Functiones et Approximatio
XXXIX.1 (2008), 113–131

THE ADDITIVE UNIT STRUCTURE OF PURE QUARTIC COMPLEX FIELDS

Alan Filipin, Robert Tichy, Volker Ziegler

Dedicated to Prof. Władysław Narkiewicz
on the occasion of his 70th birthday

Abstract: All complex, pure quartic fields with maximal orders generated by their units are determined. Furthermore, a quantitative version of the unit sum number problem is considered.
Keywords: Additive structure of units, unit sum number problem

1. Introduction

In the 1960's Jacobson [7] observed that the rings of integers of the number fields $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{5})$ have the property, that each algebraic integer can be written as the sum of distinct units. Some years later Śliwa [10] proved that these are the only quadratic fields with this property. Moreover, Belcher [2], [3] investigated cubic and quartic number fields.

A similar question is, to ask which number fields admit representations of integers as sums of units, or more generally, which rings are additively generated by their units. The first result in this direction was found by Zelinsky [13] who investigated endomorphism rings of vector spaces. Investigations of this kind led Goldsmith et. al. [6] to the following definition.

Definition 1.1. *Let R be a ring (with identity). An element r is called k-good if $r = e_1 + \cdots + e_k$, with units $e_1, \ldots, e_k \in R^*$. If every element of R is k-good we call also the ring k-good.*

The unit sum number $u(R)$ is defined as $\min\{k : R \text{ is } k-\text{good}\}$. If the minimum does not exist but the units generate R additively we set $u(R) = \omega$. If the units do not generate R we set $u(R) = \infty$.

In the case of algebraic integers Ashrafi and Vàmos [1] showed that quadratic, cubic and certain cyclotomic fields have no finite unit sum number. Later this result was succeeded by Jarden and Narkiewicz [8] who proved that no ring of

1991 Mathematics Subject Classification: primary: 11R16, secondary: 11R27, 11N37.
The authors are supported by the Austrian Science Fund FWF (project numbers: S9611 and S9603).

algebraic integers has finite unit sum number. However, the question remains which rings of algebraic integers are generated by their units. This question has been solved independently for quadratic fields by Belcher [2] and Ashrafi and Vàmos [1] and for cubic fields by Tichy and Ziegler [11].

Another question which arises (see for instance [8, Problem C] or [14, Section 2]) is, how many integers with bounded norm do exist so that they can be written as a sum of exactly m units. The simple case of imaginary quadratic fields has been discussed by Ziegler [14].

In this paper we investigate the additive unit structure of pure quartic complex fields $\mathbb{Q}(\sqrt[4]{D})$, where $0 > D \in \mathbb{Z}$ is not -4. Note that without loss of generality we may assume $D = -ab^2c^3$, with

1. $a \neq -1$, b and c, square-free and pairwise relatively prime;
2. b and c positive;
3. $|a| \geq c$ if a is odd;
4. c is odd;
5. $D \neq -4$.

We prove the following theorem:

Theorem 1.1. *Let $0 > D \in \mathbb{Z}$ and assume D fulfills the restrictions made above. Let $K = \mathbb{Q}(\sqrt[4]{D})$ and let $\mathfrak{O}_D$ be the maximal order of K. Define the following sets:*

$$\begin{aligned}
S_1 &:= \{ -(2(2n+1)^2 \pm 4)^2 + 16 \,:\, n \in \mathbb{Z},\ 2n+1 \textit{ and } (2n+1)^2 \pm 4 \textit{ square-free},\\
&\qquad -(2(2n+1)^2 \pm 4)^2 + 16 < 0\};\\
S_2 &:= \{ -1, -3, -7, -36, -100, -135, -129735\};\\
S_3 &:= \{ -4\left((2n+1)^2 \pm 4\right)^2 \,:\, n \in \mathbb{Z},\ 2\left((2n+1)^2 \pm 4\right) \textit{ square-free}\};\\
S_4 &:= \{ -(2n^2 \pm 1)^2 \,:\, n \in \mathbb{Z},\ 2n^2 \pm 1 \textit{ square-free}\};\\
S_5 &:= \{ -3b_n^2 \,:\, b_{n+3} = 15b_{n+2} - 15b_{n+1} + b_n,\ n \geq 0,\ b_0 = 1,\ b_1 = 8,\ b_2 = 105,\\
&\qquad 3 \nmid b_n,\ b_n \textit{ square-free}\};\\
S_6 &:= \{ -3b_n^2 \,:\, b_{n+3} = 15b_{n+2} - 15b_{n+1} + b_n,\ n \geq 0,\ b_0 = 7,\ b_1 = 104,\ b_2 = 1455,\\
&\qquad 3 \nmid b_n,\ b_n \textit{ square-free}\}.
\end{aligned}$$

Moreover, let $S := S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6$. Then $\mathfrak{O}_D$ is generated by its units if and only if $D \in S$. In particular, there exists a unit $\epsilon \in \mathfrak{O}_D^*$ such that

- *$\{1, \epsilon, \epsilon^2, \epsilon^3\}$ is a basis of $\mathfrak{O}_K$ if $D \in S_1 \cup S_2$;*
- *$\{1, \sqrt{-1}, \epsilon, \epsilon\sqrt{-1}\}$ is a basis of $\mathfrak{O}_K$ if $D \in S_3 \cup S_4 \cup \{-36, -100\}$;*
- *$\{1, \rho, \epsilon, \epsilon\rho\}$ is a basis of $\mathfrak{O}_K$ if $D \in S_5 \cup S_6 \cup \{-36\}$;*

where $\rho = \frac{-1+\sqrt{-3}}{2}$.

Note that the proof of Theorem 1.1 is constructive, i.e. for each case we compute the unit ϵ whose existence is claimed in the theorem.

The paper is organized as follows. In the next section we discuss properties of the ring of integers of $K = \mathbb{Q}(\sqrt[4]{D})$. In order to prove Theorem 1.1 completely we have to consider several cases. First, we study the case $D \neq -b^2, -3b^2$ in section 3. Then we investigate the special cases $D = -3b^2$ (section 4) and $D = -b^2$ (section 5). In the last section we consider a quantitative version of the unit sum number problem for fields of the form $\mathbb{Q}(\sqrt[4]{-b^2})$.

2. Rings of integers

In this section we state a result of Funakura [5], who computed the integral bases of pure quartic number fields. In the real case this result has been established by Ljunggren [9] but without proof.

Throughout the rest of the paper we assume the restrictions on D made in Theorem 1.1. Furthermore we put $\alpha = \sqrt{D}/bc$, $\beta = \sqrt[4]{D}$ and $\gamma = \sqrt[4]{D^3}/bc^2$. Obviously, $\{1, \alpha, \beta, \gamma\}$ is a $\mathbb{Q}$-basis for $K = \mathbb{Q}(\sqrt[4]{D})$. With this notation Funakura [5] proved the following result:

Theorem 2.1 (Funakura). *The numbers $1, \lambda, \mu$ and ν given in Table 1 form an integral basis of $\mathfrak{O}_D$.*

Table 1: Integral basis for $\mathfrak{O}_D$

D	1	λ	μ	ν
$1 \mod 8$	1	$\frac{1+\alpha}{2}$	β	$\frac{-ab+\alpha+b\beta+\gamma}{4}$
$2 \mod 4$ $3 \mod 4$	1	α	β	γ
$4 \mod 16$ $5 \mod 8$	1	$\frac{1+\alpha}{2}$	β	$\frac{\beta+\gamma}{2}$
$12 \mod 32$	1	α	$\frac{1+\alpha+\beta}{2}$	$\frac{\beta+\gamma}{2}$
$28 \mod 32$	1	α	$\frac{1+\alpha+\beta}{2}$	$\frac{4\alpha+b\beta+2\gamma}{8}$

Note that Table 1 contains all cases since by assumption $8 \nmid D$. Furthermore, we remark that $\mathfrak{O}_D \subseteq \frac{1}{4}\mathbb{Z}[\alpha, \beta, \gamma]$. Indeed, the only not obvious case is $d \equiv 28 \mod 32$, but in this case $4|D$ and therefore $2|b$, hence $\nu \in \frac{1}{4}\mathbb{Z}[\alpha, \beta, \gamma]$.

Next, let us consider the roots of unity in the field $K = \mathbb{Q}(\sqrt[4]{D})$.

