was done in the proof of the Theorem, one can see that the restriction of the reduction $K_{2n+1}(F(\mu_\infty)) \longrightarrow K_{2n+1}(\kappa_{\tilde{v}})$ to the torsion, is surjective.

Acknowledgements. The author gratefully acknowledges financial support of the Alexander von Humboldt foundation during a visit to the Institut für Experimentalle Mathematik at the Universität Duisburg-Essen in summer 2006. He thanks the Max-Planck-Institut für Mathematik in Bonn for hospitality and financial support during a visit in the first half of 2007. It is a pleasure to thank Grzegorz Banaszak for stimulating discussions.

References

[1] D. Arlettaz, G. Banaszak, *On the non-torsion elements in the algebraic K-theory of rings of integers*, J. reine angew. Math. **461** (1995), 63–79.
[2] A. Borel, *Stable real cohomology of arithmetic groups*, Ann. École Normal Sup. **7** (1974), 235–272.
[3] W. Gajda, *On cyclotomic numbers and the reduction map for the K-theory of the integers*, K-theory, **23** (2001), 323–343.
[4] B. Harris, G. Segal, *K_i of rings of algebraic integers*, Annals of Math. **101** (1975), 20–33.
[5] D. Quillen, *On the cohomology and K-theory of the general linear group over a finite field*, Annals of Math. **96** (1972), 552–586.
[6] D. Quillen, *Finite generation of the groups K_i of rings of algebraic integers*, Lecture Notes in Math. **341** (1973), publ Springer Verlag, 179–198.

Address: Department of Mathematics, Adam Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland
E-mail: gajda@amu.edu.pl
Received: 17 September 2007; **revised:** 7 January 2008