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RUMELY'S LOCAL GLOBAL PRINCIPLE FOR WEAKLY P$\mathcal{S}$C FIELDS OVER HOLOMORPHY DOMAINS

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Dedicated to Professor Władyslaw Narkiewicz

Abstract: Let K be a global field, $\mathcal{V}$ an infinite proper subset of the set of all primes of K, and $\mathcal{S}$ a finite subset of $\mathcal{V}$. Denote the maximal Galois extension of K in which each $\mathfrak{p} \in \mathcal{S}$ totally splits by $K_{\mathrm{tot},\mathcal{S}}$. Let M be an algebraic extension of K. Let $\mathcal{V}_M$ (resp. $\mathcal{S}_M$) be the set of primes of M which lie over primes in $\mathcal{V}$ (resp. $\mathcal{S}$). For each $\mathfrak{q} \in \mathcal{V}_M$ let $\hat{\mathcal{O}}_{M,\mathfrak{q}} = \{x \in \hat{M}_\mathfrak{q} \mid |x|_\mathfrak{q} \le 1\}$, where $\hat{M}_\mathfrak{q}$ is a completion of M at $\mathfrak{q}$, and let $\mathcal{O}_{M,\mathcal{V}} = \{x \in M \mid |x|_\mathfrak{q} \le 1 \text{ for each } \mathfrak{q} \in \mathcal{V}_M\}$.

For $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_e) \in \mathrm{Gal}(K)^e$, let $K_s(\boldsymbol{\sigma}) = \{x \in K_s \mid \sigma_i(x) = x,\ i = 1, \dots, e\}$. Then, for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ (with respect to the Haar measure), the field $M = K_s(\boldsymbol{\sigma}) \cap K_{\mathrm{tot},\mathcal{S}}$ satisfies the following local global principle: Let $V \subseteq \mathbb{A}^n$ be an affine absolutely irreducible variety defined over M. Suppose that there exist $\mathbf{x}_\mathfrak{q} \in V(\hat{\mathcal{O}}_{M,\mathfrak{q}})$ for each $\mathfrak{q} \in \mathcal{V}_M \smallsetminus \mathcal{S}_M$ and $\mathbf{x}_\mathfrak{q} \in V_{\mathrm{simp}}(\hat{\mathcal{O}}_{M,\mathfrak{q}})$ for each $\mathfrak{q} \in \mathcal{S}_M$ such that $|x_{i,\mathfrak{q}}|_\mathfrak{q} < 1$, $i = 1, \dots, n$, for each archimedean prime $\mathfrak{q} \in \mathcal{V}_M$. Then $V(\mathcal{O}_{M,\mathcal{V}}) \neq \emptyset$.

Keywords: local global principle, weakly PSC fields, global fields, totally S-adic numbers

Introduction

Hilbert's tenth problem asked for the existence of an algorithm to solve **diophantine equations**, that is equations with coefficients in $\mathbb{Z}$ whose solutions are sought in $\mathbb{Z}$. The development of recursion theory since 1930 and works of Martin Davis, Hilary Putnam, and Julia Robinson finally led Juri Matijasevich in 1972 to a negative answer to that problem. This invoked Julia Robinson to ask whether Hilbert's tenth problem has a positive solution over the ring $\tilde{\mathbb{Z}}$ of all algebraic integers. Indeed, on page 367 of her joint paper [4] with Davis and Matijasevich she guessed that there should be one.

Using capacity theory, Rumely [24], [25] proved in 1987 a local global principle for $\tilde{\mathbb{Z}}$: If an absolutely irreducible affine variety V over $\tilde{\mathbb{Q}}$ has an integral point over every completion of $\tilde{\mathbb{Q}}$, then V has a point with coordinates in $\tilde{\mathbb{Z}}$. This led Rumely to an algorithm for solving diophantine problems over $\tilde{\mathbb{Z}}$.

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In 1988-89 Moret-Bailly [16], [17], [18] reproved Rumely's theorem with rationality conditions using methods of algebraic geometry.

Green, Pop, and Roquette [6] consider a global field K, a set $\mathcal{V}$ of primes of K which does not include all of the primes of K, and a finite subset $\mathcal{S}$ of $\mathcal{V}$. Each $\mathfrak{p} \in \mathcal{V}$ is an equivalence class of absolute values of K. Let $|\,|_\mathfrak{p}$ be an absolute value representing $\mathfrak{p}$. If $\mathfrak{p}$ is archimedean and complex, let $K_\mathfrak{p}$ be the algebraic closure $\tilde{K}$ of K. If $\mathfrak{p}$ is archimedean and real, let $K_\mathfrak{p}$ be a real closure of K at $\mathfrak{p}$. If $\mathfrak{p}$ is nonarchimedean, let $K_\mathfrak{p}$ be a Henselian closure of K at $\mathfrak{p}$. Now let $N = K_{\mathrm{tot},\mathcal{S}} = \bigcap_{\mathfrak{p}\in\mathcal{S}} \bigcap_{\tau\in\mathrm{Gal}(K)} K_\mathfrak{p}^\tau$ be the field of **totally $\mathcal{S}$-adic numbers**. It is the maximal Galois extension of K in which each $\mathfrak{p} \in \mathcal{S}$ totally decomposes. Here $\mathrm{Gal}(K) = \mathrm{Gal}(K_s/K)$ is the absolute Galois group of K. Consider the subset $\mathcal{O}_{N,\mathcal{V}}$ of N consisting of all $x \in N$ such that $|x|_\mathfrak{q} \le 1$ for each prime $\mathfrak{q}$ of N whose restriction to K lies in $\mathcal{V}$. The main result of [6] is a local-global principle for $\mathcal{O}_{N,\mathcal{V}}$: If an affine absolutely irreducible variety V defined over K has a $K_\mathfrak{p}$-rational point $\mathbf{x}_\mathfrak{p}$ with $|\mathbf{x}_\mathfrak{p}|_\mathfrak{p} \le 1$ for each $\mathfrak{p} \in \mathcal{V}$ such that $\mathbf{x}_\mathfrak{p}$ is simple if $\mathfrak{p} \in \mathcal{S}$ and $|\mathbf{x}_\mathfrak{p}|_\mathfrak{p} < 1$ if $\mathfrak{p}$ is archimedean, then V has a simple $\mathcal{O}_{N,\mathcal{V}}$-rational point. The language of proof of [6] is that of the theory of algebraic function fields of one variable.

The present work generalizes the methods of [6] and proves a local-global principle and an approximation theorem for diophantine equations of subrings of fields lying "deeper" than $K_{\mathrm{tot},\mathcal{S}}$. To explain the latter objects recall that $\mathrm{Gal}(K)$ is equipped with a Haar measure. For each $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ we denote the fixed field of $\sigma_1, \ldots, \sigma_e$ in $K_{\mathrm{tot},\mathcal{S}}$ by $M = K_{\mathrm{tot},\mathcal{S}}(\boldsymbol{\sigma})$. As in the preceeding paragraph, let $\mathcal{O}_{M,\mathcal{V}}$ be the set of all $x \in M$ with $|x|_\mathfrak{q} \le 1$ for each prime $\mathfrak{q}$ of M lying over some $\mathfrak{p} \in \mathcal{V}$. For a fixed integer $e \ge 0$ we prove that for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ the field $M = K_{\mathrm{tot},\mathcal{S}}(\boldsymbol{\sigma})$ satisfies a weak approximation theorem: Let V be an affine absolutely irreducible variety defined over K and let $\mathcal{T}$ be a finite subset of $\mathcal{V}$. For each $\mathfrak{p} \in \mathcal{S} \cap \mathcal{T}$ let $\mathbf{x}_\mathfrak{p}$ be a simple $K_\mathfrak{p}$-rational point of V and for each $\mathfrak{p} \in \mathcal{T} \smallsetminus \mathcal{S}$ let $\mathbf{x}_\mathfrak{p}$ be a $K_\mathfrak{p}$-rational point of V (which need not be simple). Finally let $\varepsilon > 0$. Then there exists $\mathbf{x} \in V(M)$ such that $|\mathbf{x} - \mathbf{x}_\mathfrak{p}|_\mathfrak{p} < \varepsilon$ for each $\mathfrak{p} \in \mathcal{T}$. The local-global principle for $\mathcal{O}_{M,\mathcal{V}}$ follows from the weak approximation theorem. Another corollary is that M is P$\mathcal{S}$C. This means that if V is an absolutely irreducible variety defined over M with a simple $K_\mathfrak{p}^\tau$-rational point for each $\mathfrak{p} \in \mathcal{S}$ and every $\tau \in \mathrm{Gal}(K)$, then V has an M-rational point.

The exact formulation of our results appears in Section 3, where we prove a "strong approximation theorem" from which all other results follow. The proof follows quite closely the proof of the analogous results in [12]. In that work we handled only the case where $\mathcal{V}$ contains no archimedean primes and where $K_{\mathrm{tot},\mathcal{S}}(\boldsymbol{\sigma})$ is replaced by its maximal purely inseparable extension $K_{\mathrm{tot},\mathcal{S}}(\boldsymbol{\sigma})_{\mathrm{ins}}$. The advantage of the latter field is that it is perfect. In our case we have to face the possibility that prime divisors of algebraic function fields over $K_{\mathrm{tot},\mathcal{S}}(\boldsymbol{\sigma})$ ramify when going over to the algebraic closure. This brings in extra technical complications into the proof.

The extra difficulty in the proof of our results over that of [6] is that we have to find $K_{\mathrm{tot},\mathcal{S}}(\boldsymbol{\sigma})$-rational points on varieties and not only $K_{\mathrm{tot},\mathcal{S}}$-rational points. In addition, $K_{\mathrm{tot},\mathcal{S}}(\boldsymbol{\sigma})$, unlike $K_{\mathrm{tot},\mathcal{S}}$, is not Galois over K. But note that $K_{\mathrm{tot},\mathcal{S}}(\boldsymbol{\sigma}) =$

$K_s(\boldsymbol{\sigma}) \cap K_{\mathrm{tot},\mathcal{S}}$, where $K_s(\boldsymbol{\sigma})$ is the fixed field in K_s of $\sigma_1, \ldots, \sigma_e$. What makes our proof work is Corollary 1.6 of [13] which says that for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ the field $K_s(\boldsymbol{\sigma})$ is "PAC over $\mathcal{O}_{K,\mathcal{V}}$". This implies that $K_{\mathrm{tot},\mathcal{S}}(\boldsymbol{\sigma})$ is "weakly-P$\mathcal{S}$C over $O_{K,\mathcal{V}}$". Both notions are explained in Section 1 along with the exact citation from [13].

One of the main ingredients of the proof of our result is the solvability of each "$(\mathcal{S},\mathcal{V})$-Skolem density problem for $K_{\mathrm{tot},\mathcal{S}}(\boldsymbol{\sigma})$" (explained in Section 1). The PAC over $\mathcal{O}_{K,\mathcal{V}}$ property of $K_s(\boldsymbol{\sigma})$ suffices in [13] to prove not only that for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ each $(\mathcal{S},\mathcal{V})$-Skolem density problem for $K_{\mathrm{tot},\mathcal{S}}(\boldsymbol{\sigma})$ is solvable but that each $(\mathcal{S},\mathcal{V})$-Skolem density problem for $K_{\mathrm{tot},\mathcal{S}}[\boldsymbol{\sigma}]$ is solvable. The latter field is the intersection of $K_{\mathrm{tot},\mathcal{S}}$ with the maximal Galois extension $K_s[\boldsymbol{\sigma}]$ of K in $K_s(\boldsymbol{\sigma})$. However, as is shown in [1], the field $K_s[\boldsymbol{\sigma}]$ is not PAC over $O_{K,\mathcal{V}}$, hence we can not deduce that $K_{\mathrm{tot},\mathcal{S}}[\boldsymbol{\sigma}]$ is weakly-P$\mathcal{S}$C over $O_{K,\mathcal{V}}$. Unfortunately the weakly-P$\mathcal{S}$C over $O_{K,\mathcal{V}}$ property enters again in the proof of the present work in Section 8. Thus, we are unable to prove that for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ the local-global principle holds for the fields $K_{\mathrm{tot},\mathcal{S}}[\boldsymbol{\sigma}]$. That question remains open.

1. Weakly P$\mathcal{S}$C fields over holomorphy domains

The objects of our results are defined over a global field K. The property of the field $M = K_{\mathrm{tot},\mathcal{S}}(\boldsymbol{\sigma})$ (when $\boldsymbol{\sigma}$ is taken at random in $\mathrm{Gal}(K)^e$) that lies behind the local-global principle is being "weakly P$\mathcal{S}$C over $\mathcal{O}_{M,\mathcal{V}}$". We introduce the notion and quote all results we need about this notion from [13].

Data 1.1. We fix the following data for the rest of this work:

(a) K is a global field.

(b) K_s is the separable closure and $\tilde{K}$ is the algebraic closure of K.

(c) $\mathrm{Gal}(K) = \mathrm{Gal}(K_s/K)$ is the absolute Galois group of K. For each nonnegative integer e the group $\mathrm{Gal}(K)^e$ has a unique Haar measure μ such that $\mu(\mathrm{Gal}(K)^e) = 1$. Given $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$, we write $K_s(\boldsymbol{\sigma})$ for the fixed field of $\sigma_1, \ldots, \sigma_e$ in K_s.

(d) $\mathbb{P} = \mathbb{P}_K$ is the set of all primes (finite and infinite) of K. A **finite** (resp. **infinite**) **prime** $\mathfrak{p}$ of a field E is an equivalence class of nonarchimedean (resp. archimedean) absolute values $|\ |_\mathfrak{p}$ of E. The unit disc $D_\mathfrak{p}$ is the $\mathfrak{p}$-open set $\{x \in E \mid |x|_\mathfrak{p} \le 1\}$ (resp. $\{x \in E \mid |x|_\mathfrak{p} < 1\}$) if $\mathfrak{p}$ is finite (resp. infinite). We denote the set of all finite primes of K by $\mathbb{P}_{\mathrm{fin}}$ and the set of all infinite primes of K by $\mathbb{P}_{\mathrm{inf}}$. Thus, $\mathbb{P}_{\mathrm{inf}} = \emptyset$ and $\mathbb{P}_{\mathrm{fin}} = \mathbb{P}$ when $\mathrm{char}(K) > 0$. For each $\mathfrak{p} \in \mathbb{P}$ we choose an absolute value $|\ |_\mathfrak{p}$ which belongs to $\mathfrak{p}$.

(e) $\mathcal{V}$ is a proper subset of $\mathbb{P}$.