Proposition 2.1. *Let $K = \mathbb{Q}(\sqrt[4]{D})$ and assume $D < 0$ fulfills the assumptions of Theorem 1.1. Then*

- $\zeta_4 \in K$ *if and only if* $D = -b^2$*;*
- $\zeta_6 \in K$ *if and only if* $D = -3b^2$ *or* $D = -36$*;*
- $\zeta_8 \in K$ *if and only if* $D = -1$*;*
- $\zeta_{12} \in K$ *if and only if* $D = -36$.

Proof. The assertions of the proposition are standard facts which can be verified easily. ■

Because of Proposition 2.1 we will consider the cases $D = -b^2$ and $D = -3b^2$ separately. We also know $\mathbb{Q}(\sqrt[4]{D})$ is not Galois except for the case $D = -b^2$. In this case we have $\mathbb{Q}(\sqrt[4]{D}) = \mathbb{Q}(\sqrt{-1}, \sqrt{2b})$ and we use the following corollary of Theorem 2.1 (cf. [5]):

Corollary 2.1 (Funakura). *Let $l \geq 2$ be a square free integer. An integral basis of $\mathbb{Q}(\sqrt{-1}, \sqrt{l})$ is given by*

$$\begin{array}{llll} 1,\ \sqrt{-1}, & \frac{1+\sqrt{l}}{2}, & \frac{\sqrt{-1}+\sqrt{-l}}{2}, & \textit{if } l \equiv 1 \bmod 4; \\ 1,\ \sqrt{-1}, & \frac{\sqrt{l}+\sqrt{-l}}{2}, & \frac{\sqrt{l}-\sqrt{-l}}{2}, & \textit{if } l \equiv 2 \bmod 4; \\ 1,\ \sqrt{-1}, & \frac{1+\sqrt{-l}}{2}, & \frac{\sqrt{-1}-\sqrt{l}}{2}, & \textit{if } l \equiv 3 \bmod 4. \end{array}$$

Now we establish the main result of this section:

Proposition 2.2. *Let $D < 0$ and assume $\mathfrak{O}_D$ is generated by its units. Then there exists a unit $\epsilon \in \mathfrak{O}_D^*$ such that*

- *The basis $\{1, \epsilon, \epsilon^2, \epsilon^3\}$ generates $\mathfrak{O}_D$ if $D \neq -3b^2, -b^2$;*
- *The basis $\{1, \zeta_6, \epsilon, \epsilon\zeta_6\}$ generates $\mathfrak{O}_D$ if $D = -3b^2$ or $D = -36$;*
- *The basis $\{1, \zeta_4, \epsilon, \epsilon\zeta_4\}$ generates $\mathfrak{O}_D$ if $D = -b^2$.*

Moreover, in each case the bases with ϵ replaced by ϵ^{-1} also generate $\mathfrak{O}_D$.

In the proof of Proposition 2.2 we will need the following lemma.

Lemma 2.1. *Let $\zeta \in \mathfrak{O}_D$ be an n-th root of unity, with n maximal. Then the ring $\mathfrak{O}_D$ is generated by its units if and only if it is generated by $\{1, \zeta, \zeta^2, \ldots, \zeta^{n-1}, \epsilon, \zeta\epsilon, \ldots, \zeta^{n-1}\epsilon^3\}$, where ϵ is the fundamental unit of $\mathfrak{O}_D$.*

Proof of Lemma 2.1. By Dirichlet's unit theorem we may assume that $\mathfrak{O}_D$ is generated by

$$\{\zeta^{k_1}\epsilon^{l_1}, \zeta^{k_2}\epsilon^{l_2}, \zeta^{k_3}\epsilon^{l_3}, \zeta^{k_4}\epsilon^{l_4}\}.$$

Since ϵ is an algebraic integer of degree at most four, $\zeta^k\epsilon^l$ can be written as a linear combination of $\zeta^k, \zeta^k\epsilon, \zeta^k\epsilon^2, \zeta^k\epsilon^3$, which already proves the lemma. ∎

Proof of Proposition 2.2. In the first case our assertion follows from Proposition 2.1 and Lemma 2.1.

In the case of $D = -3b^2$ we have $D \equiv 4 \bmod 16$ or $D \equiv 5 \bmod 8$. Note that the case $8|D$ is excluded. Therefore $1, \lambda = (1+\alpha)/2, \mu = \beta$ and $\nu = (\beta+\gamma)/2$ form an integral basis (see Theorem 2.1). By Dirichlet's unit theorem and Lemma 2.1 we know if $\mathfrak{O}_D$ is generated by its units, then there exists a unit ϵ and a subset $\mathcal{B} \subset \{1, \zeta_6, \epsilon, \epsilon\zeta_6, \epsilon^2, \epsilon^2\zeta_6, \epsilon^3, \epsilon^3\zeta_6\}$ with four elements that is a basis for $\mathfrak{O}_D$. Assume $\epsilon = x + y\lambda + z\mu + w\nu$. Note that a subset $\mathcal{B} = \{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$ is a basis of $\mathfrak{O}_D$, if and only if the matrix M corresponding to the base change from $\mathcal{B}$ to $\{1, \lambda, \mu, \nu\}$ has determinant ± 1. Therefore for each possible subset $\mathcal{B}$ we compute the corresponding determinant, using for computations the equality $\zeta_6 = \frac{1+\alpha}{2} = \lambda$. Each determinant contains the factor $w^2 + wz + z^2$ and the determinant corresponding to $\{1, \zeta_6, \epsilon, \epsilon\zeta_6\}$ is exactly $w^2 + wz + z^2$. This implies, whenever $\mathcal{B}$ is a basis then $w^2+wz+z^2=\pm 1$, hence $\{1,\zeta_6,\epsilon,\epsilon\zeta_6\}$ is also a basis. Thus the second case is settled.

The case of $D=-b^2$ is similar. We have to consider the base change from the base proposed by Corollary 2.1 to a subset $\mathcal{B}$ as above, where we have to use ζ_4 instead of ζ_6. Moreover, we have to handle three subcases which are indicated by Corollary 2.1. But in each subcase the determinant corresponding to the subset $\{1,\zeta_4,\epsilon,\epsilon\zeta_4\}$ is exactly z^2+w^2 and this is a factor for each other subset in each subcase. Note that we write $\epsilon=x+y\zeta_4+z\lambda+w\mu$, where λ respectively μ is the third (respectively fourth) basis element in Corollary 2.1. ∎

If $D\in\{-1,-36\}$ then $K=\mathbb{Q}(\sqrt[4]{D})$ is cyclotomic, so the assertion of Theorem 1.1 is evident, as every integer of K is a $\mathbb{Z}$-linear combination of roots of unity. Henceforth, we shall not consider these cases in the sequel.

3. The case $D\neq -b^2,-3b^2$

If $D\neq -b^2,-3b^2$ then Proposition 2.1 shows that $\mathfrak{O}_D$ is generated by its units if and only if there exists a unit ϵ such that $\mathfrak{O}_D=\mathbb{Z}[\epsilon]$. Since $\mathfrak{O}_D\subset\frac{1}{4}\mathbb{Z}[\alpha,\beta,\gamma]$ we write

$$\epsilon=\frac{x+y\alpha+z\beta+w\gamma}{4}=\xi+\eta\lambda+\zeta\mu+\tau\nu,$$

with $\xi,\eta,\zeta,\tau,x,y,z,w\in\mathbb{Z}$. As mentioned above $\mathfrak{O}_D=\mathbb{Z}[\epsilon]$ holds if and only if the matrix transforming $\mathcal{A}=\{1,\lambda,\mu,\nu\}$ to $\mathcal{C}=\{1,\epsilon,\epsilon^2,\epsilon^3\}$ has determinant ±1. In order to simplify computations we perform first the base change from $\mathcal{A}$ to $\mathcal{B}=\{1,\alpha,\beta,\gamma\}$ (represented by the matrix M_1) and then the base change from $\mathcal{B}$ to $\mathcal{C}$ (represented by the matrix M_2). Then we have to show that $\det M_1\det M_2=\pm1$.