(f) Let $\mathcal{R}$ be a subset of $\mathbb{P}$. Set $\mathcal{R}_{\mathrm{fin}} = \mathcal{R} \cap \mathbb{P}_{\mathrm{fin}}$ and $\mathcal{R}_{\mathrm{inf}} = \mathcal{R} \cap \mathbb{P}_{\mathrm{inf}}$. For each algebraic extension L of K let $\mathcal{R}_L$ be the set of all primes of L which lie over primes in $\mathcal{R}$. For $L = \tilde{K}$ we set $\tilde{\mathcal{R}} = \mathcal{R}_{\tilde{K}}$. If $\mathfrak{q} \in \mathbb{P}_L$ lies over $\mathfrak{p} \in \mathbb{P}$, we write $\mathfrak{q}|\mathfrak{p}$ and $\mathfrak{p} = \mathfrak{q}|_K$. We denote the unique absolute value which represents $\mathfrak{q}$ and extends $|\ |_\mathfrak{p}$ by $|\ |_\mathfrak{q}$.

If L is a normal extension of K, then $\mathrm{Aut}(L/K)$ acts on $\mathcal{R}_L$ according to the rule

$$|x|_{\mathfrak{p}^\sigma} = |x^{\sigma^{-1}}|_{\mathfrak{p}}, \text{ for } \mathfrak{p} \in \mathcal{R}_L \text{ and } x \in L.$$

We may choose a subset $\mathcal{R}_0$ of $\mathcal{R}_L$ which contains exactly one extension of each prime in $\mathcal{R}$. Then, for each $\mathfrak{q} \in \mathcal{R}_L$ there are $\mathfrak{p} \in \mathcal{R}_0$ and $\sigma \in \mathrm{Aut}(L/K)$ with $\mathfrak{q} = \mathfrak{p}^\sigma$. We say that $\mathcal{R}_0$ **represents** $\mathcal{R}_L$ **over** K.

We call $\mathcal{O}_{L,\mathcal{R}} = \{x \in L \mid |x|_{\mathfrak{q}} \le 1 \text{ for each } \mathfrak{q} \in \mathcal{R}_L\}$ the **$\mathcal{R}$-holomorphy domain** of L. It is closed under multiplication. If $\mathcal{R} \subseteq \mathbb{P}_{\mathrm{fin}}$, then $\mathcal{O}_{L,\mathcal{R}}$ is a ring.

We call

$$D_{L,\mathcal{R}} = \bigcap_{\mathfrak{q} \in \mathcal{R}_L} D_{\mathfrak{q}} = \{x \in L \mid \ |x|_{\mathfrak{q}} \le 1 \ \forall \mathfrak{q} \in \mathcal{R}_{\mathrm{fin},L} \text{ and } |x|_{\mathfrak{q}} < 1 \ \forall \mathfrak{q} \in \mathcal{R}_{\mathrm{inf},L}\}$$

the **open $\mathcal{R}$-holomorphy domain** of L.

For $\mathbf{a} = (a_1, \ldots, a_n) \in L^n$ let $|\mathbf{a}|_{\mathcal{R}} = \max\limits_{\mathfrak{q} \in \mathcal{R}_{K(\mathbf{a})}} \max\limits_{1 \le i \le n} |a_i|_{\mathfrak{q}} = \max\limits_{\tilde{\mathfrak{q}} \in \tilde{\mathcal{R}}} \max\limits_{1 \le i \le n} |a_i|_{\tilde{\mathfrak{q}}}$.

For $f(X) = \sum_{i=0}^n a_i X^i \in L[X]$, we set $|f|_{\mathcal{R}} = |(a_0, \ldots, a_n)|_{\mathcal{R}}$.

Proposition 1.2 (Strong approximation theorem [2, p. 67]). *Let $\mathcal{T}$ be a finite subset of $\mathcal{V}$. For each $\mathfrak{p} \in \mathcal{T}$ consider an element $a_{\mathfrak{p}}$ of K and let ε be a positive real number. Then there exists $x \in \mathcal{O}_{K,\mathcal{V} \smallsetminus \mathcal{T}}$ such that $|x - a_{\mathfrak{p}}|_{\mathfrak{p}} < \varepsilon$ for each $\mathfrak{p} \in \mathcal{T}$.*

Data 1.3. We add the following data to Data 1.1 and fix it for the rest of this work:

(a) Let $\mathfrak{p} \in \mathbb{P}$.
$\tilde{\mathfrak{p}}$ is a fixed extension of $\mathfrak{p}$ to a prime of $\tilde{K}$. If $\tilde{\mathfrak{q}} \in \tilde{\mathbb{P}}$ and $\tilde{\mathfrak{q}}|\mathfrak{p}$, then there is a $\sigma \in \mathrm{Gal}(K)$ such that $\tilde{\mathfrak{q}} = \tilde{\mathfrak{p}}^\sigma$.
$\hat{K}_{\mathfrak{p}}$ is the completion of K at $\mathfrak{p}$ inside the completion of $\tilde{K}$ at $\tilde{\mathfrak{p}}$. Then $|\ |_{\mathfrak{p}}$ uniquely extends to an absolute value $|\ |_{\mathfrak{p}}$ of $\hat{K}_{\mathfrak{p}}$ and then uniquely to an absolute value of $\tilde{K}\hat{K}_{\mathfrak{p}}$. The restriction of the latter to $\tilde{K}$ coincides with $|\ |_{\tilde{\mathfrak{p}}}$.
If $\mathfrak{p} \in \mathbb{P}_{\mathrm{inf}}$, then either $\hat{K}_{\mathfrak{p}} \cong \mathbb{R}$ or $\hat{K}_{\mathfrak{p}} \cong \mathbb{C}$; in the former case $\mathfrak{p}$ is **real**, in the latter case $\mathfrak{p}$ is **complex**.
$K_{\mathfrak{p}} = K_s \cap \hat{K}_{\mathfrak{p}}$. It is well defined up to a K-isomorphism. If $\mathfrak{p} \in \mathbb{P}_{\mathrm{fin}}$, then $K_{\mathfrak{p}}$ is an Henselian closure of K at $\mathfrak{p}$. Since $\hat{K}_{\mathfrak{p}}/K_{\mathfrak{p}}$ is a separable extension [8, Lemma 2.2], so is $\hat{K}_{\mathfrak{p}}/K$. If $\mathfrak{p} \in \mathbb{P}_{\mathrm{inf}}$ is real, then $K_{\mathfrak{p}}$ is a real closure of K at $\mathfrak{p}$; if $\mathfrak{p} \in \mathbb{P}_{\mathrm{inf}}$ is complex, then $K_{\mathfrak{p}} = K_s$.
$K_{\mathrm{t}\mathfrak{p}} = \bigcap_{\sigma \in \mathrm{Gal}(K)} K_{\mathfrak{p}}^\sigma$.
(b) $\mathcal{S}$ is a finite subset of $\mathcal{V}$.
(c) $N = K_{\mathrm{tot},\mathcal{S}} = \bigcap_{\mathfrak{p} \in \mathcal{S}} K_{\mathrm{t}\mathfrak{p}}$. This is the maximal Galois extension of K in which each $\mathfrak{p} \in \mathcal{S}$ totally decomposes. If $\mathcal{S} = \emptyset$, we let $N = K_s$.
Note that if L is a subextension of N/K, then $L_{\mathrm{tot},\mathcal{S}_L} = N$.
(d) For each $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ we put $K_{\mathrm{tot},S}(\boldsymbol{\sigma}) = K_s(\boldsymbol{\sigma}) \cap K_{\mathrm{tot},\mathcal{S}}$.

Definition 1.4. *Let M be an algebraic extension of K and $\mathcal{O}$ a subset of M. We say that M is **PAC over** $\mathcal{O}$ if for every absolutely irreducible polynomial $h \in M[T, X]$ and every nonzero $g \in M[T]$ there exists $(a, b) \in \mathcal{O} \times M$ such that $h(a, b) = 0$ and $g(a) \neq 0$.*

Let M be a subextension of N/K and $\mathcal{O}$ a subset of M. We say that M is **weakly P$\mathcal{S}$C over** $\mathcal{O}$ if for every absolutely irreducible polynomial $h \in M[T, X]$ monic in X such that all of the roots of $h(0, X)$ are simple and belong to N and every $g \in M[T]$ with $g(0) \neq 0$, there exists $(a, b) \in \mathcal{O} \times M$ such that $h(a, b) = 0$ and $g(a) \neq 0$.

Remark 1.5. If $\mathcal{O} \subseteq \mathcal{O}' \subseteq M$ and M is weakly P$\mathcal{S}$C over $\mathcal{O}$, then it is also weakly P$\mathcal{S}$C over $\mathcal{O}'$.

By [13, Cor. 1.6], $K_s(\boldsymbol{\sigma})$ is PAC over $O_{K,\mathcal{V}}$ for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$. This implies the following result.

Lemma 1.6 ([13, Lemma 1.12 (a)]). *For almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ the field $K_{\mathrm{tot},\mathcal{S}}(\boldsymbol{\sigma})$ is weakly P$\mathcal{S}$C over $\mathcal{O}_{K,\mathcal{V}}$.*

Lemma 1.7 (Quasi uniform approximation [13, Lemma 1.14]). *Let M be a subextension of N/K which is weakly P$\mathcal{S}$C over $\mathcal{O}_{M,\mathcal{V}}$ and let $\mathcal{T}$ be a finite subset of $\mathcal{V}$ which contains $\mathcal{S}$. Let $x \in N$ and $\varepsilon > 0$. Then M has a finite subset B (depending on $\mathcal{T}, x, \varepsilon$) such that for each $\tilde{\mathfrak{q}} \in \tilde{\mathcal{T}}$ there is $b \in B$ with $|b - x|_{\tilde{\mathfrak{q}}} < \varepsilon$.*

Lemma 1.8 ([13, Prop. 1.15 and Remark 1.11 (b)]). *Let M be a subextension of N/K which is weakly P$\mathcal{S}$C over $\mathcal{O}_{M,\mathcal{V}}$. Let $\mathfrak{p} \in \mathcal{V} \smallsetminus \mathcal{S}$ and $\tilde{\mathfrak{q}}$ an extension of $\mathfrak{p}$ to $\tilde{K}$. Suppose $\tilde{\mathfrak{q}} = \tilde{\mathfrak{p}}^\sigma$ for $\sigma \in \mathrm{Gal}(K)$. Then $K_{\mathfrak{p}}^\sigma M = K_s$ and M is $\tilde{\mathfrak{q}}$-dense in $\tilde{K}$.*

Lemma 1.9 ([21, Lemma 3.5]). *Let M be a subextension of N/K and suppose that M is weakly P$\mathcal{S}$C over a subset $\mathcal{O}$. Let Γ be an absolutely irreducible projective curve defined over M, let F be the function field of Γ over M, and let t be an element in $F \smallsetminus M$ whose zeros are simple and belong to $\Gamma_{\mathrm{simp}}(N)$. Also, let A be a finite subset of $M^\times$. Then there exists $P \in \Gamma_{\mathrm{simp}}(M)$ such that $t(P) \in \mathcal{O} \smallsetminus A$.*

Definition 1.10. *A data for an $(\mathcal{S}, \mathcal{V})$-**Skolem density problem** for an algebraic extension M of K consists of a quadruple $(\mathcal{T}, \mathbf{f}, \mathbf{a}, \gamma)$ in which*

(a) $\mathcal{T}$ is a finite subset of $\mathcal{V}$ containing $\mathcal{S}$;
*(b) $\mathbf{f} = (f_1, \ldots, f_m)$ and $f_i \in \tilde{K}[X_1, \ldots, X_n]$ is $\mathfrak{p}$-**primitive**, i.e. $|f_i|_{\mathfrak{p}} = 1$, for each $\mathfrak{p} \in \tilde{\mathcal{V}}_{\mathrm{fin}} \smallsetminus \tilde{\mathcal{T}}$, $i = 1, \ldots, m$;*
(c) a point $\mathbf{a} = (a_1, \ldots, a_n) \in M^n$; and
(d) a positive real number γ.

*A **solution** is a point $\mathbf{x} \in \mathcal{O}^n_{M,\mathcal{V}_{\mathrm{fin}} \smallsetminus \mathcal{T}}$ with $|\mathbf{x} - \mathbf{a}|_{\mathcal{T}} < \gamma$ and $f_i(\mathbf{x}) \in \mathcal{O}^\times_{\tilde{K},\mathcal{V}_{\mathrm{fin}} \smallsetminus \mathcal{T}}$, $i = 1, \ldots, m$.*

*M is called **an $\mathcal{S}$-Skolem field with respect to** $\mathcal{V}$ if every $(\mathcal{S}, \mathcal{V})$-Skolem density problem for M has a solution.*

Proposition 1.11 ([13, Thm. 3.7]). *If M is a subextension of N/K which is weakly P$\mathcal{S}$C over $\mathcal{O}_{M,\mathcal{V}}$, then M is an $\mathcal{S}$-Skolem field with respect to $\mathcal{V}$.*

2. Rumely's local global principle

We give an exact formulation of the major results of this work. This requires a variety of data and notation in addition to those introduced in Section 1.

Data 2.1. We introduce data and notation, and keep the assumptions we make here for the rest of this work:

(a) For each $\mathfrak{q} \in \mathcal{V}_N$ we fix an extension $\tilde{\mathfrak{q}} \in \tilde{\mathcal{V}}$. For each subextension L of N/K let $\hat{L}_\mathfrak{q}$ be the completion of L at $\mathfrak{q}|_L$ inside the completion of $\tilde{K}$ at $\tilde{\mathfrak{q}}$. The latter completion has a unique absolute value $|\,|_\mathfrak{q}$ which coincides with $|\,|_{\mathfrak{q}|_K}$ on K. Let $L_\mathfrak{q} = K_s \bigcap \hat{L}_\mathfrak{q}$ and $D_{L,\mathfrak{q}} = \{x \in L_\mathfrak{q} \mid |x|_\mathfrak{q} \le 1\}$ if $\mathfrak{q} \in \mathcal{V}_{\mathrm{fin},N}$ and $D_{L,\mathfrak{q}} = \{x \in L_\mathfrak{q} \mid |x|_\mathfrak{q} < 1\}$ if $\mathfrak{q} \in \mathcal{V}_{\mathrm{inf},N}$. Then $L_\mathfrak{q} \subseteq N_\mathfrak{q}$ and $L_\mathfrak{q}$ is the fixed field in K_s of the decomposition group $\{\tau \in \mathrm{Gal}(L) \mid \tilde{\mathfrak{q}}^\tau = \tilde{\mathfrak{q}}\}$. If $\mathfrak{q} \in \mathcal{V}_{\mathrm{fin},N}$, then $L_\mathfrak{q}$ is a Henselian closure of L at $\mathfrak{q}|_L$ and $D_{L,\mathfrak{q}}$ is its valuation ring. We extend each $\sigma \in \mathrm{Gal}(N/L)$ to $\tilde{\sigma} \in \mathrm{Aut}(\tilde{L}/L)$ which satisfies $\tilde{\mathfrak{q}}^{\tilde{\sigma}} = \widetilde{\mathfrak{q}^\sigma}$. Then $\tilde{\sigma}$ maps $L_\mathfrak{q}$ isomorphically onto $L_{\mathfrak{q}^\sigma}$ and $D_{L,\mathfrak{q}}$ onto $D_{L,\mathfrak{q}^\sigma}$. If L is a finite subextension of N/K and $\mathfrak{q}|_L$ is non-complex, then $\mathrm{Aut}(L_\mathfrak{q}/L) = 1$ [7, Prop. 14.5 and Prop. 15.6].