We distinguish between three cases: $D\equiv 1 \mod 8$ or $D\equiv 28 \mod 32$ (case I), $D\equiv 5\mod 8$, $D\equiv 4\mod 16$ or $D\equiv 12\mod 32$ (case II) and $D\equiv 2,3\mod 4$ (case III). Since the treatment of these three cases is nearly the same we give the details only for the first case. Because of Theorem 2.1 we write

$$\begin{aligned}
\epsilon&=\frac{x+y\alpha+z\beta+w\gamma}{4} &&\text{case I;}\\
\epsilon&=\frac{x+y\alpha+z\beta+w\gamma}{2} &&\text{case II;}\\
\epsilon&=x+y\alpha+z\beta+w\gamma &&\text{case III;}
\end{aligned}$$

with $x,y,z,w\in\mathbb{Z}$. Moreover, we deduce that $\det M_2=8,4,1$ according to the cases I, II, and III.

Now let us compute $\det M_1$. We have

$$\det M_1=\frac{\det M'}{m}=\frac{1}{m}\det\begin{pmatrix}1&0&0&0\\x&y&z&w\\x_2&y_2&z_2&w_2\\x_3&y_3&z_3&w_3\end{pmatrix},\tag{3.1}$$

where $m = 4096$ in case I. Moreover, x_2, y_2, z_2, w_2 (respectively x_3, y_3, z_3, w_3) are the coefficients in ϵ^2 (respectively ϵ^3) of $1, \alpha, \beta, \gamma$. In particular, we have

$$\begin{aligned}
x_2 &= x^2 - ac(y^2 + 2bzw), \quad & y_2 &= 2xy + b(cz^2 - aw^2),\\
z_2 &= 2(xz - ayw), & w_2 &= 2(cyz + xw),
\end{aligned}$$
$$\begin{aligned}
x_3 &= x^3 + 3a^2bcyw^2 - 3ac(xy^2 + bz(cyz + 2xw)),\\
y_3 &= 3x(bcz^2 + xy) - a(3bxw^2 + c(y^3 + 6byzw)),\\
z_3 &= 3x^2z + a^2bw^3 - 3a(2xyw + cz(y^2 + bzw)),\\
w_3 &= 3x^2w - 3cy(ayw - 2xz) + bcz(cz^2 - 3aw^2).
\end{aligned}$$

Now we compute

$$\det M' = -(cz^2 + aw^2)\left(8abcy^2zw + b^2(cz^2 - aw^2)^2 - 4acy^4\right),$$

and therefore we obtain

$$(cz^2 + aw^2)(8abcy^2zw + b^2(cz^2 - aw^2)^2 - 4acy^4) + 512e_1 = 0, \tag{3.2}$$

where $e_1 = \pm 1$ according to the sign of $\det M$. Moreover, we deduce $cz^2 + aw^2 | 512$. Since $c, a > 0$, we have $|z|, |w| \leq \sqrt{512} < 23$.

Next we observe that ϵ is a unit, hence $\mathrm{N}_{K/\mathbb{Q}}(\epsilon) = 1$. Note that the norm in K is always positive since K has an imaginary quadratic subfield. Thus we have $\mathrm{N}_{K/\mathbb{Q}}(4\epsilon) = 256$. In terms of x, y, z and w we obtain

$$\begin{aligned}
x^4 + a^3b^2cw^4 + a^2c\left(cy^4 - 4bcy^2zw + 2b(bcz^2 + 2xy)w^2\right)&\\
+ac\left(b^2c^2z^4 + 2y^2x^2 + 4bxz(xw - cyz)\right) - 256 = 0.&
\end{aligned} \tag{3.3}$$

Let $\epsilon^{-1} = \frac{x_{-1} + y_{-1}\alpha + z_{-1}\beta + w_{-1}\gamma}{4}$. Then we find

$$z_{-1} = \frac{acy^2z - x^2z - abcz^2w - 2axyw - a^2bw^3}{16}, \tag{3.4}$$

$$w_{-1} = \frac{2cxyz - bc^2z^3 + acy^2w - x^2w - abczw^2}{16}. \tag{3.5}$$

Note that with $1, \epsilon, \epsilon^2$ and ϵ^3 also $1, \epsilon^{-1}, \epsilon^{-2}$ and ϵ^{-3} generate $\mathfrak{O}_D$. Therefore equation (3.2) is satisfied with x, y, z and w replaced by x_{-1}, y_{-1}, z_{-1} and w_{-1}. In particular we have $cz_{-1}^2 + aw_{-1}^2 | 512$, hence $|z_{-1}|, |w_{-1}| \leq \sqrt{512} < 23$.

We have thus reduced our problem to the Diophantine system consisting of the equations (3.2), (3.3), (3.4) and (3.5) with unknowns a, b, c, x, y. Note that for the other quantities there are only finitely many possibilities. In order to solve this system we distinguish between three further cases, i.e. $w = 0$, $z = 0$ and $zw \neq 0$. As noted above we restrict our considerations to case I. Cases II and III can be settled by similar arguments; for instance, the value of m in formula (3.1) has to be changed to 64 in case II and to 1 in case III.

3.1. The case $w = 0$

Under the assumption $w = 0$ we obtain $cz^2|512$. Since c is odd and positive we obtain $c = 1$. Moreover $z \in \{\pm 16, \pm 8, \pm 4, \pm 2, \pm 1\}$. From (3.2), (3.3), (3.4) and (3.5) we obtain the system

$$\begin{aligned} 0 &= z^2(b^2z^4 - 4ay^4) + 512e_1, \\ 0 &= a^2y^4 + x^4 + a(b^2z^4 - 4bz^2xy + 2x^2y^2) - 256, \\ 0 &= ay^2z - x^2z - 16z_{-1}, \\ 0 &= 2xyz - bz^3 - 16w_{-1}. \end{aligned} \tag{3.6}$$

Since 512 is not a square we may assume $y \neq 0$. Now let us assume $w_{-1} = 0$. Solving the third and the last equation of (3.6) for a (respectively b) under the assumption $w_{-1} = 0$ we obtain (note that we assume $y \neq 0$)

$$a = \frac{x^2z + 16z_{-1}}{y^2z}, \qquad b = \frac{2xy}{z^2}.$$

We insert these results into the first and the second equation. After some manipulations we have

$$z_{-1}^2 - z^2 = 0, \qquad 8e_1 - y^2zz_{-1} = 0.$$

The first equation implies $z = \pm z_{-1}$, and therefore the second equation turns into $8 = \pm z^2y^2$, yielding a contradiction. Thus we conclude $w_{-1} \neq 0$.

Assuming $w = 0$, $yw_{-1} \neq 0$ and solving the third and fourth equation of (3.6) for a (respectively b) we obtain

$$a = \frac{x^2z + 16z_{-1}}{y^2z}, \qquad b = 2\frac{xyz - 8w_{-1}}{z^3}.$$

Furthermore we derive the equations

$$\begin{aligned} 0 &= 4w_{-1}^2 - yz(z_{-1}y + xw_{-1}) + 8e_1, \\ 0 &= -y^2z^3 + y^2zz_{-1}^2 + 16z_{-1}w_{-1}^2 + x^2zw_{-1}^2. \end{aligned}$$

Solving the first equation for x we get

$$x = \frac{8e_1 - y^2zz_{-1} + 4w_{-1}^2}{yzw_{-1}},$$

and the equation

$$(2z^2z_{-1}^2 - z^4)y^4 - (16e_1zz_{-1} - 8zz_{-1}w_{-1}^2)y^2 + 16(4 + 4e_1w_{-1}^2 + 16w_{-1}^4) = 0.$$

As mentioned above we have only finitely many possibilities for e_1, z, z_{-1} and w_{-1}. Solving the equation for y in all cases and inserting into the expression for x, a and b, we obtain all solutions to system (3.6). After discarding all solutions that yield non integral values for x, y, a or b or negative values for a or b we obtain the solutions $a = c = 1$, $b = 24$ and $a = 4$, $b = 3$, $c = 1$. Since b (respectively a) are not square-free there is no admissible solution.