(b) For an abstract absolutely irreducible variety W defined over K and for each extension L of K we let $W(L)$ (resp. $W_{\mathrm{simp}}(L)$) be the set of all L-rational (resp. simple L-rational) points of W. Whenever we say that W is an affine absolutely irreducible variety we also mean that W is embedded in some affine space. Then, if D is a subset of L, a D**-rational point** of W is an L-rational point of W whose coordinates lie in D. We denote the set of all D-rational points of W by $W(D)$. Similar notation is imposed for closed subsets of W.

(c) M is a subextension of N/K which is weakly P$\mathcal{S}$C over $\mathcal{O}_{M,\mathcal{V}}$.

(d) $\mathcal{W}$ is a finite subset of $\mathcal{V}$ which contains $\mathcal{S}$.

(e) Let V be an affine absolutely irreducible variety defined over K. Then $V_{K,\mathcal{S},\mathcal{W}}$ is the set of all points $(\mathbf{z}_\mathfrak{q})_{\mathfrak{q}\in\mathcal{W}_N} \in \prod_{\mathfrak{q}\in\mathcal{W}_N} V_{\mathrm{simp}}(N_\mathfrak{q})$ for which

(1) there exists a finite subextension L of M/K such that $\mathbf{z}_\mathfrak{q} \in V_{\mathrm{simp}}(L_\mathfrak{q})$ and $\mathbf{z}_{\mathfrak{q}^\sigma} = \mathbf{z}_\mathfrak{q}^\sigma$ for each $\mathfrak{q} \in \mathcal{W}_N$ and $\sigma \in \mathrm{Gal}(N/L)$.

Each $(\mathbf{z}_\mathfrak{q})_{\mathfrak{q}\in\mathcal{W}_N}$ that satisfies (1) is said to be L**-rational**.

(f) $V_{D,\mathcal{S},\mathcal{W}} = V_{K,\mathcal{S},\mathcal{W}} \cap \prod_{\mathfrak{q}\in\mathcal{W}_N} V_{\mathrm{simp}}(D_{N,\mathfrak{q}})$.

We will extend these data in the sequel by more data and assumptions, as necessary.

Here is our main theorem.

Theorem 2.2 (Strong approximation theorem). *Let V be an affine absolutely irreducible variety defined over K and embedded in $\mathbb{A}^n$. Consider a $(\mathbf{z}_\mathfrak{q})_{\mathfrak{q}\in\mathcal{W}_N} \in V_{K,\mathcal{S},\mathcal{W}}$ and an $\varepsilon > 0$. Then:*

(a) There exists $\mathbf{z} \in V(M)$ such that $|\mathbf{z} - \mathbf{z}_\mathfrak{q}|_\mathfrak{q} < \varepsilon$ for each $\mathfrak{q} \in \mathcal{W}_N$.

(b) If $V(D_{N,\mathfrak{q}}) \ne \emptyset$ for each $\mathfrak{q} \in \mathcal{V}_N \smallsetminus \mathcal{W}_N$, then there exists $\mathbf{z} \in V(M)$ such that $|\mathbf{z} - \mathbf{z}_\mathfrak{q}|_\mathfrak{q} < \varepsilon$ for each $\mathfrak{q} \in \mathcal{W}_N$ and $\mathbf{z} \in D_{N,\mathfrak{q}}^n$ for each $\mathfrak{q} \in \mathcal{V}_N \smallsetminus \mathcal{W}_N$.

(c) If $V(D_{N,\mathfrak{q}}) \ne \emptyset$ for each $\mathfrak{q} \in \mathcal{V}_N \smallsetminus \mathcal{W}_N$ and $\mathbf{z}_\mathfrak{q} \in D_{N,\mathfrak{q}}^n$ for each $\mathfrak{q} \in \mathcal{W}_N$, then there exists $\mathbf{z} \in V(D_{M,\mathcal{V}})$ such that $|\mathbf{z} - \mathbf{z}_\mathfrak{q}|_\mathfrak{q} < \varepsilon$ for each $\mathfrak{q} \in \mathcal{W}_N$.

Part (c) is an interesting special case of Part (b). In Section 9 we first prove (c), and then conclude (b) and (a).

Remark 2.3. (a) We may replace K in Data 2.1 by any finite subextension L of M/K and extend all the objects that have been defined over K to L. Then the assumptions made on them remain true and N does not change. It follows that Theorem 2.2 for K implies the theorem for L. Also, we may start from a variety V which is defined over M and then replace K by a finite subextension of M/K over which V is already defined.

(b) It suffices to prove Theorem 2.2 only for points $(\mathbf{z}_\mathfrak{q})_{\mathfrak{q}\in\mathcal{W}_N}$ which are K-rational. Indeed, if $(\mathbf{z}_\mathfrak{q})_{\mathfrak{q}\in\mathcal{W}_N}$ is L-rational for some finite subextension L of M/K, then we may apply the theorem in its restricted form to L rather than to K and approximate $(\mathbf{z}_\mathfrak{q})_{\mathfrak{q}\in\mathcal{W}_N}$ by a point in $V(M)$ as (a), (b), (c) of the theorem require.

(c) Let $\mathfrak{q} \in \mathcal{V}_N$. If $\mathfrak{q} \in \mathcal{S}_N$, then $K_\mathfrak{q} \subseteq M_\mathfrak{q} \subseteq N_\mathfrak{q} \subseteq K_\mathfrak{q}$, so $M_\mathfrak{q} = N_\mathfrak{q}$. If $\mathfrak{q} \notin \mathcal{S}_N$, then $M_\mathfrak{q} = N_\mathfrak{q} = K_s$ (Lemma 1.8).

(d) We use the assumption $\mathcal{S}_N \subseteq \mathcal{W}_N$ (Data 2.1(d)) only to simplify notation. In applications that do not make this assumption we use Lemma 9.1 to restore it.

The strong approximation theorem yields a weak one, which we prove in Section 9.

Theorem 2.4 (Weak approximation theorem). *Let $\mathcal{T}$ be a finite subset of $\mathcal{V}_M$ and let V be an affine absolutely irreducible variety defined over M.*

(a) If $V_{\rm simp}(D_{M,\mathfrak{p}}) \neq \emptyset$ for each $\mathfrak{p} \in \mathcal{S}_M$ and $V(D_{M,\mathfrak{p}}) \neq \emptyset$ for each $\mathfrak{p} \in \mathcal{V}_M \smallsetminus \mathcal{S}_M$, then each point in

$$\prod_{\mathfrak{p}\in\mathcal{T}\cap\mathcal{S}_M} V_{\rm simp}(D_{M,\mathfrak{p}}) \times \prod_{\mathfrak{p}\in\mathcal{T}\smallsetminus\mathcal{S}_M} V(D_{M,\mathfrak{p}})$$

can be approximated by a point in $V(D_{M,\mathcal{V}})$.

(b) If $V_{\rm simp}(M_\mathfrak{p}) \neq \emptyset$ for each $\mathfrak{p} \in \mathcal{S}_M$, then $V_{\rm simp}(M)$ is dense in

$$\prod_{\mathfrak{p}\in\mathcal{T}\cap\mathcal{S}_M} V_{\rm simp}(M_\mathfrak{p}) \times \prod_{\mathfrak{p}\in\mathcal{T}\smallsetminus\mathcal{S}_M} V(M_\mathfrak{p})\,.$$

Taking $\mathcal{T}$ in Theorem 2.4 to be nonempty gives a local-global principle.

Theorem 2.5 (Local-global principle). *Let V be an affine absolutely irreducible variety defined over M. Suppose $V_{\rm simp}(D_{M,\mathfrak{p}}) \neq \emptyset$ for each $\mathfrak{p} \in \mathcal{S}_M$ and $V(D_{M,\mathfrak{p}}) \neq \emptyset$ for each $\mathfrak{p} \in \mathcal{V}_M \smallsetminus \mathcal{S}_M$. Then $V(D_{M,\mathcal{V}}) \neq \emptyset$.*

Remark 2.6 ([12, Remark 1.6]). It is possible to replace each $M_\mathfrak{p}$ in Theorems 2.4 and 2.5 by its completion $\hat{M}_\mathfrak{p}$.

An algebraic extension L of K is said to be P$\mathcal{S}$C, if every absolutely irreducible variety V defined over L with a simple $L_\mathfrak{p}$-rational point for each $\mathfrak{p} \in \mathcal{S}_L$ has an L-rational point.

Taking $\mathcal{T}$ in Theorem 2.4 to be a finite nonempty subset of $\mathcal{S}_M$ and assuming that $V_{\mathrm{simp}}(M_\mathfrak{p}) \neq \emptyset$ for each $\mathfrak{p} \in \mathcal{S}_M$, we get that $V_{\mathrm{simp}}(M)$ is dense in the nonempty set $\prod_{\mathfrak{p}\in\mathcal{T}} V_{\mathrm{simp}}(M_\mathfrak{p})$. Therefore, $V_{\mathrm{simp}}(M) \neq \emptyset$. This proves the following result:

Corollary 2.7. *The field M is P$\mathcal{S}$C.*

Corollary 2.8. *Let $\mathcal{V} \subseteq \mathbb{P}_{\mathrm{fin}}$ be a proper subset of $\mathbb{P}$ and let V be an affine absolutely irreducible variety defined over M. If $V_{\mathrm{simp}}(O_{N,\mathcal{V}})$ is nonempty, then so is $V(O_{M,\mathcal{V}})$.*

Examples of weakly P$\mathcal{S}$C fields are given by Lemma 1.6.

Corollary 2.9. *For almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$, the field $M = K_{\mathrm{tot},\mathcal{S}}(\boldsymbol{\sigma})$ satisfies the consequences of Theorems 2.2, 2.4, and 2.5, and Corollary 2.7. In particular, M is P$\mathcal{S}$C.*

3. Restatement of the approximation theorem for integral points on curves

This section starts the long proof of the strong approximation theorem for $D_{M,\mathcal{V}}$-integral points on a curve (Theorem 2.2(c) for $\dim(V) = 1$), from which all of the other results follow. We first reformulate the theorem in this case in terms of function fields, state a somewhat stronger result and finally describe the five steps needed to prove the stronger result. To fix notation we add additional data to Data 1.1, 1.3, and 2.1.

Data 3.1. The following data and notation remain in force until the end of Section 8.

C	is an absolutely irreducible affine curve in $\mathbb{A}^n$ defined over K,
$\mathbf{x} = (x_1, \ldots, x_n)$	is a generic point of C over K and over each completion $\hat{K}_\mathfrak{p}$,
$F_0 = K(\mathbf{x})$	is the function field of C over K,
$F = MF_0 = M(\mathbf{x})$	is the function field of C, considered as a curve over M,
$\mathrm{genus}(F/M)$	is the genus of F/M,
s	$= 2\,\mathrm{genus}(F/M) + 2$ is a useful constant,
Γ	is a normal projective model of F/M,
M'	is a field that contains M and is linearly disjoint from F,
$F' = M'F$	is the function field obtained by extension of scalars to M',
$\Gamma(M')$	is the set of all M'-rational points of Γ,
$\Gamma(F'/M')$	is the set of all prime divisors of F'/M',
$\mathrm{Div}(F'/M')$	is the group of divisors of F'/M',
$P_1^*, \ldots, P_e^*$	are the distinct poles of $x_1, \ldots, x_n$ in $\Gamma(F/M)$,
$P_{i,1}^*, \ldots, P_{i,d_i}^*$	are the distinct prime divisors of $\tilde{K}F/\tilde{K}$ which lie over P_i^*,

e_i is the ramification index of $P^*_{i,j}$ over P^*_i,
D^* $= P^*_1 + \cdots + P^*_e$
ε_0 is a positive real number.

Remark 3.2. (a) For each divisor A of F'/M' we consider the vector space

$$\mathcal{L}_{M'}(A) = \{f \in F' \mid \operatorname{div}(f) + A \geq 0\}$$

over M'. It has a finite dimension, which is denoted by $\dim_{M'}(A)$. The group $\operatorname{Div}(F/M)$ naturally embeds in $\operatorname{Div}(F'/M')$. If M'/M is separable, then since M' is linearly disjoint from F over M, a basis of $\mathcal{L}_M(A)$ is also a basis of $\mathcal{L}_{M'}(A)$ and $\operatorname{genus}(F'/M') = \operatorname{genus}(F/M)$ [3, p. 132]. Thus $\dim_M(A) = \dim_{M'}(A)$ and we can drop the reference to the ground field from the dimension of A.

(b) $P^*_i = e_i(P^*_{i1} + \cdots + P^*_{i,d_i})$, hence $\deg(P^*_i) = d_i e_i$

Remark 3.3. Suppose M'/M is separable. Then the extension of Γ to M' is still normal [14, p. 147, Cor.]. We identify each point of $\Gamma(M')$ with a prime divisor P of F'/M' of degree 1. If $f \in F'$, then $f(P)$ is the value of the rational function f of Γ at P, if we view P as a point on the curve, or the value of the place associated with P at the element f of F', if we view P as a prime divisor of F'/M'. In both cases $f(P)$ is an element of $M' \cup \{\infty\}$. This element is ∞ exactly when P is a pole of f. Thus, if $P \in \Gamma(\tilde{K})$ does not belong to $\{P^*_{ij} \mid i = 1, \ldots, e;\ j = 1, \ldots, d_i\}$, then $\mathbf{x}(P) = (x_1(P), \ldots, x_n(P))$ is a point in $C(\tilde{K})$.

Now suppose that M' is equipped with an absolute value $|\ |_\mathfrak{p}$. The $\mathfrak{p}$-adic topology of M' induces a topology on $\Gamma(M')$ whose basis consists of the sets

$$\{P \in \Gamma(M') \mid |f_1(P)|_\mathfrak{p} < 1, \ldots, |f_m(P)|_\mathfrak{p} < 1\}$$

with $f_1, \ldots, f_m \in F'$. Here we make the convention that $|\infty|_\mathfrak{p} = \infty$. This is actually the weakest topology on $\Gamma(M')$ such that each $f \in F'$ defines a continuous function

$$f\colon \Gamma(M') \to M' \cup \{\infty\}, \qquad P \mapsto f(P),$$

where the neighborhoods of ∞ are, as usual, the complements of the closed neighborhoods of 0.

Next suppose that for each $\mathfrak{p} \in \mathcal{V}_N$ we are given a point $\mathbf{z}_\mathfrak{p} \in C(D_{N,\mathfrak{p}})$ such that $(\mathbf{z}_\mathfrak{q})_{\mathfrak{q}\in\mathcal{W}_N} \in C_{D,\mathcal{S},\mathcal{W}}$ (Data 2.1(f)). Our goal is to approximate $(\mathbf{z}_\mathfrak{q})_{\mathfrak{q}\in\mathcal{W}_N}$ by an element of $C(D_{M,\mathcal{V}})$. If $\mathfrak{p} \in \mathcal{W}_N$, then $\mathbf{z}_\mathfrak{p} \in C_{\mathrm{simp}}(D_{N,\mathfrak{p}})$. Hence, there exists a unique $P_\mathfrak{p} \in \Gamma(N_\mathfrak{p})$ such that $\mathbf{x}(P_\mathfrak{p}) = \mathbf{z}_\mathfrak{p}$ [10, p. 457, Cor. A3]. If $\mathfrak{p} \in \mathcal{V}_N \smallsetminus \mathcal{W}_N$, then $N_\mathfrak{p} = K_s$ (Remark 2.3(c)) and we may choose $P_\mathfrak{p} \in \Gamma(N_\mathfrak{p})$ such that $\mathbf{x}(P_\mathfrak{p}) = \mathbf{z}_\mathfrak{p}$. In all cases, $\mathbf{x}(P_\mathfrak{p}) \in D^n_{N,\mathfrak{p}}$.

By definition, there exists a finite subextension L of M/K such that $\mathbf{z}_{\mathfrak{q}^\sigma} = \mathbf{z}^\sigma_\mathfrak{q}$ for each $\mathfrak{q} \in \mathcal{W}_N$ and each $\sigma \in \operatorname{Gal}(N/L)$. By Data 3.1, $L_\mathfrak{q}F_0$ is a regular extension of $L_\mathfrak{q}$ and therefore it is linearly disjoint from $N_\mathfrak{q}$ over $L_\mathfrak{q}$. Hence, each $\sigma \in \operatorname{Gal}(N/L)$ extends to an isomorphism $\sigma\colon N_\mathfrak{q}F_0 \to N_{\mathfrak{q}^\sigma}F_0$ that maps $L_\mathfrak{q}$ onto $L_{\mathfrak{q}^\sigma}$ and fixes each element of F_0 (Use Data 2.1(a)). It follows that $\mathbf{x}(P_{\mathfrak{q}^\sigma}) = \mathbf{z}_{\mathfrak{q}^\sigma} = \mathbf{z}^\sigma_\mathfrak{q} = \mathbf{x}(P_\mathfrak{q})^\sigma = \mathbf{x}(P^\sigma_\mathfrak{q})$. Therefore, $P_{\mathfrak{q}^\sigma} = P^\sigma_\mathfrak{q}$.

By remark 2.3(b), we may assume that $L = K$ and conclude that Theorem 2.2(c) for $V = C$ is equivalent to the following theorem:

Theorem 3.4 (Approximation theorem for function fields of one variable). *Suppose that for each $\mathfrak{p} \in \mathcal{V}_N$ there exists $P_\mathfrak{p} \in \Gamma(N_\mathfrak{p})$ such that $\mathbf{x}(P_\mathfrak{p}) \in D^n_{N,\mathfrak{p}}$. Assume that $P_{\mathfrak{q}^\sigma} = P_\mathfrak{q}^\sigma$ for each $\mathfrak{q} \in \mathcal{W}_N$ and $\sigma \in \mathrm{Gal}(N/K)$. Then, there exists $P \in \Gamma(M)$ such that $\mathbf{x}(P) \in D^n_\mathfrak{p}$ for all $\mathfrak{p} \in \mathcal{V}_N$ and $|\mathbf{x}(P) - \mathbf{x}(P_\mathfrak{q})|_\mathfrak{q} < \varepsilon_0$ for each $\mathfrak{q} \in \mathcal{W}_N$. In particular, $P \notin \{P_1^*, \ldots, P_e^*\}$.*

Our method of proof forces us to prove a stronger theorem than Theorem 3.4.

Theorem 3.5. *Suppose for each $\mathfrak{p} \in \mathcal{V}_N$ there exists $P_\mathfrak{p} \in \Gamma(N_\mathfrak{p})$ such that $\mathbf{x}(P_\mathfrak{p}) \in D^n_{N,\mathfrak{p}}$. Assume that $P_{\mathfrak{q}^\sigma} = P_\mathfrak{q}^\sigma$ for each $\mathfrak{q} \in \mathcal{W}_N$ and each $\sigma \in \mathrm{Gal}(N/K)$. Then there exists a nonzero function $f \in F$ with the following properties:*

(1a) There exists a positive integer m (which can be chosen to be arbitrarily large) such that $\mathrm{div}_\infty(f) = mD^*$.

*(1b) Each of the zeros of f is N-rational and **simple**, that is $\mathrm{div}_0(f) = \sum_{i=1}^l P_i$ with distinct $P_i \in \Gamma(N)$.*

(1c) For all $\mathfrak{p} \in \mathcal{V}_N$ we have $\mathbf{x}(P_i) \in D^n_{N,\mathfrak{p}}$, $i = 1, \ldots, l$, and

(1d) $|\mathbf{x}(P_i) - \mathbf{x}(P_\mathfrak{q})|_\mathfrak{q} \le \varepsilon_0$, $i = 1, \ldots, l$, if $\mathfrak{q} \in \mathcal{W}_N$.

Moreover, one of the zeros of f is M-rational.

To prove Theorem 3.5 we fix the data of the assumption of the theorem:

Data and Assumption 3.6. For each $\mathfrak{p} \in \mathcal{V}_N$ we fix a point $P_\mathfrak{p}$ of $\Gamma(N_\mathfrak{p})$ such that $\mathbf{x}(P_\mathfrak{p}) \in D^n_{N,\mathfrak{p}}$. We assume that $P_{\mathfrak{q}^\sigma} = P_\mathfrak{q}^\sigma$ for each $\mathfrak{q} \in \mathcal{W}_N$ and each $\sigma \in \mathrm{Gal}(N/K)$. This data will remain in force until the end of Section 8.

Definition 3.7 (Admissible functions). *Let $\mathfrak{p} \in \mathcal{V}_N$. A function $f \in NF$ is $\mathfrak{p}$-**admissible** if*

*(2a) there exists a positive integer m such that $\mathrm{div}_\infty(f) = mD^*$ (we say that f is of **level** m),*

(2b) each zero of f is simple and belongs to $\Gamma(N_\mathfrak{p})$,

(2c) $\mathbf{x}(P) \in D^n_{N,\mathfrak{p}}$ for each zero $P \in \Gamma(N_\mathfrak{p})$ of f, and

(2d) if $\mathfrak{p} \in \mathcal{W}_N$, then $|\mathbf{x}(P) - \mathbf{x}(P_\mathfrak{p})|_\mathfrak{p} < \varepsilon_0$ for each zero $P \in \Gamma(N_\mathfrak{p})$ of f.

*Let $\mathcal{T}$ be a subset of $\mathcal{V}_N$. We say that f is $\mathcal{T}$-**admissible** if f is $\mathfrak{p}$-admissible for each $\mathfrak{p} \in \mathcal{T}$. In this case we also say that f is **admissible along** $\mathcal{T}$.*

Definition 3.8 (Small sets). *A subset $\mathcal{T}$ of $\mathcal{V}_N$ is **small** if it satisfies one of the following equivalent conditions:*

(3a) $\mathcal{T}|_L$ is a finite set for each finite subextension L of N/K.

(3b) $\mathcal{T}$ is contained in a set $\mathcal{T}' = \bigcup_{a \in A} \{\mathfrak{p} \in \mathcal{V}_N \mid |a|_\mathfrak{p} > 1\}$ for some nonempty finite subset A of N.

*Thus, for each finite subextension L of N/K there is a finite subset $\mathcal{T}_0$ of $\mathcal{T}$ which contains exactly one extension of each element of $\mathcal{T}|_L$. So, $\mathcal{T} \subseteq \{\mathfrak{q}^\sigma \mid \mathfrak{q} \in \mathcal{T}_0$ and $\sigma \in \mathrm{Gal}(L)\}$. We say that $\mathcal{T}_0$ **represents** $\mathcal{T}|_L$. If $\mathcal{T} = \{\mathfrak{q}^\sigma \mid \mathfrak{q} \in \mathcal{T}_0$ and $\sigma \in \mathrm{Gal}(L)\}$, we say that $\mathcal{T}$ is L-**rational**.*

Starting with an arbitrary small set $\mathcal{T}$ as above, we may enlarge A to a finite set which is invariant under $\mathrm{Gal}(K)$. Then the set $\mathcal{T}'$ of (3b) becomes K-rational. Thus, each small subset of $\mathcal{V}_N$ is contained in a K-rational small subset of $\mathcal{V}_N$.

*Finally, an (L-**rational) big subset** of $\mathcal{V}_N$ is the complement of an (L-rational) small set.*

The proof of Theorem 3.5 constructs f in five steps. In each of them f is admissible along a set $\mathcal{T}$ which is larger than the set of the preceding step. Of course, f is changed from one step to the next one. Thus, in each step we actually construct not only one function, but a family of functions, which are close to each other in the "$\mathcal{T}$-topology". Our construction follows the construction of Roquette et al. [23] over $\tilde{K}$. We use Remark 2.3(c) to approximate functions in NF by admissible functions in F.

The headings of the steps below describe the set $\mathcal{T}$ along which f is admissible.

1. A SINGLE VALUATION. To construct a function $f \in NF$ which is $\mathfrak{p}$-admissible for a single valuation $\mathfrak{p} \in \mathcal{V}_N$ we use the Rumely-Jacobi existence theorem for algebraic functions and the theorem about the continuity of the zeros of algebraic functions. The former forces us to assume that the completion of K at $\mathfrak{p}|_K$ is a local field. The latter holds over $N_\mathfrak{p}$. We prove that if f' is sufficiently $\mathfrak{p}$-close to f, then it is also $\mathfrak{p}$-admissible. Then we use the $\mathfrak{p}$-density of M in N to choose $f \in F$.
2. FINITELY MANY VALUATIONS. We use the weak approximation theorem.
3. SMALL SETS. An essential tool in this step is Lemma 1.7.
4. A BIG SET OF VALUATIONS. We use here the theory of good reduction.
5. THE WHOLE SET $\mathcal{V}_N$. In order to combine the big set of valuations with its complement (which is small) we use Proposition 1.11.

Finally we use Lemma 1.9 in order to choose f with an M-rational zero.

4. Finitely many valuations

The existence of an admissible function at a single valuation is a consequence of the Jacobi-Rumely-Pop existence theorem. We use the principle of variation of constants (Corollary 4.4) to approximate several functions, each admissible at a single valuation, by a function which is admissible at each of these valuations.

Before we do that, we fix further data and make more assumptions on the top of those already made in Data 2.1, Data 3.1, and Data 3.6.

Data and Assumption 4.1. We choose a finite extension K_1 of K which is contained in M and over which Γ is defined. Then $F_1 = K_1(\mathbf{x})$ is the function field of C and of Γ over K_1 and $F = MF_1$. Since D^* is M-rational, we may assume in addition that D^* is K_1-rational.

Let $\sigma \in \mathrm{Gal}(K_1)$. Since $\tilde{K}$ and F_1 are linearly disjoint over K_1, σ extends uniquely to an element of $\mathrm{Aut}(\tilde{K}F/F_1)$ which we also denote by σ. This σ acts on $\Gamma(\tilde{K})$ such that $f^\sigma(P^\sigma) = f(P)^\sigma$ for all $f \in \tilde{K}F$ and $P \in \Gamma(\tilde{K})$. Extend the action of σ to the group of divisors of $\tilde{K}F/\tilde{K}$ by linearity. Then $\mathrm{div}(f)^\sigma = \mathrm{div}(f^\sigma)$ for each $f \in \tilde{K}F$. Assumption 4.1 implies that $(D^*)^\sigma = D^*$.

Lemma 4.2. *Let E/L be an algebraic function field of one variable (in particular we assume that E/L is regular) and A a positive divisor of E/L of degree l. Suppose A decomposes in $\tilde{L}E$ as a sum of l distinct prime divisors: $A = \sum_{i=1}^{l} \tilde{P}_i$. For each i let P_i be the restriction of $\tilde{P}_i$ to L. Then the residue field of E at P_i is separable over L.*

Proof. We have to prove only the case where $p = \mathrm{char}(L) > 0$. Extending L to its separable closure does not change the degree of A [3, p. 126, Thm.] nor its factorization over L_s. So, we may assume that $L = L_s$ and we have to prove that $\deg(P_i) = 1$, $i = 1, \ldots, l$.

Let $A = \sum_{i=1}^{k} a_i Q_i$ be the factorization of A in E into a sum of prime divisors with distinct $Q_1, \ldots, Q_k$ and positive integers $a_1, \ldots, a_k$. Since $\tilde{L}/L$ is purely inseparable, each Q_i extends uniquely to a prime divisor $\tilde{Q}_i$ of $\tilde{L}E/\tilde{L}$ whose ramification index q_i is a power of p [3, p. 111]. Thus, $Q_i = q_i \tilde{Q}_i$ and $A = \sum_{i=1}^{k} a_i q_i \tilde{Q}_i$ is a factorization of A over $\tilde{L}E$. Comparing the two factorization of A, we find that $k = l$ and after relabeling, $P_i = Q_i$, $\tilde{P}_i = \tilde{Q}_i$, and $a_i = q_i = 1$, $i = 1, \ldots, l$. Since $l = \deg(A) = \sum_{i=1}^{l} \deg(P_i)$, we find that $\deg(P_i) = 1$, $i = 1, \ldots, l$, as contended. ■

Proposition 4.3 (Continuity of zeros of algebraic functions). *Let $(M', |\ |_\mathfrak{p})$ be an absolute valued field which is separable over K_1 and let $F' = M'F_1$. Suppose $(M', |\ |_\mathfrak{p})$ is Henselian, real closed or algebraically closed. Consider an element $0 \neq f \in F'$, set $A = \mathrm{div}_\infty(f)$, and suppose $\mathrm{div}_0(f) = \sum_{i=1}^{l} P_i$, where P_i are distinct prime divisors of F'/M' of degree 1. Write $f = \sum_{j=1}^{d} c_j u_j$, where $c_j \in M'$ and $u_1, \ldots, u_d$ form a basis for the M'-vector space $\mathcal{L}_{M'}(A)$. For each $1 \le i \le l$ let $U_i \subseteq \Gamma(M')$ be a $\mathfrak{p}$-open neighborhood of P_i. Then there exists $\varepsilon > 0$ such that if $c_1', \ldots, c_d' \in M'$ satisfy $|c_j' - c_j|_\mathfrak{p} < \varepsilon$, $j = 1, \ldots, d$ and $f' = \sum_{j=1}^{d} c_j' u_j$, then $\mathrm{div}_\infty(f') = A$ and $\mathrm{div}_0(f') = \sum_{i=1}^{l} P_i'$ with $P_i' \in U_i$.*

Proof. The result for the the algebraically closed case appears in [20, Thm. 1.1] and in [6, Thm. 7.1]. The proofs rely on the fact that the statements of the proposition are elementary in the language of absolute valued fields. Unfortunately, no proof or reference is given in those articles to that fact, although it is highly nontrivial. The interested reader may find the missing proof in [9, Part G of the proof of Proposition 3.5].