3.2. The case $z = 0$

This case runs along similar lines as case $w = 0$. Note that we have to consider the cases $y = 0$ and $z_{-1} = 0$ separately again. Moreover, we conclude $a = 1$ or $a = 2$ only, and we have to distinguish between these cases. In particular, the following Diophantine system must be solved:

$$\begin{aligned} 0 &= 512e_1 + aw^2(a^2b^2w^4 - 4acy^4), \\ 256 &= 2acy^2x^2 + x^4 + a^3b^2cw^4 + a^2c(cy^4 + 4bxyw^2), \\ 0 &= 16z_{-1} + 2axyw + a^2bw^3, \\ 0 &= 16w_{-1} - acy^2w + x^2w. \end{aligned} \tag{3.7}$$

Let us consider the case $y = 0$ and thus $b^2a^3w^6 = \pm 512$. This is possible only for $a = 2$, $b = 1$ and $w = \pm 2$. If we insert this in the second equation, we find $128c - 256 = -x^4$. Since c is odd we derive that $2^7 = 128$ is the exact power of 2 that divides x^4. Since 128 is not a fourth power we obtain a contradiction, showing that $y \neq 0$.

Now we investigate the case $z_{-1} = 0$. Solving the last two equations of (3.7) we derive

$$b = -\frac{2xy}{aw^2}, \qquad c = \frac{x^2w + 16w_{-1}}{ay^2w}.$$

Moreover, the first two equations can be written as

$$0 = 8e_1 - ay^2ww_{-1}, \qquad 0 = w_{-1}^2 - w^2.$$

The only integral solutions to this system are $a = 2$, $w = \pm w_{-1} = \pm 1$, $y = \pm 2$ and $a = 2$, $w = \pm w_{-1} = \pm 2$, $y = \pm 1$. If we insert this result in the expressions for b and c we obtain for the first solution $b = \pm 2x$ and $c = \frac{\pm x^2 \pm 16}{8}$ and for the second $b = \pm \frac{x}{4}$ and $c = \frac{\pm x^2 \pm 16w_{-1}}{2}$. In the first case we get a contradiction because a and b are both even. In the second case $4|x$ since b is an integer, but therefore $8|c$, which contradicts the fact that c is square-free. Thus we have shown $z_{-1} \neq 0$ and we may assume from now on $yz_{-1} \neq 0$.

Analogously to the case $w = 0$ and $yw_{-1} \neq 0$ we get the only positive solutions $a = 1$, $b = 24$, $c = 1$ and $a = 1$, $b = 3$, $c = 4$. These solutions do not yield solutions for our problem since in any case either b or c has a square factor.

3.3. The case $zw \neq 0$

By equation (3.2) we know $cz^2 + aw^2 | 512$. Since $a, c > 0$ we have only finitely many possibilities for the quadruple (a, c, z, w). Since with $1, \epsilon, \epsilon^2, \epsilon^3$ also $1, \epsilon^{-1}, \epsilon^{-2}, \epsilon^{-3}$ generate $\mathfrak{O}_D$ we also have $cz_{-1}^2 + aw_{-1}^2 | 512$. Therefore we obtain only 14288 possibilities for the sextuple $(a, c, z, w, z_{-1}, w_{-1})$. Inserting each possibility for a, c, z, w, z_{-1} and w_{-1} in the system obtained from (3.2), (3.3), (3.4) and (3.5) yields systems in the three variables b, x and y. Solving all these systems, e.g. by

Table 2: Integral solutions

a	b	c	x	y	z	w	z_{-1}	w_{-1}
7	1	1	∓ 3	± 1	± 1	± 1	± 1	± 1
7	1	1	± 3	∓ 1	± 1	± 1	± 1	± 1
1	6	1	0	± 2	± 1	∓ 1	∓ 1	± 1
1	6	1	0	∓ 2	± 1	∓ 1	∓ 1	± 1
1	10	1	0	± 2	± 1	± 1	± 1	± 1
1	10	1	0	∓ 2	± 1	± 1	± 1	± 1
5	1	3	± 1	∓ 1	± 1	± 1	± 1	± 1
5	1	3	∓ 1	± 1	± 1	± 1	± 1	± 1
5	31	3	± 31	∓ 1	∓ 1	± 1	∓ 1	± 1
5	31	3	∓ 31	± 1	∓ 1	± 1	∓ 1	± 1

Table 3: The unit ϵ written as linear combination of $1, \lambda, \mu$ and ν. (mixed signs)

a	b	c	D	ϵ
3	1	1	-3	$\pm\lambda \pm \mu$
7	1	1	-7	$\pm(1+\nu) \pm (3-\lambda+\nu)$
1	6	1	-36	$\pm(1-2\mu+\nu) \pm (1+\lambda-2\mu+\nu)$
1	10	1	-100	$\pm(1-2\mu+\nu) \pm (1+\lambda-2\mu+\nu)$
5	1	3	-135	$\pm(1+\nu) \pm (2-2\lambda+\nu)$
5	31	3	-129735	$\pm(26-8\mu+\nu) \pm (42-\lambda-8\mu+\nu)$

computing their Groebner bases, we find that 12760 systems have a solution but only for 20 systems we have integral solutions (see Table 2) such that $b > 0$ and b square-free.

We are left to check that these solutions yield integral units ϵ. Indeed by Theorem 2.1 we can compute the corresponding units ϵ as a linear combination of $1, \lambda, \mu$ and ν (see Table 3). Table 3 shows that these solutions yield indeed integral units. Note that in Table 3 we have included the solutions for $a = 3$ and $b = c = 1$. Since we did not find any solutions that are not listed in Theorem 1.1, the theorem is proved for $D \neq -b^2, -3b^2$.

Finally we note that simple changes give the result also in cases II and III.

4. The case $D = -3b^2$

In this case we have $D \equiv 5 \mod 8$ (respectively $D \equiv 4 \mod 16$). We use the basis $1, \lambda, \mu, \nu$ given in Theorem 2.1. By Proposition 2.2 we know that we have to find a unit ϵ such that $\{1, \zeta_6, \epsilon, \epsilon\zeta_6\}$ is a basis for $\mathfrak{O}_D$. Let us assume $\epsilon = x + y\lambda + z\mu + w\nu$ is such a unit. By the proof of Proposition 2.2 we know that $z^2 + wz + w^2 = \pm 1$. Since with $\{1, \zeta_6, \epsilon, \epsilon\zeta_6\}$ also $\{1, \zeta_6, \epsilon^{-1}, \epsilon^{-1}\zeta_6\}$ is a basis we

Table 4: Solutions for x

z	w	z_{-1}	w_{-1}	Groebner Basis
0	±1	0	±1	$(x-y)^2-3y^2+1, b+2xy+y^2$
0	∓1	0	±1	$(x-y)^2-3y^2-1, b+2xy+y^2$
±1	∓1	0	±1	$(x+2y)^2-3y^2-2, b+2xy+y^2-1$
∓1	±1	0	±1	$(x+2y)^2-3y^2+2, b+2xy+y^2+1$
±1	0	0	±1	$2\left((x+y/2)^2-3(y/2)^2\right)+1, 2b-2xy-y^2+1$
∓1	0	0	±1	$2\left((x+y/2)^2-3(y/2)^2\right)-1, 2b-2xy-y^2-1$
0	±1	±1	∓1	$(x-y)^2-3y^2-2, b+2xy+y^2+1$
0	∓1	±1	∓1	$(x-y)^2-3y^2+2, b+2xy+y^2-1$
±1	∓1	±1	∓1	$(x+2y)^2-3y^2+1, b+2xy+y^2$
∓1	±1	±1	∓1	$(x+2y)^2-3y^2-1, b+2xy+y^2$
±1	0	±1	∓1	$2\left((x+y/2)^2-3(y/2)^2\right)+1, 2b-2xy-y^2-1$
∓1	0	±1	∓1	$2\left((x+y/2)^2-3(y/2)^2\right)-1, 2b-2xy-y^2+1$
0	±1	±1	0	$(x-y)^2-3y^2-1, b+2xy+y^2+1$
0	∓1	±1	0	$(x-y)^2-3y^2+1, b+2xy+y^2-1$
±1	∓1	±1	0	$(x+2y)^2-3y^2-1, b+2xy+y^2-1$
∓1	±1	±1	0	$(x+2y)^2-3y^2+1, b+2xy+y^2+1$
±1	0	±1	0	$(x+y/2)^2-3(y/2)^2+1, 2b-2xy-y^2$
∓1	0	±1	0	$(x+y/2)^2-3(y/2)^2-1, 2b-2xy-y^2$

have $z_{-1}^2+w_{-1}z_{-1}+w_{-1}^2=\pm1$. Computing the norm of ϵ and its inverse we find

$$\begin{aligned}1 &= (x^2+xy+y^2)^2+3b(wx-yz)(w(x+2y)+(2x+y)z)\\ &\quad +3b^2(w^2+wz+z^2)^2,\\ z_{-1} &= -bw^3-2wxy-wy^2-x^2z-2xyz+bz^3,\\ w_{-1} &= -bw^3-wx^2+wy^2-3bw^2z+2xyz+y^2z-3bwz^2-2bz^3.\end{aligned} \tag{4.1}$$

We remind the reader that the Diophantine equation

$$X^2+XY+Y^2=\pm1$$

has the six solutions $(X,Y)=(\pm1,0),(0,\pm1),(\pm1,\mp1)$. Inserting all possible values for e,z,w,z_{-1} and w_{-1} we get 72 systems of equations in x,y and b. We compute the Groebner bases for these systems (see Table 4).

The first element of the Groebner basis yields a Pellian equation of the form $X^2-3Y^2=\pm1,\pm2,\pm4$. These equations have integral solutions if and only if the right side is $1,-2$ or 4. Note that $2+\sqrt{3}$ is a fundamental unit of $\mathbb{Q}(\sqrt{3})$ and is also a fundamental solution to $X^2-3Y^2=1$. Hence for all systems the solutions

Table 5: Initial values for the recurring sequences $(x_n), (y_n)$ and (b_n)

z	w	z_{-1}	w_{-1}	(x_0, x_1, x_2, x_3)	(y_0, y_1, y_2, y_3)	$(b_0, b_1, b_2, b_3, b_4)$
± 1	0	± 1	∓ 1	$\pm(1,1,3,11)$ $\pm(1,1,3,11)$	$\mp(0,1,4,15)$ $\pm(-1,0,1,4)$	$-(-1,0,7,104,1455)$ $(0,1,8,105,1456)$
± 1	0	0	± 1	$\pm(1,1,3,11)$ $\pm(1,1,3,11)$	$\mp(0,1,4,15)$ $\pm(-1,0,1,4)$	$-(0,1,8,105,1456)$ $(-1,0,7,104,1455)$
± 1	0	∓ 1	0	$\pm(0,2,8,30)$ $\pm(0,2,8,30)$	$\pm(1,1,3,11)$ $\mp(1,3,11,41)$	$(0,4,56,780,10864)$ $-(0,4,56,780,10864)$
± 1	∓ 1	± 1	0	$\pm(0,1,4,15)$ $\pm(0,1,4,15)$	$\mp(-1,0,1,4)$ $\mp(1,4,15,56)$	$(-1,0,7,104,1455)$ $(1,8,105,1456,20273)$
± 1	∓ 1	0	± 1	$\pm(1,1,3,11)$ $\pm(1,1,3,11)$	$\mp(1,3,11,41)$ $\mp(3,1,1,3)$	$(0,4,56,780,10864)$ $-(-4,0,4,56,780)$
± 1	∓ 1	∓ 1	± 1	$\pm(0,1,4,15)$ $\pm(0,1,4,15)$	$\mp(-1,0,1,4)$ $\mp(1,4,15,56)$	$(0,1,8,105,1456)$ $(0,7,104,1455,20272)$
0	± 1	± 1	0	$\pm(0,1,4,15)$ $\pm(0,1,4,15)$	$\pm(1,3,11,41)$ $\mp(1,1,3,11)$	$-(1,8,105,1456,20273)$ $(-1,0,7,104,1455)$
0	± 1	0	∓ 1	$\pm(0,1,4,15)$ $\pm(0,1,4,15)$	$\pm(1,3,11,41)$ $\mp(1,1,3,11)$	$-(0,7,104,1455,20272)$ $(0,1,8,105,1456)$
0	± 1	∓ 1	± 1	$\pm(1,1,3,11)$ $\pm(1,1,3,11)$	$\mp(-2,0,2,8)$ $\pm(0,2,8,30)$	$(-4,0,4,56,780)$ $-(0,4,56,780,10864)$

can be written in the form $(x, y, b) = (x_n, y_n, b_n)$ with

$$\begin{aligned} x_n &= a_x(2+\sqrt{3})^n + b_x(2-\sqrt{3})^n, \\ y_n &= a_y(2+\sqrt{3})^n + b_y(2-\sqrt{3})^n, \\ b_n &= a_b(7+4\sqrt{3})^n + b_b(7-4\sqrt{3})^n + c_b, \end{aligned}$$

where $a_x, a_y, a_b, b_x, b_y, b_b$ and c_b are suitable constants and $0 \leq n \in \mathbb{Z}$. Therefore we conclude that x_n and y_n fulfill the recursions $x_{n+2} = 4x_{n+1} - x_n$ and $y_{n+2} = 4y_{n+1} - y_n$, while b_n fulfills $b_{n+3} = 15b_{n+2} - 15b_{n+1} + b_n$ for $n \geq 0$. Therefore we are left to compute all possible initial values. All possibilities are listed in Table 5.

The signs in Table 5 cannot be arbitrarily mixed. That means, if we choose the upper/lower sign for z, then we must also choose the upper/lower sign for w, z_{-1} and w_{-1}. Similarly we must choose for $x_0, x_1, x_2, x_3, y_0, y_1, y_2$ and y_3 the same case (upper or lower sign). But we can choose independently the signs of z and x_0. From Table 4 above we obtain all units ϵ such that $\{1, \zeta_6, \epsilon, \epsilon\zeta_6\}$ is a basis of $\mathfrak{O}_D$. These units are listed in Table 6. Note that we only list the initial values of the sequences.

5. The case $D = -b^2$

In this case we have $K = \mathbb{Q}(\sqrt[4]{D}) = \mathbb{Q}(\sqrt{-1}, \sqrt{2b})$ as mentioned above. Let $l = b/2$ if b is even and $l = 2b$ otherwise, then l is a square-free integer and $\mathbb{Q}(\sqrt[4]{D}) = \mathbb{Q}(\sqrt{-1}, \sqrt{l})$. We use the notation of Corollary 2.1, and therefore we have to distinguish the three different cases $l = 4k+1, l = 4k+2$ and $l = 4k+3$, respectively. In order to prove Theorem 1.1 we have to find all units ϵ such that $\{1, \zeta_4, \epsilon, \epsilon\zeta_4\}$ generates $\mathfrak{O}_D$ (see Proposition 2.2). Let us write $\epsilon = x + y\lambda + z\mu + w\nu$, where $\lambda = \zeta_4$ and μ and ν are the third and fourth basis elements of the bases described in Corollary 2.1. We know by the proof of Proposition 2.2 that $z^2 + w^2 = 1$, i.e. $(z, w) = (\pm 1, 0), (0, \pm 1)$. Since with $\{1, \zeta_4, \epsilon, \epsilon\zeta_4\}$ also $\{1, \zeta_4, \epsilon^{-1}, \epsilon^{-1}\zeta_4\}$ generates $\mathfrak{O}_D$ we find $z_{-1}^2 + w_{-1}^2 = 1$, where $\epsilon^{-1} = x_{-1} + y_{-1}\lambda + z_{-1}\mu + w_{-1}\nu$. Similar as in the sections above we deduce

$$\begin{aligned}
\mathrm{N}_{K/\mathbb{Q}}(\epsilon) = {} & k^2(z^2+w^2)^2 + (y^2+x^2)((z+x)^2 + (y+w)^2) \\
& - 2k\big(z^3x + zxw(4y+w) + z^2(x^2 + y(w-y)) \\
& + w^2(-x^2 + y(y+w))\big),
\end{aligned}$$

if $l \equiv 1 \mod 4$;

$$\begin{aligned}
\mathrm{N}_{K/\mathbb{Q}}(\epsilon) = {} & (1+2k)^2z^4 + (y^2+x^2)^2 + (4+8k)z(y^2-x^2)w \\
& + (4+8k)yxw^2 + (1+2k)^2w^4 + (2+4k)z^2((1+2k)w^2 - 2yx),
\end{aligned}$$

if $l \equiv 2 \mod 4$; and finally if $l \equiv 3 \mod 4$

$$\begin{aligned}
\mathrm{N}_{K/\mathbb{Q}}(\epsilon) = {} & (1+k)^2z^4 + (2+2k)z^3x + (y^2+x^2)^2 + 2y(y^2+x^2)w \\
& + ((3+2k)y^2 - (1+2k)x^2)w^2 + (2+2k)yw^3 + (1+k)^2w^4 \\
& + 2zx(y^2 + x^2 + 4(1+k)yw + (1+k)w^2) \\
& + z^2(-(1+2k)y^2 + (3+2k)x^2 + (2+2n)yw + 2(1+k)^2w^2).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
z_{-1} &= kz^3 + zy^2 - z^2x - zx^2 - 2yxw + kzw^2 - xw^2, \\
w_{-1} &= z^2y + 2zyx - kz^2w + y^2w - x^2w + yw^2 - kw^3,
\end{aligned}$$