The Henselian case is reduced to the algebraically closed case in [6, Cor. 7.2]. That proof actually applies only in the case where M' is perfect. We modify that proof to cover all cases, including the real closed case.

The absolute value $||_\mathfrak{p}$ of M' uniquely extends to an absolute value $||_\mathfrak{p}$ of the algebraic closure $\tilde{M}$ of M'. For each $1 \le i \le l$, the inequalities that define U_i define a $\mathfrak{p}$-adic open neighborhood $\tilde{U}_i$ of P_i in $\Gamma(\tilde{M})$ such that $\tilde{U}_i \cap \Gamma(M') = U_i$ and $\tilde{U}_i^\sigma = \tilde{U}_i$ for each $\sigma \in \mathrm{Aut}(\tilde{M}/M')$. Since the P_i's are distinct and the $\mathfrak{p}$-topology is Hausdorff, we can make the U_i's smaller, if necessary, to assume that the $\tilde{U}_i$'s are disjoint. Moreover, since M'/K_1 is separable, $P_1, \ldots, P_l$ are M'-normal, hence also smooth (= nonsingular). That is, each P_i satisfies the Jacobian condition in an appropriate affine neighborhood [19, p. 233, Cor. 1]. That condition does not change under extension of the base field. Thus, $P_1, \ldots, P_l$ are also smooth over $\tilde{M}$. Therefore, we may make the U_i smaller to assume that each point of $\tilde{U}_i$ is smooth, hence $\tilde{M}$-normal. Thus, we may again identify each point of $\tilde{U}_i$ with a prime divisor of $F\tilde{M}/\tilde{M}$.

The elements $u_1, \ldots, u_d$ being linearly independent over M' remain linearly independent over $\tilde{M}$ (because F'/M' is regular). In addition, they belong to $\mathcal{L}_{\tilde{M}}(A)$, so they can be extended to a basis $u_1, \ldots, u_d, u_{d+1}, \ldots, u_r$ of $\mathcal{L}_{\tilde{M}}(A)$.

For each $1 \le i \le l$, the proposition for $(\tilde{M}, ||_\mathfrak{p})$ gives an $\varepsilon > 0$ (which can be chosen to be independent of i) such that if $c'_1, \ldots, c'_r \in \tilde{M}$ satisfy $|c'_j - c_j|_\mathfrak{p} < \varepsilon$ for $j = 1, \ldots, d$, $|c'_j|_\mathfrak{p} < \varepsilon$ for $j = d+1, \ldots, r$, and $f' = \sum_{j=1}^r c'_j u_j$, then $\mathrm{div}_\infty(f') = A$ and $\mathrm{div}_0(f') = \sum_{i=1}^l P'_i$ with $P'_i \in \tilde{U}_i$ for $i = 1, \ldots, l$. In particular, P'_i is $\tilde{M}$-normal and

$$\begin{aligned}\deg(\mathrm{div}_0(f')) &= \deg(\mathrm{div}_\infty(f'))\\ &= \deg(A) = \deg(\mathrm{div}_\infty(f)) = \deg(\mathrm{div}_0(f)) = l.\end{aligned} \tag{4.1}$$

If, in addition, $c'_1, \ldots, c'_d \in M'$ and $c'_{d+1}, \ldots, c'_r = 0$, then $f' \in M'F$. By (4.1), $\sum_{i=1}^l \deg(P'_i) = l$. In addition, $P'_1, \ldots, P'_l$ lie in disjoint sets $\tilde{U}_1, \ldots, \tilde{U}_l$, so they are distinct. By Lemma 4.2, $P'_1, \ldots, P'_l \in \Gamma(M'_s)$. Moreover, for each $\sigma \in \mathrm{Gal}(M')$, we have $(P'_i)^\sigma \in \tilde{U}_i$, hence $(P'_i)^\sigma = P'_i$. Consequently, $P'_i \in \tilde{U}_i \cap \Gamma(M') = U_i$, as desired. ■

Corollary 4.4 (Principle of variation of constants). *Let $f \in NF$ be a $\mathfrak{p}$-admissible function for a prime $\mathfrak{p} \in \mathcal{V}_N$. Set $A = \mathrm{div}_\infty(f)$, let $u_1, \ldots, u_d \in NF$ be a basis for $\mathcal{L}_N(A)$, and write $f = \sum_{j=1}^d c_j u_j$ with $c_j \in N$. Then there exists $\varepsilon > 0$ such that if $c'_1, \ldots, c'_d \in N$ satisfy $|c'_j - c_j|_\mathfrak{p} < \varepsilon$, $j = 1, \ldots, d$ and $f' = \sum_{j=1}^d c'_j u_j$, then f' is $\mathfrak{p}$-admissible and $\mathrm{div}_\infty(f') = A$.*

Proof. By assumption, $\mathrm{div}_0(f) = \sum_{i=1}^l P_i$, with $P_i \in \Gamma(N_\mathfrak{p})$ distinct and $\mathbf{x}(P_i) \in D^n_{N,\mathfrak{p}}$. Also, $|\mathbf{x}(P_i) - \mathbf{x}(P_\mathfrak{p})|_\mathfrak{p} < \varepsilon_0$ if $\mathfrak{p} \in \mathcal{W}_N$, $i = 1, \ldots, l$. Now we apply Proposition 4.3 to the case where $M' = N_\mathfrak{p}$ and the U_i are disjoint $\mathfrak{p}$-open neighborhoods of P_i which are contained in the $\mathfrak{p}$-open subset $\{P \in \Gamma(N_\mathfrak{p}) \mid \mathbf{x}(P) \in D^n_{N,\mathfrak{p}}$ and $|\mathbf{x}(P) - \mathbf{x}(P_\mathfrak{p})|_\mathfrak{p} < \varepsilon_0$ if $\mathfrak{p} \in \mathcal{W}_N\}$. ■

Proposition 4.5 (Existence theorem for a single valuation). *Let $\mathfrak{p} \in \mathcal{V}_N$. Then there exists a positive integer $m_\mathfrak{p}$ such that for each multiple m of $m_\mathfrak{p}$ there exists a $\mathfrak{p}$-admissible function $f \in F$ such that $\mathrm{div}_\infty(f) = mD^*$.*

Proof. Recall that $P_{\mathfrak{p}} \in \Gamma(N_{\mathfrak{p}}) = \Gamma(M_{\mathfrak{p}})$ (Data 3.6 and Remark 2.3(c)). Choose a finite subextension L of M/K_1 such that $P_{\mathfrak{p}}$ is $L_{\mathfrak{p}}$-rational. Let $\hat{L}$ be the completion of $L_{\mathfrak{p}}$.

Since L is a global field, $\hat{L}$ is a local field. Since $\mathbf{x}(P_{\mathfrak{p}}) \in D^n_{N,\mathfrak{p}}$ the $\mathfrak{p}$-open subset

$$U = \{P \in \Gamma(\hat{L}) \mid \mathbf{x}(P) \in D^n_{N,\mathfrak{p}} \text{ and } |\mathbf{x}(P) - \mathbf{x}(P_{\mathfrak{p}})|_{\mathfrak{p}} < \varepsilon_0 \text{ if } \mathfrak{p} \in \mathcal{W}_N\}$$

of $\Gamma(\hat{L})$ is not empty. Theorem 2.1 of [6] improves the existence theorem of Jacobi-Rumely and gives a nonconstant function $g \in \hat{L}F_1$ whose pole divisor is a multiple of D^*. (Note that by Assumption 4.1, D^* is $\hat{L}$-rational.) Moreover, the zeros, $P_1, \ldots, P_l$ of g are $\hat{L}$-rational, simple, and belong to U. By [6, Remark 2.5], there exists a positive integer $m_{\mathfrak{p}}$ such that for each multiple m of $m_{\mathfrak{p}}$ the function g can be chosen with $\mathrm{div}_\infty(g) = mD^*$

Let $u_1, \ldots, u_d \in LF_1$ be a basis for $\mathcal{L}_L(mD^*)$. Assume without loss that $\hat{L}$ is linearly disjoint from LF_1 over L. Since $\hat{L}/L$ is separable, $u_1, \ldots, u_d$ also form a basis for $\mathcal{L}_{\hat{L}}(mD^*)$. Hence, there exist $b_1, \ldots, b_d \in \hat{L}$ such that $g = \sum_{j=1}^d b_j u_j$. Use the density of L in $\hat{L}$ to choose $\mathbf{c} \in L^d \subseteq M^d$ which is $\mathfrak{p}$-close to $\mathbf{b}$. Let $f = \sum_{j=1}^d c_j u_j$. Apply Proposition 4.3 to g, f, and $\hat{L}$ instead of to f, f', and M' (choose U_i disjoint and contained in U) to conclude that $\mathrm{div}_\infty(f) = mD^*$, each of the zeros of f is simple and belongs to U. In particular, f is $\mathfrak{p}$-admissible. ■

Lemma 4.6. *Let L be an extension of K_1 in M, $\mathfrak{p} \in \mathcal{V}_N$, and $\sigma \in \mathrm{Gal}(N/L)$. Extend σ to an element of $\mathrm{Aut}(NF/LF_1)$ with the same notation. Suppose that a function $f \in NF$ is $\mathfrak{p}$-admissible. Then f^σ is $\mathfrak{p}^\sigma$-admissible. In particular, if $f \in LF_1$, then f is $\mathfrak{p}^\sigma$-admissible.*

Proof. Since LF_1 is linearly disjoint from $N_{\mathfrak{p}}$ over L, we may extend σ to an isomorphism $\sigma\colon N_{\mathfrak{p}}F \to N_{\mathfrak{p}^\sigma}F$. By assumption $\mathrm{div}(f) = \sum_{j=1}^l P_j - mD^*$, where the P_j are distinct elements of $\Gamma(N_{\mathfrak{p}})$, m is a positive integer, $\mathbf{x}(P_j) \in D^n_{N,\mathfrak{p}}$ and $|\mathbf{x}(P_j) - \mathbf{x}(P_{\mathfrak{p}})|_{\mathfrak{p}} < \varepsilon_0$ if $\mathfrak{p} \in \mathcal{W}_N$. Apply σ to get $\mathrm{div}(f^\sigma) = \sum_{j=1}^l P_j^\sigma - mD^*$, $\mathbf{x}(P_j^\sigma) \in D^n_{N,\mathfrak{p}^\sigma}$, and $|\mathbf{x}(P_j^\sigma) - \mathbf{x}(P_{\mathfrak{p}^\sigma})|_{\mathfrak{p}^\sigma} < \varepsilon_0$ if $\mathfrak{p} \in \mathcal{W}_N$. Also, $P_1^\sigma, \ldots, P_l^\sigma$ are distinct. Therefore, f^σ is $\mathfrak{p}^\sigma$-admissible. ■

Proposition 4.7 (Existence theorem for finitely many valuations). *Let $\mathcal{T}$ be a finite subset of $\mathcal{V}_N$. Then, for each m_0, there exists a $\mathcal{T}$-admissible function $f \in F$ of level $\geq m_0$.*

Proof. Let $\mathcal{T}_0$ be a subset of $\mathcal{T}$ which represents $\mathcal{T}|_M$ (Definition 3.8). For each $\mathfrak{p} \in \mathcal{T}_0$ let $m_{\mathfrak{p}}$ be the positive integer that Proposition 4.5 gives. Choose a common multiple $m \geq m_0$ of the $m_{\mathfrak{p}}$'s. For each $\mathfrak{p} \in \mathcal{T}_0$ take $f_{\mathfrak{p}} \in F$ which is $\mathfrak{p}$-admissible of level m. Let $u_1, \ldots, u_d$ be a basis for $\mathcal{L}_M(mD^*)$ and write $f_{\mathfrak{p}} = \sum_{j=1}^d c_{\mathfrak{p}j} u_j$ with $c_{\mathfrak{p}j} \in M$.

Apply the weak approximation theorem to $\mathcal{T}_0|_M$ and choose $\mathbf{c} \in M^d$ which is $\mathfrak{p}$-close to $\mathbf{c}_{\mathfrak{p}}$ for each $\mathfrak{p} \in \mathcal{T}_0$. By Corollary 4.4, $f = \sum_{j=1}^d c_j u_j$ is $\mathfrak{p}$-admissible for each $\mathfrak{p} \in \mathcal{T}_0$ and $\mathrm{div}_\infty(f) = mD^*$. By Lemma 4.6, with M replacing L, f is $\mathfrak{p}$-admissible for each $\mathfrak{p} \in \mathcal{T}$. ■

5. Small sets

We use Proposition 4.7 and the weak approximation theorem to prove an existence and density theorem for admissible functions in F along a given small set. An essential tool in this step is Lemma 1.7.

Lemma 5.1. *Let E/L be a function field of one variable and let m be an integer $\geq 2\,\mathrm{genus}(E/L)$. Consider positive divisors $D_1, \ldots, D_l$ of E/L with $\deg(D_i) = r_i$. Suppose $D_1, \ldots, D_l$ are relatively prime in pairs. Then*

$$\dim(\mathcal{L}_L(mD_i)/\mathcal{L}_L((m-1)D_i)) = r_i, \qquad i = 1, \ldots, l. \tag{5.1}$$

Let $y_{i1}, \ldots, y_{i,r_i}$ be a basis for $\mathcal{L}_L(mD_i)$ modulo $\mathcal{L}_L((m-1)D_i)$ and set $D = D_1 + \cdots + D_l$. Then, y_{ij}, $i = 1, \ldots, l$, $j = 1, \ldots, r_i$ form a basis for $\mathcal{L}_L(mD)$ modulo $\mathcal{L}_L((m-1)D)$. Moreover, if each D_i is a prime divisor, then $\mathrm{div}_\infty(y_{ij}) = mD_i$ for all i and j.