Table 6: Units generating $\mathfrak{O}_D$ (mixed signs)

(b_0, b_1, b_2)	(x_0, x_1)	(y_0, y_1)	ϵ
$(1, 8, 105)$	$(0, 1)$	$(1, 3)$	$\pm(x_n + y_n\lambda) \pm \mu$
	$(1, 4)$	$(0, -1)$	$\pm(x_n + y_n\lambda) \pm (\mu - \nu)$
	$(1, 3)$	$(-1, -4)$	$\pm(x_n + y_n\lambda) \pm \nu$
$(7, 104, 1455)$	$(1, 4)$	$(3, 11)$	$\pm(x_n + y_n\lambda) \pm \mu$
	$(4, 15)$	$(-1, -4)$	$\pm(x_n + y_n\lambda) \pm (\mu - \nu)$
	$(3, 11)$	$(-4, -15)$	$\pm(x_n + y_n\lambda) \pm \nu$

if $l \equiv 1 \mod 4$;

$$z_{-1} = z(y^2 - x^2) + (1+2k)wz^2 + 2xyw + (1+2k)w^3,$$
$$w_{-1} = (1+2k)z^3 - 2xyz + w(y^2 - x^2) + (1+2k)zw^2,$$

if $l \equiv 2 \mod 4$; and finally if $l \equiv 3 \mod 4$

$$z_{-1} = -(1+k)z^3 + zy^2 - z^2x - zx^2 - 2xyw - zw^2 - kzw^2 - xw^2,$$
$$w_{-1} = z^2y + 2xyz + (1+k)wz^2 + w(y^2 - x^2) + yw^2 + (1+k)w^3.$$

Inserting for z, w, z_{-1} and w_{-1} all possible values we derive several systems of equations. In the following we treat the subcases separately.

5.1. The case $l = 4k + 1$

As already mentioned above we have either $z = 0$ or $w = 0$. First, let us assume $z = 0$. Then our system is of the form

$$0 = z_{-1} + 2yxw + xw^2, \qquad 0 = y^2w - x^2w + yw^2 - kw^3 - w_{-1},$$
$$1 = k^2w^4 - 2kw^2(y^2 + yw - x^2) + (y^2 + x^2)(x^2 + (y+w)^2).$$

Solving the second equation for k we find

$$k = \frac{y^2w - x^2w + yw^2 - w_{-1}}{w^3} = y^2 - x^2 + wy - ww_{-1}$$

and

$$0 = z_{-1} + xw(2y + w), \qquad 1 = x^2(2y + w)^2 + w_{-1}^2.$$

Since $2y + w \neq 0$ we have either $x^2 = (2y+w)^2 = 1$ or $x = 0$. Let us treat the first case. We see $(2y+w)^2 = 1$ implies either $y = 0$ or $y = -w = \pm 1$. Therefore we have $k = 0, -1$. The case $k = -1$ yields a negative b and $k = 0$ yields $D = -4$. In the case of $x = 0$ we instantly have $z_{-1} = 0$ (first equation), $w_{-1}^2 = 1$ and $k = y^2 + wy - ww_{-1}$. Let us write $y = \pm|n|$ and $w = \pm 1$ (with the same sign), then we find $k = |n|^2 + |n| \pm 1$ where the "+" holds if $w = -w_{-1}$ and "−" holds if $w = w_{-1}$. Note that for $y = \pm(|n| - 1)$ and $w = \mp 1$ we also get $k = |n|^2 + |n| \pm 1$, with the same last sign as above if we choose a suitable sign for w_{-1}. Note that the sign of w_{-1} can be chosen independently from the signs of y and w. Furthermore, the polynomial $|n|^2 - |n| \pm 1$ takes the same values as the polynomial $|n|^2 + |n| \pm 1$. Hence we have found all possible solutions in this subcase (see Table 7).

The case $w = 0$ is similar. This time our system turns into

$$0 = z^2y + 2xyz - w_{-1}, \qquad 0 = kz^3 - z_{-1} + zy^2 - z^2x - zx^2,$$
$$1 = k^2z^4 + (y^2 + x^2)(y^2 + (z+x)^2) - 2k(z^3x + z^2(-y^2 + x^2)).$$

If we perform the change of variables $x \to -y$, $y \to x$, $z \to -w$, $w \to z$, $z_{-1} \to w_{-1}$, $w_{-1} \to -z_{-1}$, multiply the second equation by -1 and exchange the first and the second equation, then we obtain the same system as in the case above, i.e. we are done also in this case.

5.2. The case $l = 4k+3$

This case is similar to the case above. Let us assume $z = 0$ first. Then we obtain the system

$$\begin{aligned} 0 &= z_{-1} + 2xyw + xw^2, \\ 0 &= y^2w - x^2w + yw^2 + w^3 + kw^3 - w_{-1}, \\ 1 &= (y^2+x^2)^2 + 2yw(y^2+x^2) \\ &\quad + ((3+2k)y^2 - (1+2k)x^2)w^2 + 2(1+k)yw^3 + (1+k)^2w^4. \end{aligned} \tag{5.1}$$

We treat the case $x = 0$ separately. In this case the system turns into

$$\begin{aligned} 0 &= z_{-1}, \qquad 0 = y^2 + wy + 1 + k - ww_{-1}, \\ 1 &= y^4 + 2y^3w + (3+2k)y^2 + 2(1+k)yw + (1+k)^2, \end{aligned}$$

if we take into account $w = \pm 1$. The first equation yields $z_{-1} = 0$, hence $w_{-1} = \pm 1$. Solving the second equation for k yields $k = -y^2 - wy - 1 + ww_{-1}$. Since k has to be ≥ 0 we find $k = 0$ and $0 = -y^2 - wy$ (note $-y^2 - wy - 2 < 0$). Therefore we have $y = 0$ or $y = -w$. This yields the solution $b = 6$ (see Table 7).

Now we assume $x \neq 0$ and we solve the first equation of (5.1) for y. We get $y = -\frac{x+z_{-1}}{2wx} = -w/2 \pm 1/2x, -w/2$ if we take into account that $w^2 = 1$. Since y is an integer and $w = \pm 1$ we conclude $x = \pm 1$. Then we have either $y = 0$ or $y = -w$. Inserting this into the second equation yields $k = \pm 1$ if $y = -w$ and $k = 1, 3$ if $y = 0$. But for each case we obtain a contradiction to the third equation. Hence we have no solution in the case $x \neq 0$.

The case $w = 0$ yields by the change of variables $x \to -y$, $y \to x$, $z \to -w$, $w \to z$, $z_{-1} \to w_{-1}$, $w_{-1} \to -z_{-1}$ the same system as in the case $z = 0$. Therefore we are done also in this case.

5.3. The case $l = 4k+2$

In this case we obtain the system

$$\begin{aligned} 0 &= -z_{-1} + z(y^2 - x^2) + (2k+1)w(z^2+w^2) + 2xyw, \\ 0 &= -w_{-1} + (2k+1)z(z^2+w^2) - 2xyz + w(y^2 - x^2), \\ 1 &= (1+2k)^2(z^4+w^4) + (x^2+y^2)^2 + 4(1+2n)zw(y^2-x^2) \\ &\quad + 4(1+2k)xyw^2 + 2(1+2k)z^2((1+2k)w^2 - 2xy). \end{aligned}$$

By the same reasons as above we have to treat only the case $z = 0$. Therefore the system turns into

$$\begin{aligned} 0 &= z_{-1} + (2k+1)w + 2xyw, \qquad 0 = w_{-1} + w(y^2 - x^2), \\ 1 &= (1+2k)^2 + (x^2+y^2)^2 + 4(1+2k)xy, \end{aligned}$$

if we take into account $w^2 = 1$. The second equation can be written in the form $y^2 - x^2 = \pm 1, 0$. In the case of $y^2 - x^2 = \pm 1$ we have $(x, y) = \pm(1, 0), \pm(0, 1)$ and inserting this into the third equation we find $(1+2k)^2 = 0$ yielding a contradiction, since k is integral. In the case $y^2 - x^2 = 0$ the third equation turns into

$$(1+2k)^2 \pm 4(1+2k)x^2 + 4x^4 = ((1+2k) \pm 2x^2)^2 = 1.$$

Solving for k we obtain $k = \pm x^2 - 1$ or $k = \pm x^2$. Since k has to be non-negative we find $k = x^2 - 1$ or $k = x^2$, i.e. in the equation above the $-$ sign holds and therefore $y = -x$. Let us put $x = n$, with $n \in \mathbb{Z}$. Then we have parameterized all solutions in this case (see Table 7).

5.4. List of units

We have listed all $D = -b^2$ for which $\mathfrak{O}_D$ is generated by its units. According to Proposition 2.2 in each case there exists a unit ϵ such that $\{1, \zeta_4, \epsilon, \epsilon\zeta_4\}$ is a basis of $\mathfrak{O}_D$. These units have been explicitly computed in the paragraphs above. Table 7 lists all possible units.

Table 7: Units generating $\mathfrak{O}_D$, with $D = -b^2$, and $t = 0, 1, 2, 3$.

k	l	b	ϵ
$\lvert n\rvert^2 + \lvert n\rvert \pm 1$	1	$2((2\lvert n\rvert+1)^2 \pm 4)$	$\zeta_4^t\left(\lvert n\rvert + \frac{1+\sqrt{(2\lvert n\rvert+1)^2\pm 1}}{2}\right)$ $\zeta_4^t\left(\lvert n\rvert - 1 - \frac{1+\sqrt{(2\lvert n\rvert+1)^2\pm 1}}{2}\right)$
0	3	6	$\zeta_4^t\left(1 - \frac{1-\sqrt{-3}}{2}\right)$ $\zeta_4^t\left(\frac{1-\sqrt{-3}}{2}\right)$ $\zeta_4^t\left(1 \pm i\left(1 - \frac{1-\sqrt{-3}}{2}\right)\right)$ $\zeta_4^t\left(1 + \frac{i+\sqrt{3}}{2}\right)$
n^2	2	$(2n^2+1)$	$\zeta_4^t\left(n(1-\zeta_4) \pm \frac{\sqrt{4n^2+2}-\sqrt{-4n^2-2}}{2}\right)$
n^2-1	2	$(2n^2-1)$	$\zeta_4^t\left(n(1-\zeta_4) \pm \frac{\sqrt{4n^2-2}-\sqrt{-4n^2+2}}{2}\right)$

6. Quantitative results

In this last section we want to investigate a quantitative version of the unit sum number problem. In particular we want to know how many non-associated integers with norm $\leq x$ exist that admit a representation as sum of less than m units. For the simple case of imaginary quadratic fields Ziegler [14] found asymptotic formulas. To give a precise statement we need the following definition.

Definition 6.1. *We define the counting function $u_K(m;x)$ as the number of classes of associated integers α such that*

$$|\mathrm{N}_{K/\mathbb{Q}}(\alpha)| \le x, \qquad \alpha = \sum_{i=1}^{m} \epsilon_i, \qquad \epsilon_i \in \mathfrak{O}_K^*,$$

and no subsum vanishes.

Note that the counting function $u_K(m;x)$ is well defined since $|\mathrm{N}_{K/\mathbb{Q}}(\alpha)| = |\mathrm{N}_{K/\mathbb{Q}}(\beta)|$ if $\alpha \backsim \beta$.

In this section we find an asymptotic expansion for $u_K(m;x)$ in the case of $K = \mathbb{Q}(\sqrt[4]{-b^2})$ and m small. Note that in our case the norm is always positive. Before we state our results we investigate the unit group of $K = \mathbb{Q}(\sqrt[4]{-b^2})$.

Proposition 6.1. *Let $K = \mathbb{Q}(\sqrt[4]{-b^2})$, $\eta > 1$ the fundamental unit of the quadratic field $\mathbb{Q}(\sqrt{2b})$ and $|\epsilon| > 1$ the fundamental unit of K. Then either $\eta = \zeta\epsilon$ or $\eta = \zeta\epsilon^2$, where ζ is a suitable root of unity.*

Proof. Let us note that K is a so called CM-field, this is a totally complex field, which is the quadratic extension of a totally real field. For these fields we know that $Q := [U : WU^+] = 1, 2$ where U is the unit group of K, U^+ the unit group of the maximal real subfield of K and W the roots of unity of K (see [12, Theorem 4.12]). In the case $Q = 1$ obviously η is the fundamental unit. In the case of $Q = 2$, $\zeta\epsilon^2$ (with suitable root of unity ζ) generates the unit group of $\mathbb{Q}(\sqrt{2b})$, hence $\eta = \zeta\epsilon^2$. ∎

For the rest of the paper we use the notation $Q := [U : WU^+]$, where U, U^+ and W are defined as in the proof above. Now let us state the main theorem of this section.

Theorem 6.1. *Let $K = \mathbb{Q}(\sqrt[4]{-b^2})$, $b \neq 2$ square-free and $\eta > 1$ the fundamental unit of $\mathbb{Q}(\sqrt{2b})$. For $m \le \frac{\eta^{1/Q}}{2}$ we have the following asymptotics*

$$u_K(m;x) = \frac{1}{(m-1)!}\left(\frac{2Q\log x}{\log \eta}\right)^{m-1} + O\left((\log x)^{m-2}\right).$$

Note that the constants implied by the O-term depend only on m and η.

In order to prove Theorem 6.1, we need following inequalities:

Lemma 6.1. *Let $0 < x \le x_i \le X$ for $i = 1, \dots, n$ and write $C_1 = x_1 + \dots + x_n$ and $C_{-1} = \frac{1}{x_1} + \dots + \frac{1}{x_n}$. Then*

$$C_1 + xXC_{-1} \le n(x+X) \tag{6.1}$$

and

$$C_1C_{-1} \le \frac{n^2(x+X)^2}{4xX}, \tag{6.2}$$

respectively

Proof. Let us consider the inequality

$$\frac{(X - x_i)(x - x_i)}{x_i} = \frac{xX}{x_i} + x_i - (X + x) \leq 0.$$

If we sum up all these inequalities for $i = 1, \ldots, n$ we obtain (6.1). The second inequality follows from (6.1) and the arithmetic-geometric mean inequality. ■

Proof of Theorem 6.1. Let $\alpha = \eta_1 + \cdots + \eta_m$ be an algebraic integer with norm $\leq x$ that admits a representation as sum of m units η_i, $i = 1, \ldots, m$. Because we consider equivalence classes we may assume $\alpha = 1 + \eta_2 + \cdots + \eta_m$, where $\eta_i = \zeta_4^{l_i} \epsilon^{k_i}$, with $|\epsilon| > 1$ and ϵ the fundamental unit of K, $l_i = 0, 1, 2, 3$ and $0 = k_1 \leq \cdots \leq k_m$. Let σ be the automorphism induced by $\sqrt{2b} \mapsto -\sqrt{2b}$ and $\zeta_4 \mapsto \zeta_4$. Because $|\sigma(\epsilon)| = |\epsilon^{-1}|$, we have

$$\begin{aligned} \mathrm{N}_{K/\mathbb{Q}}(1 + \eta_2 + \ldots + \eta_m) &= |(1 + \eta_2 + \ldots + \eta_m)\sigma(1 + \eta_2 + \ldots + \eta_m)|^2 = \\ &|(1 + \zeta_4^{l_2}\epsilon^{k_2} + \ldots + \zeta_4^{l_m}\epsilon^{k_m})(1 + \zeta_4^{l_2'}\epsilon^{-k_2} + \ldots + \zeta_4^{l_m'}\epsilon^{-k_m})|^2, \end{aligned}$$

with $l_i' = 0, 1, 2, 3$.

The idea of the proof is as follows. First, we find an upper bound N_1 for the largest exponent, such that $k_m \leq N_1$ implies $\mathrm{N}_{K/\mathbb{Q}}(\alpha) \leq x$. Next, we find a lower bound N_2 for the largest exponent, such that $k_m > N_2$ implies $\mathrm{N}_{K/\mathbb{Q}}(\alpha) > x$. Therefore we conclude, if α has a representation as sum of m units, then $k_m \leq N_2$. By these bounds we deduce lower and upper bounds for $u_K(m; x)$. Note that every representation of an integer as sum of m units is unique, if $m \leq |\epsilon/2|$. This is easy to see if we interpret the representation as a digit expansion with basis ϵ and digit set $\{-m, -m+1, \ldots, m\}$.