Proof. By Riemann-Roch, (5.1) above and (5.2) below are true:

$$\dim(\mathcal{L}_L(mD)/\mathcal{L}_L((m-1)D)) = \deg(D). \tag{5.2}$$

Since $y_{ij} \in \mathcal{L}_L(mD)$, it suffices to prove that they are linearly independent modulo $\mathcal{L}_L((m-1)D)$. Indeed, suppose

$$\sum_{i=1}^{l}\sum_{j=1}^{r_i} a_{ij}y_{ij} \equiv 0 \mod \mathcal{L}_L((m-1)D) \tag{5.3}$$

with $a_{ij} \in L$. Write $D_i = \sum_{k=1}^{l_i} e_{ik}P_{ik}$ with positive integers e_{ik} and distinct prime divisors P_{ik} of E/L. Denote the normalized valuation of E/L corresponding to P_{ik} by v_{ik}. Then $v_{ik}(y_{i'j}) \geq 0$ if $i' \neq i$. It follows from (5.3) that $v_{ik}(\sum_{j=1}^{r_i} a_{ij}y_{ij}) \geq -(m-1)e_{ik}$. Hence, $\sum_{j=1}^{r_i} a_{ij}y_{ij}$ belongs to $\mathcal{L}_L((m-1)D_i)$. By the choice of the y_{ij}, this implies that $a_{ij} = 0$ for $j = 1, \ldots, r_i$.

Finally, if D_i is a prime divisor, then in the above notation, $v_{i1}(y_{ij}) \geq -m$ and $v_{i1}(y_{ij}) \not\geq -m+1$. Hence, $v_{i1}(y_{ij}) = -m$ and $\mathrm{div}_\infty(y_{ij}) = mD_i$. ■

We use Lemma 5.1 to construct a basis for $\mathcal{L}_M(mD^*)$ modulo $\mathcal{L}_M((m-1)D^*)$ which will belong to a finitely generated subgroup of $F^\times$ that does not depend on m. This requires an additional data.

Data 5.2. Let $s = 2\mathrm{genus}(F/M) + 2$. By Remark 3.2(b), $\deg(P_i^*) = d_ie_i$ for each $1 \leq i \leq e$. Then for each $s \leq r \leq 2s-1$ let

$$\begin{aligned} B_{ir} = \{u_{ijkr} \mid j = 1, \ldots, d_i,\ k = 1, \ldots, e_i\} &\text{ be a basis for } \mathcal{L}_M(rP_i^*) \\ \text{modulo } \mathcal{L}_M((r-1)P_i^*) &\text{ in particular } \mathrm{div}_\infty(u_{ijkr}) = rP_i^* \end{aligned}$$

Now write each $m \geq s$ as $m = qs + r$ with $q \geq 0$ and $s \leq r \leq 2s-1$. Then let

$$\begin{aligned}
u_{ijkm} &= u_{i11s}^q u_{ijkr} \\
B_{im} &= \{u_{ijkm} \mid j = 1, \ldots, d_i,\ k = 1, \ldots, e_i\} \\
B_m &= B_{1m} \cup \cdots \cup B_{em} \\
B_0 &= \text{basis for } \mathcal{L}_M((s-1)D^*) \text{ which contains } 1 \\
K_2 &= \text{a finite subextension of } M/K_1 \text{ such that } B_0 \cup B_s \cup \cdots \cup B_{2s-1} \subseteq K_2F_1 \\
F_2 &= K_2F_1
\end{aligned}$$

Lemma 5.3. *Let $m \geq s$. Then:*

(a) $\mathrm{div}_\infty(u_{ijkm}) = mP_i^*$ and B_{im} is a basis for $\mathcal{L}_M(mP_i^*)$ modulo $\mathcal{L}_M((m-1)P_i^*)$, $i = 1, \ldots, e$, $j = 1, \ldots, d_i$, $k = 1, \ldots, e_i$.
(b) B_m is a basis of $\mathcal{L}_M(mD^*)$ modulo $\mathcal{L}_M((m-1)D^*)$.
(c) F_2 contains a basis for $\mathcal{L}_M(mD^*)$.

Proof of (a). Let $m = qs + r$ as in Data 5.2. Then $u_{i11s}^q u_{ijkr}$, $j = 1, \ldots, d_i$, $k = 1, \ldots, e_i$, belong to $\mathcal{L}_M(mP_i^*)$ and are linearly independent over M modulo $\mathcal{L}_M((m-1)P_i^*)$. We conclude from (5.1) applied to P_i^* rather than to D_i that these elements form a basis for $\mathcal{L}_M(mP_i^*)$ modulo $\mathcal{L}_M((m-1)P_i^*)$.

Proof of (b) Apply Lemma 5.1 to $D^* = P_1^* + \cdots + P_e^*$ rather than to $D = D_1 + \cdots + D_l$.

Proof of (c) By our choice of B_0 and by (b), $B_0 \cup B_s \cup \cdots \cup B_m \subseteq F_2$ and is a basis for $\mathcal{L}_M(mD^*)$. ∎

Notation 5.4. Following Lemma 5.3, we set for each $m \geq s-1$

$$\nu_m = \dim(\mathcal{L}_M(mD^*))$$

By Riemann-Roch, $\nu_m \geq 2$. Then we list the elements of $B_0 \cup B_s \cup B_{s+1} \cup B_{s+2} \cup \cdots$ as $u_1, u_2, u_3, \ldots$ such that $u_1 = 1$, $B_0 = \{u_1, \ldots, u_{\nu_{s-1}}\}$, and

$$B_m = \{u_{\nu_{m-1}+1}, \ldots, u_{\nu_m}\} = \{u_{ijkm} \mid i = 1, \ldots, e,\ j = 1, \ldots, d_i,\ k = 1, \ldots, e_i\}$$

for $m \geq s$. By Data 5.2, all of the u_i belong to F_2 and $\{u_1, \ldots, u_{\nu_m}\}$ is a basis for $\mathcal{L}_{M'}(mD^*)$ for each separable algebraic extension M' of M.

Proposition 5.5. *Let $\mathcal{T}$ be a small subset of $\mathcal{V}_N$. Then, for each m_0 there exists a $\mathcal{T}$-admissible function $f \in F$ of level $m \geq m_0$.*

Moreover, write $f = \sum_{i=1}^{\nu_m} c_i u_i$ with $c_i \in M$. Then there exists $\varepsilon > 0$ such that if $\mathbf{c}' \in N^{\nu_m}$ satisfies $|\mathbf{c}' - \mathbf{c}|_{\mathcal{T}} < \varepsilon$, then $f' = \sum_{i=1}^{\nu_m} c_i' u_i$ is also a $\mathcal{T}$-admissible function of level m.

Proof. Let $\mathcal{T}_2$ be a finite subset of $\mathcal{T}$ which represents $\mathcal{T}|_{K_2}$ (Definition 3.8). By Proposition 4.7, there exists a $\mathcal{T}_2$-admissible function $g \in F$ of level $m \geq \max(m_0, s-1)$. Write $g = \sum_{i=1}^{\nu_m} a_i u_i$ with $a_i \in M$. By Corollary 4.4, there exists $\delta > 0$ such that for each $\mathfrak{q} \in \mathcal{T}_2$ if $\mathbf{a}' \in N^{\nu_m}$ satisfies $|\mathbf{a}' - \mathbf{a}|_\mathfrak{q} < \delta$, then $g' = \sum_{i=1}^{\nu_m} a_i' u_i$ is $\mathfrak{q}$-admissible of level m.

Let K_2' be a finite Galois subextension of N/K_2 which contains $a_1, \ldots, a_{\nu_m}$. Then $A = \{\mathbf{a}^\sigma \mid \sigma \in \mathrm{Gal}(N/K_2)\} = \{\mathbf{a}^\sigma \mid \sigma \in \mathrm{Gal}(K_2'/K_2)\}$ is a finite subset of N. We have not assumed M to be normal over K. Hence, A need not be a subset of M. However, by Lemma 1.7, M^{ν_m} has a finite subset B with the following property: For all $\mathfrak{q} \in \mathcal{T}_2$, $\tau \in \mathrm{Gal}(N/K_2)$, and $\mathbf{a}' \in A$ there exists $\mathbf{b}_{\mathfrak{q}^\tau,\mathbf{a}'} \in B$ such that $|\mathbf{b}_{\mathfrak{q}^\tau,\mathbf{a}'} - \mathbf{a}'|_{\mathfrak{q}^\tau} < \delta$. Choose a finite subextension K_3 of M/K_2 such that $B \subseteq K_3^{\nu_m}$.

Now let $\mathfrak{p} \in \mathcal{T}$. Then there exists $\sigma \in \mathrm{Gal}(N/K_2)$ and $\mathfrak{q} \in \mathcal{T}_2$ such that $\mathfrak{p} = \mathfrak{q}^\sigma$. Since $\mathbf{a}' = \mathbf{a}^\sigma$ belongs to A, we have $|\mathbf{b}_{\mathfrak{p},\mathbf{a}'} - \mathbf{a}^\sigma|_{\mathfrak{q}^\sigma} < \delta$, so $|\mathbf{b}_{\mathfrak{p},\mathbf{a}'}^{\sigma^{-1}} - \mathbf{a}|_\mathfrak{q} < \delta$. Hence, by the first paragraph, $\sum_{i=1}^{\nu_m} b_{\mathfrak{p},\mathbf{a}',i}^{\sigma^{-1}} u_i$ is a $\mathfrak{q}$-admissible function of level m. Since $u_i \in F_2$ (Notation 5.4), we have $u_i^\sigma = u_i$, $i = 1, \ldots, \nu_m$. Hence, by Lemma 4.6, with K_2 rather than L, the element $f_\mathfrak{p} = \sum_{i=1}^{\nu_m} b_{\mathfrak{q},\mathbf{a}',i} u_i$ of K_3F_1 is a $\mathfrak{p}$-admissible function of level m.

Next choose a finite subset $\mathcal{T}_3$ of $\mathcal{T}$ which represents $\mathcal{T}|_{K_3}$. By the preceding paragraph, for each $\mathfrak{q} \in \mathcal{T}_3$ there exists a $\mathfrak{q}$-admissible function $f_\mathfrak{q} = \sum_{i=1}^{\nu_m} c_{\mathfrak{q},i} u_i$ of level m with $c_{\mathfrak{q},i} \in K_3$. By Corollary 4.4, there exists $\varepsilon > 0$ such that if $\mathfrak{q} \in \mathcal{T}_3$ and $\mathbf{c}' \in N^{\nu_m}$ satisfy $|\mathbf{c}' - \mathbf{c}_\mathfrak{q}|_\mathfrak{q} < \varepsilon$, then $f' = \sum_{i=1}^{\nu_m} c_i' u_i$ is a $\mathfrak{q}$-admissible function of level m.

By the weak approximation theorem there exists $\mathbf{c} \in K_3^{\nu_m}$ such that $|\mathbf{c} - \mathbf{c}_\mathfrak{q}|_\mathfrak{q} < \varepsilon$ for each $\mathfrak{q} \in \mathcal{T}_3$. Then $f = \sum_{i=1}^{\nu_m} c_i u_i$ is $\mathfrak{q}$-admissible of level m for each $\mathfrak{q} \in \mathcal{T}_3$. For each $\sigma \in \mathrm{Gal}(N/K_3)$ we have $f^\sigma = f$. Hence, by Lemma 4.6, f is $\mathfrak{q}^\sigma$-admissible. It follows that f is $\mathcal{T}$-admissible.

Finally suppose that $\mathbf{c}' \in N^{\nu_m}$ and $|\mathbf{c}' - \mathbf{c}|_\mathfrak{p} < \varepsilon$ for each $\mathfrak{p} \in \mathcal{T}$. Write $\mathfrak{p} = \mathfrak{q}^\sigma$ with $\mathfrak{q} \in \mathcal{T}_3$ and $\sigma \in \mathrm{Gal}(N/K_3)$. Then $|(\mathbf{c}')^{\sigma^{-1}} - \mathbf{c}|_\mathfrak{q} < \varepsilon$ and hence $\sum_{i=1}^{\nu_m} (c_i')^{\sigma^{-1}} u_i$ is $\mathfrak{q}$-admissible of level m. Consequently, by Lemma 4.6, $f' = \sum_{i=1}^{\nu_m} c_i' u_i$ is $\mathfrak{p}$-admissible of level m. ■

6. Good reduction

Consider a **finite prime** $\mathfrak{p}$ of $\tilde{K}F$ such that $\mathfrak{p}|_K \in \mathcal{V} \setminus \mathcal{S}$. Thus, $\mathfrak{p}$ is an equivalence class of valuations of $\tilde{K}F$ and $|\ |_\mathfrak{p}$ is a multiplicative representative of $\mathfrak{p}$. In other words, $|\ |_\mathfrak{p}$ is a map of $\tilde{K}F$ into an ordered multiplicative Abelian group satisfying the usual rules of an absolute value. We use a bar over objects associated with $\tilde{K}F$ to denote their reduction modulo $\mathfrak{p}$.

The function field $\tilde{K}F/\tilde{K}$ has a **good reduction** at $\mathfrak{p}$ if the following conditions hold:

(1a) There exists $f \in \tilde{K}F$ which is **$\mathfrak{p}$-regular**. That is, $|f|_\mathfrak{p} = 1$, $\bar{f} \in \overline{\tilde{K}F}$ is transcendental over $\overline{\tilde{K}}$, and $[\tilde{K}F : \tilde{K}(f)] = [\overline{\tilde{K}F} : \overline{\tilde{K}}(\bar{f})]$. Thus $\overline{\tilde{K}F}$ is a function field of one variable over $\overline{\tilde{K}}$.

(1b) $\mathrm{genus}(\tilde{K}F/\tilde{K}) = \mathrm{genus}(\overline{\tilde{K}F}/\overline{\tilde{K}})$.

In this case we also say that $\mathfrak{p}$ is a **good extension** of $\mathfrak{p}|_{\tilde K}$ to $\tilde KF$. Note that if $g \in \tilde KF$ and $\bar g$ is transcendental over $\bar{\tilde K}$, then g is $\mathfrak{p}$-regular if and only if $\deg(\mathrm{div}_0(g)) = \deg(\mathrm{div}_0(\bar g))$ or, equivalently, $\deg(\mathrm{div}_\infty(g)) = \deg(\mathrm{div}_\infty(\bar g))$.

The **support** of a divisor A of $\tilde KF/\tilde K$ is the set $P_1,\ldots,P_l$ of distinct prime divisors of $\tilde KF/\tilde K$ such that $A = \sum_{i=1}^l k_iP_i$ with nonzero integers k_i.