Let r be the number of η_i's such that $k_i = 0$, and let s be the number of η_i's such that $k_i = k_m$. Then

$$\mathrm{N}_{K/\mathbb{Q}}(\alpha)^{1/2} \leq \left|(r + C_1 + s|\epsilon|^{k_m})(r + C_{-1} + s|\epsilon|^{-k_m})\right|,$$

where $C_1 = \sum_{i=r+1}^{m-s} |\epsilon|^{k_i}$, $C_{-1} = \sum_{i=r+1}^{m-s} |\epsilon|^{-k_i}$. Using Lemma 6.1 we get

$$\begin{aligned} \mathrm{N}_{K/\mathbb{Q}}(\alpha)^{\frac{1}{2}} &\leq r^2 + s^2 + \frac{r+s}{|\epsilon|}(m - r - s) + \frac{rs}{|\epsilon|} + \left(\frac{m - r - s}{2}\right)^2 \frac{(|\epsilon| + |\epsilon|^{k_m - 1})^2}{|\epsilon|^{k_m}} \\ &\quad + (r + s)(m - r - s)|\epsilon|^{k_m - 1} + rs|\epsilon|^{k_m} \\ &< 2rs|\epsilon|^{k_m} < m^2|\epsilon|^{k_m} < |\epsilon|^{k_m + 2}. \end{aligned}$$

Therefore $|\epsilon|^{2k_m + 4} \leq x$ implies $\mathrm{N}_{K/\mathbb{Q}}(\alpha) \leq x$ and we may choose $N_1 = \frac{\log x}{2 \log |\epsilon|} - 2$.

On the other hand we have

$$\begin{aligned}
\mathrm{N}_{K/\mathbb{Q}}(\alpha)^{1/2} &\geq \left(\frac{s|\epsilon|^{k_m}}{\sqrt{2}} - C_1 - r\right)\left(\frac{r}{\sqrt{2}} - C_{-1} - s|\epsilon|^{k_m}\right)\\
&\geq \left(\frac{s|\epsilon|^{k_m}}{\sqrt{2}} - (m-r-s)|\epsilon|^{k_m-1} - r\right)\left(\frac{r}{\sqrt{2}} - \frac{m-r-s}{|\epsilon|} - s|\epsilon|^{-k_m}\right)\\
&> \frac{\left(\frac{s|\epsilon|^{k_m}}{\sqrt{2}} - m|\epsilon|^{k_m-1}\right)\left(\frac{r|\epsilon|^{k_m}}{\sqrt{2}} - m|\epsilon|^{k_m-1}\right)}{|\epsilon|^{k_m}} \geq \left(\frac{\sqrt{2}-1}{2}\right)^2 |\epsilon|^{k_m}.
\end{aligned}$$

Therefore we take $N_2 = \frac{\log x}{2\log|\epsilon|} + \frac{2(\log 2 - \log(\sqrt{2}-1))}{\log|\epsilon|}$.

It remains to prove that there are $\frac{(4N)^{m-1}}{(m-1)!} + O(N^{m-2})$ algebraic integers α with $k_m \leq N$. We choose the exponents (k_i, l_i) of $\eta_i = \zeta_4^{l_i}\epsilon^{k_i}$ in the following way:

1. There is a number r with $1 \leq r \leq m$ such that $k_1 = \cdots = k_r = 0$. For these units we have $2r-1$ possibilities to choose admissible exponents $l_1, \ldots, l_r$ for ζ_4.
2. Assume that we have chosen $1 \leq j \leq m-r$ different pairs (k,l) for the remaining exponents. Then we have $\binom{m-r-1}{j-1}$ possibilities to choose for each unit η out of the remaining $m-r$ units a pair (k,l), such that each pair corresponds to at least one unit.
3. It remains to determine how many possibilities we have to choose j pairs (k,l) of admissible exponents.
   - For a fixed exponent k we can choose at most two different exponents l, i.e. we have to choose $\lceil j/2\rceil \leq n \leq j$ exponents k.
   - For the n distinct exponents k we have $\binom{N}{n}$ possibilities.
   - We have $\binom{j}{n}$ possibilities to choose which exponents k are attached to two different exponents l.
   - If there is only one exponent l attached to a fixed exponent k we have 4 possibilities to choose this exponent. If there are two exponents l attached to k we have 4 possibilities to choose these two exponents l.

All together we have

$$\begin{aligned}
\sum_{r=1}^{m}(2r-1)\sum_{j=1}^{m-r}\binom{m-r-1}{j-1}&\sum_{n=\lceil j/2\rceil}^{j}\binom{j}{n}\binom{N}{n}4^n\\
&= \frac{(4N)^{m-1}}{(m-1)!} + O(N^{m-2})
\end{aligned}$$

possibilities for α. ∎

Remark. The restriction $m \leq \frac{\eta^{1/Q}}{2}$ is essential, because otherwise the term ϵ^{k_m} may not be the dominant summand. In a forthcoming paper [4] we will show how to handle the case of m large.

Acknowledgements. The authors are grateful to the anonymous referee for suggesting several improvements in the presentation of the paper. We are also grateful to Władysław Narkiewicz for drawing our attention to problems concerning the additive unit structure in number fields.

References

[1] N. Ashrafi, P. Vámos, *On the unit sum number of some rings*, Q. J. Math. **56** (2005), no. 1, 1–12.
[2] P. Belcher, *Integers expressible as sums of distinct units*, Bull. Lond. Math. Soc. **6** (1974), 66–68.
[3] P. Belcher, *A test for integers being sums of distinct units applied to cubic fields*, J. Lond. Math. Soc., II. Ser. **12** (1976), 141–148.
[4] A. Filipin, R. Tichy, V. Ziegler, *On the quantitative unit sum number problem – An application of the subspace theorem*, to appear in Acta Arith.
[5] T. Funakura, *On integral bases of pure quartic fields*, Math. J. Okayama Univ. **26** (1984), 27–41.
[6] B. Goldsmith, S. Pabst, A. Scott, *Unit sum numbers of rings and modules*, Q. J. Math., Oxf. II. Ser. **49** (1998), no. 195, 331–344.
[7] B. Jacobson, *Sums of distinct divisors and sums of distinct units*, Proc. Am. Math. Soc. **15** (1964), 179–183.
[8] M. Jarden, W. Narkiewicz, *On sums of units* Monatsh. Math. **150** (2007), no. 4, 327–336.
[9] W. Ljunggren, *Eigenschaften der Einheiten reeller quadratischer und rein biquadratischer Zahlkörper mit Anwendung auf die Lösung einer Klasse unbestimmter Gleichungen vierten Grades*. Skr. Norske Vid.-Akad. **12** (1936), 1–73.
[10] J. Śliwa, *Sums of distinct units* Bull. Acad. Pol. Sci. **22** (1974), 11–13.
[11] R. Tichy, V. Ziegler, *Units generating the ring of integers of complex cubic fields*, Colloquium Mathematicum, **109**, (2007), 71–83.
[12] L. C. Washington, *Introduction to cyclotomic fields*, Graduate Texts in Mathematics, vol. 83, Springer-Verlag, New York, second edition, (1997).
[13] D. Zelinsky, *Every linear transformation is a sum of nonsingular ones*, Proc. Am. Math. Soc. **5** (1954), 627–630.
[14] V. Ziegler, *The unit sum number problem for rings of interest in number theory*, preprint available at http://www.finanz.math.tugraz.at/ ziegler/ Publications/ElUnits.pdf,.

Addresses: A. Filipin Graz University of Technology Institute of analysis and computational number theory, Steyrergasse 30, A-8010 Graz, Austria
R. Tichy Graz University of Technology Institute of analysis and computational number theory, Steyrergasse 30, A-8010 Graz, Austria
V. Ziegler Graz University of Technology Institute of analysis and computational number theory, Steyrergasse 30, A-8010 Graz, Austria
E-mail: filipin@finanz.math.tugraz.at, tichy@tugraz.at, ziegler@finanz.math.tugraz.at
Received: 30 May 2007; **revised:** 16 August 2007