Corollary 6.2 connects regularity and admissibility of functions. It relies on a sort of a reciprocity lemma:

Lemma 6.1 ([22, Cor. 3.9]). *Suppose that $\tilde KF/\tilde K$ has a good reduction at a finite prime $\mathfrak{p}$. Let f,g be elements of $\tilde KF$ such that f is $\mathfrak{p}$-regular and $|g|_\mathfrak{p} = 1$. Then, for each $P \in \Gamma(\tilde K)$*

$$\mathrm{Supp}(\mathrm{div}_\infty(g)) \subseteq \mathrm{Supp}(\mathrm{div}_\infty(f)) \textit{ and } f(P) = 0 \textit{ imply } |g(P)|_\mathfrak{p} \le 1.$$

We extend each finite prime $\mathfrak{p} \in \mathcal{V}_N \smallsetminus \mathcal{W}_N$ to the Henselian closure $N_\mathfrak{p} = K_s$ (recall that by Data 2.1(d), $\mathcal{S}_N \subseteq \mathcal{W}_N$) and then, in the unique possible way, to $\tilde K$. In this way we regard $\mathfrak{p}$ also as a prime of $\tilde K$.

Corollary 6.2. *Let $\mathfrak{p} \in \mathcal{V}_N \smallsetminus \mathcal{W}_N$ be a finite prime with a good extension to $\tilde KF$. Suppose $|x_i|_\mathfrak{p} = 1$ if $x_i \ne 0$, for $i = 1,\ldots,n$. Let $f \in NF$ be a $\mathfrak{p}$-regular function of level m (Definition 3.7). Suppose each of the zeros of f is simple. Then f is $\mathfrak{p}$-admissible.*

Proof. Since $N_\mathfrak{p} = K_s$, we have to verify only Condition (2c) of Definition 3.7. By assumption $\mathrm{div}_\infty(f) = mD^*$. Hence, by Data 3.1,

$$\mathrm{Supp}(\mathrm{div}_\infty(f)) = \bigcup_{i=1}^n \mathrm{Supp}(\mathrm{div}_\infty(x_i)).$$

By assumption, $|x_i|_\mathfrak{p} = 1$ if $x_i \ne 0$. Hence, if $P \in \Gamma(\tilde K)$ is a zero of f, then $|x_i(P)|_\mathfrak{p} \le 1$ (Lemma 6.1). If $x_i = 0$, then $|x_i(P)|_\mathfrak{p} = 0 < 1$. Consequently, f is $\mathfrak{p}$-admissible. ∎

In the remaining of this section we explore when functions are regular. This depends on the following extension of the reduction map of elements modulo $\mathfrak{p}$ to divisors.

Proposition 6.3 ([22, p. 247]). *Suppose $\tilde KF/\tilde K$ has a good reduction at $\mathfrak{p}$. Then there is a natural homomorphism $A \mapsto \bar A$ of $\mathrm{Div}(\tilde KF/\tilde K)$ into $\mathrm{Div}(\overline{\tilde KF}/\bar{\tilde K})$ with the following properties:*

(a) $\deg(A) = \deg(\bar A)$.
(b) $A \ge 0$ *implies* $\bar A \ge 0$.
(c) $|f|_\mathfrak{p} = 1$ *implies* $\overline{\mathrm{div}(f)} = \mathrm{div}(\bar f)$.

Lemma 6.4 ([12, Lemma 5.4]). *Suppose $\tilde KF/\tilde K$ has a good reduction at $\mathfrak{p}$ and let f be an element of $\tilde KF$ such that $\bar f$ is transcendental over $\bar M$. Then $\mathrm{div}_0(\bar f) \le \overline{\mathrm{div}_0(f)}$ and $\mathrm{div}_\infty(\bar f) \le \overline{\mathrm{div}_\infty(f)}$. Equality holds if and only if f is $\mathfrak{p}$-regular.*

Lemma 6.5 ([12, Lemma 5.5]). *Suppose $\tilde{K}F/\tilde{K}$ has a good reduction at $\mathfrak{p}$. Let A be a positive divisor of $\tilde{K}F/\tilde{K}$. For each i between 1 and l let m_i be a positive integer and let $f_i \in \tilde{K}F$ be a $\mathfrak{p}$-regular function such that $\mathrm{div}_\infty(f_i) = m_i A$. Let $m = m_1 + \cdots + m_l$. Then $f = f_1 \cdots f_l$ is also $\mathfrak{p}$-regular and $\mathrm{div}_\infty(f) = mA$.*

The following result is a well known consequence of the Bertini-Noether theorem. For example, it appears in [23] without a proof. See [12, Prop. 5.7] for a sketch of the proof.

Proposition 6.6. *Let $t_1, \ldots, t_l$ be nonconstant functions of $\tilde{K}F$ and let $P_1, \ldots, P_m$ be distinct prime divisors of $\tilde{K}F/\tilde{K}$. Then there exists a finite subset A of $K^\times$ such that if $\mathfrak{p} \in \tilde{\mathcal{V}}$ satisfies $|a|_\mathfrak{p} = 1$ for each $a \in A$, then $\mathfrak{p}$ has a good extension to $\tilde{K}F$ which we also denote by $\mathfrak{p}$ such that t_i is $\mathfrak{p}$-regular, $i = 1, \ldots, l$, and the reduced primes $\bar{P}_1, \ldots, \bar{P}_m$ are distinct.*

7. Criteria for regularity

We give here two criteria for regularity of functions of $\tilde{K}F$. The first one is formulated in terms of a basis of $\mathcal{L}_{\tilde{K}}(me_i P_{ij}^*)$ modulo $\mathcal{L}_{\tilde{K}}((me_i - 1)P_{ij}^*)$ (Data 3.1). Here it is important that $\deg(P_{ij}^*) = 1$. The second one, which is built on the first one, is formulated in terms of a basis of $\mathcal{L}_{\tilde{K}}(mP_i^*)$ modulo $\mathcal{L}_{\tilde{K}}((m-1)P_i^*)$. In both criteria m has to be large.

In the following lemma we use the integers e and e_i from Data 3.1, but the lemma is valid for arbitrary positive integers e and e_i.

Lemma 7.1. *Let E/L be an algebraic function field of one variable, Q_{ij}, $i = 1, \ldots, e$, $j = 1, \ldots, d_i$, distinct prime divisors of E/L, and m a positive integer. Set $D = \sum_{i=1}^{e} \sum_{j=1}^{d_i} e_i Q_{ij}$, and $C = \sum_{i=1}^{e} \sum_{j=1}^{d_i} (me_i - 1) Q_{ij}$. For all i and j let $t_{ij} \in E$ and $c_{ij} \in L$. Let $g \in \mathcal{L}_L(C)$ and set $f = \sum_{i=1}^{e} \sum_{j=1}^{d_i} c_{ij} t_{ij} + g$. Suppose $\mathrm{div}_\infty(t_{ij}) = me_i Q_{ij}$ for all i, j. Then, $\mathrm{div}_\infty(f) = mD$ if and only if $c_{ij} \neq 0$ for all i, j.*

Proof. First suppose $c_{ij} \neq 0$ for all i, j. Denote the normalized valuation of E/L associated with Q_{ij} by v_{ij}. Then $v_{ij}(t_{ij}) = -me_i$ and $v_{ij}(t_{i'j'}) \geq 0$ if $(i', j') \neq (i, j)$. In addition, $v_{ij}(g) \geq -me_i + 1$. Hence, $v_{ij}(f) = -me_i$. Finally, $v(f) \geq 0$ for each valuation v of E/L which is different from all of the v_{ij}'s. Consequently, $\mathrm{div}_\infty(f) = mD$.

Conversely, if $c_{ij} = 0$ for some i, j, then $\mathrm{div}_\infty(f) \leq mD - Q_{ij}$. ■

Lemma 7.2 (First criterion for regularity). *Let m be an integer $\geq 2\,\mathrm{genus}\,(F/M)$, and let t_{ij} be an element of $\tilde{K}F$ such that $\mathrm{div}_\infty(t_{ij}) = me_i P_{ij}^*$, $i = 1, \ldots, e$, $j = 1, \ldots, d_i$. Set $C = \sum_{i=1}^{e} \sum_{j=1}^{d_i} (me_i - 1) P_{ij}^*$. Suppose $\tilde{K}F/\tilde{K}$ has good reduction at a finite prime $\mathfrak{p}$ such that the reduced primes $\overline{P_{ij}^*}$ are distinct and the t_{ij} are $\mathfrak{p}$-regular. Let*

$$f = \sum_{i=1}^{e} \sum_{j=1}^{d_i} c_{ij} t_{ij} + g \tag{7.1}$$

with $c_{ij} \in \tilde{K}$ such that $|c_{ij}|_{\mathfrak{p}} = 1$ and $g \in \mathcal{L}_{\tilde{K}}(C)$ with $|g|_{\mathfrak{p}} \le 1$. Then
(a) $\{t_{ij} \mid i = 1,\dots,e;\ j = 1,\dots,d_i\}$ is a basis for $\mathcal{L}_{\tilde{K}}(mD^*)$ modulo $\mathcal{L}_{\tilde{K}}(C)$, and
(b) f is $\mathfrak{p}$-regular of level m.

Proof of (a). By assumption,

$$me_i > me_i - 1 \ge 2\,\mathrm{genus}(F/M) - 1 \ge 2\,\mathrm{genus}(\tilde{K}F/\tilde{K}) - 1$$

[3, p. 132, Thm. 1]. In addition, $\deg(P_{ij}^*) = 1$, because $\tilde{K}$ is algebraically closed. Hence, by Riemann-Roch

$$\dim(\mathcal{L}_{\tilde{K}}(me_iP_{ij}^*)) - \dim(\mathcal{L}_{\tilde{K}}((me_i-1)P_{ij}^*)) = \deg(me_iP_{ij}^*) - \deg((me_i-1)P_{ij}^*) = 1.$$

By Data 3.1, $D^* = \sum_{i=1}^e P_i^*$. By Remark 3.2(b), $P_i^* = \sum_{j=1}^{d_i} e_iP_{ij}^*$, $i = 1,\dots,e$. Since the P_{ij}^* are distinct, Lemma 5.1 applied to $\{P_{ij}^* \mid j = 1,\dots,d_i\}$ and me_i rather than to $\{D_1,\dots,D_l\}$ and m, implies that $\{t_{ij} \mid j = 1,\dots,d_i\}$ form a basis for $\mathcal{L}_{\tilde{K}}(mP_i^*)$ modulo $\mathcal{L}_{\tilde{K}}(\sum_{j=1}^{d_i}(me_i-1)P_{ij}^*)$, $i = 1,\dots,e$.

By Riemann-Roch again,

$$\dim(\mathcal{L}_{\tilde{K}}(mD^*)/\mathcal{L}_{\tilde{K}}(C)) = \sum_{i=1}^e\sum_{j=1}^{d_i} me_i - \sum_{i=1}^e\sum_{j=1}^{d_i}(me_i-1) = \sum_{i=1}^e d_i\,.$$

Since $t_{ij} \in \mathcal{L}_{\tilde{K}}(mD^*)$, it suffices to prove that they are linearly independent modulo $\mathcal{L}_{\tilde{K}}(C)$. Indeed, suppose $\sum_{i=1}^e\sum_{j=1}^{d_i} a_{ij}t_{ij} \equiv 0 \mod \mathcal{L}_{\tilde{K}}(C)$ with $a_{ij} \in \tilde{K}$. Denote the normalized valuation of $\tilde{K}F/\tilde{K}$ corresponding to P_{ik}^* by v_{ik}. Then $v_{ik}(t_{i'j}) \ge 0$ if $i' \ne i$. Since $v_{ik}(\sum_{i'=1}^e\sum_{j=1}^{d_{i'}} a_{i'j}t_{i'j}) \ge -(me_i-1)$, we have $v_{ik}(\sum_{j=1}^{d_i} a_{ij}t_{ij}) \ge -(me_i-1)$. Hence, $\sum_{j=1}^{d_i} a_{ij}t_{ij} \in \mathcal{L}_{\tilde{K}}(\sum_{j=1}^{d_i}(me_i-1)P_{ij}^*)$. By the preceding paragraph, this implies that $a_{ij} = 0$ for $j = 1,\dots,d_i$.

Proof of (b). By Lemma 7.1, $\mathrm{div}_\infty(f) = mD^*$.

Now reduce (7.1) modulo $\mathfrak{p}$ to obtain $\bar{f} = \sum_{i=1}^e\sum_{j=1}^{d_i}\bar{c}_{ij}\bar{t}_{ij} + \bar{g}$. By assumption $A = \mathrm{div}(g) + C \ge 0$. If $|g|_{\mathfrak{p}} < 1$, then $\bar{g} = 0$. Otherwise, $|g|_{\mathfrak{p}} = 1$ and $\mathrm{div}(\bar{g}) + \bar{C} = \bar{A} \ge 0$ (Proposition 6.3). Hence, in both cases $\bar{g} \in \mathcal{L}_{\overline{\tilde{K}}}(\bar{C})$. Since t_{ij} is $\mathfrak{p}$-regular, $\mathrm{div}_\infty(\bar{t}_{ij}) = me_i\overline{P_{ij}^*}$ (Lemma 6.4). By assumption, $\bar{c}_{ij} \ne 0$ for all i,j. Hence, we may apply Lemma 7.1 to $\overline{\tilde{K}F}/\overline{\tilde{K}}$ and conclude that $\mathrm{div}_\infty(\bar{f}) = m\overline{D^*} = \overline{\mathrm{div}_\infty(f)}$. Thus, by Lemma 6.4, f is $\mathfrak{p}$-regular of level m. ■

Data 7.3. We write each $m \ge s = 2\,\mathrm{genus}(F/M) + 2$ as $m = qs + r$ with $q \ge 0$ and $s \le r \le 2s-1$.

(a) We use the Riemann-Roch theorem to choose $t_{ijr} \in \tilde{K}F$ which satisfy $\mathrm{div}_\infty(t_{ijr}) = re_iP_{ij}^*$, $i = 1,\dots,e$, $j = 1,\dots,d_i$, $r = s,\dots,2s-1$.
(b) Let $t_{ijm} = t_{ijs}^q t_{ijr}$, $i = 1,\dots,e$, $j = 1,\dots,d_i$.
(c) By Remark 3.2(b), $P_i^* = e_i(P_{i1}^* + \cdots + P_{id_i}^*)$. By Riemann-Roch, $\{t_{ijr}\}$ is a basis of $\mathcal{L}_{\tilde{K}}(re_iP_{ij}^*)$ modulo $\mathcal{L}_{\tilde{K}}((re_i-1)P_{ij}^*)$. Hence, by Lemma 5.1, $\{t_{ijr} \mid j = 1,\dots,d_i\}$ is a basis for $\mathcal{L}_{\tilde{K}}(rP_i^*)$ modulo $\mathcal{L}_{\tilde{K}}(\sum_{j=1}^{d_i}(re_i-1)P_{ij}^*)$. According to

Data 5.2, $\mathrm{div}_\infty(u_{ijkr}) = rP_i^*$, so $u_{ijkr} \in \mathcal{L}_{\tilde{K}}(rP_i^*)$. Thus, there exist unique $b_{ijkj'r} \in \tilde{K}$ such that

$$u_{ijkr} \equiv \sum_{j'=1}^{d_i} b_{ijkj'r} t_{ij'r} \mod \mathcal{L}_{\tilde{K}}\Big(\sum_{l=1}^{d_i}(re_i - 1)P_{il}^*\Big). \tag{7.2}$$

By Lemma 7.1, $b_{ijkj'r} \neq 0$.

(d) We set $\mathbf{Y}_i = (Y_{i11}, \ldots, Y_{i,d_i,e_i})$, $i = 1, \ldots, e$ and consider the linear form

$$\lambda_{ilr}(\mathbf{Y}_i) = \sum_{j=1}^{d_i}\sum_{k=1}^{e_i} Y_{ijk} b_{ijklr}, \qquad l = 1, \ldots, d_i$$ ■

Lemma 7.4 (Second criterion for regularity). *Let $m \geq s = 2\,\mathrm{genus}(F/M)+2$ and let $a_{ijk}, a_\mu \in \tilde{K}$, $i = 1, \ldots, e$, $j = 1, \ldots, d_i$, $k = 1, \ldots, e_i$, $\mu = 1, \ldots, \nu_{m-1}$, and let u_μ, $\mu = 1, 2, 3, \ldots$, be as in Notation 5.4. Consider the element*

$$f = \sum_{i=1}^{e}\sum_{j=1}^{d_i}\sum_{k=1}^{e_i} a_{ijk} u_{ijkm} + \sum_{\mu=1}^{\nu_{m-1}} a_\mu u_\mu \tag{7.3}$$

of $\tilde{K}F$. Suppose $\tilde{K}F$ has a good reduction at a finite prime $\mathfrak{p}$ such that the following conditions are satisfied:

(a) The $\overline{P_{ij}^*}$ are distinct,
(b) t_{ijr} is $\mathfrak{p}$-regular, in particular $|t_{ijr}|_\mathfrak{p} = 1$,
(c) $|u_\mu|_\mathfrak{p} \leq 1$,
(d) $|b_{ijkj'r}|_\mathfrak{p} = 1$,
(e) $|a_{ijk}|_\mathfrak{p} \leq 1$ and $|a_{\mu'}|_\mathfrak{p} \leq 1$, and
(f) $|\lambda_{ilr}(\mathbf{a}_i)|_\mathfrak{p} = 1$, where $\mathbf{a}_i = (a_{i11}, \ldots, a_{i,d_i,e_i})$,

for $\mu = 1, \ldots, \nu_{2s-1}$, $\mu' = 1, \ldots, \nu_{m-1}$, $i = 1, \ldots, e$, $j, j', l = 1, \ldots, d_i$, $k = 1, \ldots, e_i$, and $r = s, \ldots, 2s-1$. Then f is $\mathfrak{p}$-regular of level m.

Proof. Let $C = \sum_{i=1}^{e}\sum_{j=1}^{d_i}(me_i - 1)P_{ij}^*$. Write $m = qs + r$ with $q \geq 0$ and $s \leq r \leq 2s-1$. By (b) and Data 7.3(a), t_{ijs} is $\mathfrak{p}$-regular with $\mathrm{div}_\infty(t_{ijs}) = se_iP_{ij}^*$ and t_{ijr} is $\mathfrak{p}$-regular with $\mathrm{div}_\infty(t_{ijr}) = re_iP_{ij}^*$. Hence, by Lemma 6.5, $t_{ijm} = t_{ijs}^q t_{ijr}$ is $\mathfrak{p}$-regular with $\mathrm{div}_\infty(t_{ijm}) = me_iP_{ij}^*$, for $i = 1, \ldots, e$, $j = 1, \ldots, d_i$.

By Data 5.2 and by (7.2)

$$\begin{aligned} u_{ijkm} &= u_{i11s}^q u_{ijkr} \\ &\equiv \Big(\sum_{j'=1}^{d_i} b_{i11j's} t_{ij's}\Big)^q \Big(\sum_{j'=1}^{d_i} b_{ijkj'r} t_{ij'r}\Big) \mod \mathcal{L}_{\tilde{K}}\Big(\sum_{l=1}^{d_i}(me_i - 1)P_{il}^*\Big). \end{aligned} \tag{7.4}$$

A general term of the expansion of the right hand side of (7.4) has the form bt, where $b = b_{i,1,1,j_1',s} \cdots b_{i,1,1,j_q',s} b_{i,j,k,j_{q+1}',r}$ and $t = t_{i,j_1',s} \cdots t_{i,j_q',s} t_{i,j_{q+1}',r}$ and $1 \leq j_1', \ldots, j_{q+1}' \leq d_i$. For each l between 1 and d_i denote the normalized valuation of $\tilde{K}F/\tilde{K}$ associated with P_{il}^* by w_{il}. Then, $w_{ij}(t_{ijr}) = -re_i$, $w_{ij}(t_{ijs}) = -se_i$, and $w_{il}(t_{ijr}) \geq 0$, $w_{il}(t_{ijs}) \geq 0$, if $l \neq j$. Hence,

$$w_{il}(bt) = w_{il}(t_{ij_1's}) + \cdots + w_{il}(t_{ij_q's}) + w_{il}(t_{ij_{q+1}'r}) \geq -(qs+r)e_i = -me_i$$

and equality holds if and only if $j_1' = \cdots = j_{q+1}' = l$. If the condition $j_1' = \cdots = j_{q+1}' = l$ is satisfied for no l, then for each $1 \leq l \leq d_i$ we have $w_{il}(bt) \geq -me_i + 1$ and $w(bt) \geq 0$ for each valuation $w \neq w_{i1}, \ldots, w_{id_i}$ of $\tilde{K}F/\tilde{K}$. This implies that $bt \in \mathcal{L}_{\tilde{K}}(\sum_{l=1}^{d_i}(me_i - 1)P_{il}^*) \subseteq \mathcal{L}_{\tilde{K}}(C)$. If $j_1' = \cdots = j_{q+1}' = l$ for some l between 1 and d_i, then $bt = b_{i11ls}^q b_{ijkj'r} t_{ilm}$ (Data 7.3(b)). It follows that

$$u_{ijkm} \equiv \sum_{l=1}^{d_i} b_{i11ls}^q b_{ijklr} t_{ilm} \mod \mathcal{L}_{\tilde{K}}(C).$$

In addition, by Notation 5.4, $u_1, \ldots, u_{\nu_{m-1}} \in \mathcal{L}_{\tilde{K}}((m-1)D^*) \subseteq \mathcal{L}_{\tilde{K}}(C)$. Hence, by (7.3),

$$\begin{aligned} f &\equiv \sum_{i=1}^{e}\sum_{j=1}^{d_i}\sum_{k=1}^{e_i} a_{ijk}u_{ijkm} \equiv \sum_{i=1}^{e}\sum_{j=1}^{d_i}\sum_{k=1}^{e_i}\sum_{l=1}^{d_i} a_{ijk}b_{i11ls}^q b_{ijklr}t_{ilm} \qquad (7.5)\\ &\equiv \sum_{i=1}^{e}\sum_{l=1}^{d_i} b_{i11ls}^q \Big(\sum_{j=1}^{d_i}\sum_{k=1}^{e_i} a_{ijk}b_{ijklr}\Big)t_{ilm} \equiv \sum_{i=1}^{e}\sum_{l=1}^{d_i} b_{i11ls}^q \lambda_{ilr}(\mathbf{a}_i)t_{ilm}\\ &\equiv \sum_{i=1}^{e}\sum_{l=1}^{d_i} c_{il}t_{ilm} \mod \mathcal{L}_{\tilde{K}}(C), \end{aligned}$$

with $c_{il} = b_{i11ls}^q \lambda_{ilr}(\mathbf{a}_i)$. By (d) and (f), $|c_{il}|_{\mathfrak{p}} = 1$, $i = 1, \ldots, e$, $l = 1, \ldots, d_i$. By (c), $|u_\mu|_{\mathfrak{p}} \leq 1$, $\mu = 1, \ldots, \nu_{2s-1}$. Hence, by Notation 5.4, $|u_{ijkr}|_{\mathfrak{p}} \leq 1$ for $i = 1, \ldots, e$, $j = 1, \ldots, d_i$, $k = 1, \ldots, e_i$, and $r = s, \ldots, 2s-1$. By Data 5.2 and Notation 5.4, for each $\kappa \geq s$ the function u_κ is a product of functions which belong to the set $\{u_{ijks}, \ldots, u_{ijk,2s-1} \mid i = 1, \ldots, e, j = 1, \ldots, d_i, k = 1, \ldots, e_i\}$. Hence $|u_\kappa|_{\mathfrak{p}} \leq 1$. In particular $|u_{ijkm}|_{\mathfrak{p}} \leq 1$. Hence, by (7.3) and (e), $|f|_{\mathfrak{p}} \leq 1$. Therefore, by (7.5) and (b), $g = f - \sum_{i=1}^{e}\sum_{l=1}^{d_i} c_{il}t_{ilm}$ belongs to $\mathcal{L}_{\tilde{K}}(C)$ and satisfies $|g|_{\mathfrak{p}} \leq 1$. We conclude from Lemma 7.2 that f is $\mathfrak{p}$-regular of level m. ∎

8. Admissible functions along $\mathcal{V}_N$

To create a $\mathcal{V}_N$-admissible function we first use Proposition 6.6 to define a big subset $\mathcal{U}$ of $\mathcal{V}_N$ which takes into account all conditions of Lemma 7.4 which do not concern $\mathbf{a}$. Then, for $\mathcal{T} = \mathcal{V}_N \smallsetminus \mathcal{U}$, we select f of the form (7.3) of Section 7, such that f is $\mathcal{T}$-admissible. The final step is to use Proposition 1.11, Lemma 7.4, and Corollary 6.2 to change the a_{ijk}'s such that f becomes also $\mathcal{U}$-admissible (and hence $\mathcal{V}_N$-admissible) and then to use Lemma 1.9 to change the a_μ's such that in addition f has an M-rational zero.

Data 8.1. We extend each finite prime $\mathfrak{p} \in \mathcal{V}_N \smallsetminus \mathcal{W}_N$ to a prime of the Henselian closure $N_\mathfrak{p} = K_s$ (Remark 2.3(c)) with the same name. We use Proposition 6.6 to choose a big subset $\mathcal{U}$ of $\mathcal{V}_{\mathrm{fin},N} \smallsetminus \mathcal{W}_N$ (which may be empty if $\mathcal{V}_{\mathrm{fin}}$ is finite) such that the following statements hold for each $\mathfrak{p} \in \mathcal{U}$ and for $s = 2\,\mathrm{genus}(F/M) + 2$, $i = 1, \dots, e$, $r = s, \dots, 2s-1$, $j, j' = 1, \dots, d_i$, $k = 1, \dots, e_i$, $\mu = 1, \dots, \nu_{2s-1}$, and $\nu = 1, \dots, n$:

(1a) $\mathfrak{p}$ has a good extension to $\tilde{K}F$ named $\mathfrak{p}$,
(1b) The $\overline{P_{ij}^*}$ are distinct,
(1c) t_{ijr} is $\mathfrak{p}$-regular,
(1d) $|u_\mu|_\mathfrak{p} \le 1$ (Notation 5.4),
(1e) $|b_{ijkj'r}|_\mathfrak{p} = 1$,
(1f) $|x_\nu|_\mathfrak{p} = 1$ if $x_\nu \ne 0$.

Note that $b_{ijkj'r} \ne 0$ (Data 7.3(c)). So, we may achieve condition (1e). Make $\mathcal{U}$ smaller, if necessary, to assume that $\mathcal{U}$ is K-rational (Definition 3.8). Then, $\mathcal{T} = \mathcal{V}_N \smallsetminus \mathcal{U}$ is a K-rational small subset of $\mathcal{V}_N$ which contains $\mathcal{W}_N$.

Notation 8.2. For each positive integer $m \ge s = 2\,\mathrm{genus}\,(F/M) + 2$ we denote the space $\mathbb{A}^{\nu_{m-1}} \times \prod_{i=1}^{e} \mathbb{A}^{d_i} \times \mathbb{A}^{e_i}$ by A_m. The zero coordinate of a point $\mathbf{a} \in A_m$ is a ν_{m-1}-tuple $\mathbf{a}_0 = (a_1, \dots, a_{\nu_{m-1}})$ and for each $i \ge 1$ the ith coordinate is a $(d_i \times e_i)$-matrix $\mathbf{a}_i = (a_{ijk})_{1 \le j \le d_i,\, 1 \le k \le e_i}$.

Proposition 8.3 (Density of admissible functions).
Let $m_0 \ge s = 2\,\mathrm{genus}(F/M)+2$. Then there exists $m \ge m_0$, a point $\mathbf{c} \in A_m(M)$, and $\varepsilon > 0$ with the following property: If $\mathbf{a} \in A_m(N)$ satisfies

(2a) $|\mathbf{a} - \mathbf{c}|_\mathcal{T} < \varepsilon$ and
(2b) $|\mathbf{a}|_\mathfrak{p} \le 1$ and $|\lambda_{ilr}(\mathbf{a}_i)|_\mathfrak{p} = 1$ for each $\mathfrak{p} \in \mathcal{U}$, for $i = 1, \dots, e$, $l = 1, \dots, d_i$, and $r = s, \dots, 2s-1$,

then the function

(3) $\quad f = \sum_{i=1}^{e} \sum_{j=1}^{d_i} \sum_{k=1}^{e_i} a_{ijk} u_{ijkm} + \sum_{\mu=1}^{\nu_{m-1}} a_\mu u_\mu$

is $\mathcal{V}_N$-admissible of level m.

Proof. Rename the function f that Proposition 5.5 supplies as h and rewrite h in the form
$$h = \sum_{i=1}^{e} \sum_{j=1}^{d_i} \sum_{k=1}^{e_i} c_{ijk} u_{ijkm} + \sum_{\mu=1}^{\nu_{m-1}} c_\mu u_\mu,$$
with $\mathbf{c} \in A_m(M)$. Retain also the role of m and ε from Proposition 5.5.

Now suppose that $\mathbf{a} \in A_m(N)$ satisfies Condition (2) and f is as in (3). By Proposition 5.5, f is $\mathcal{T}$-admissible of level m. In particular, each of the zeros of f is N-rational and simple. By Data 8.1, (2b), and Lemma 7.4, f is $\mathfrak{p}$-regular of level m for each $\mathfrak{p} \in \mathcal{U}$. By Data 8.1, $\tilde{K}F/\tilde{K}$ has a good reduction at each $\mathfrak{p} \in \mathcal{U}$. Since f is of level m and $|x_i|_\mathfrak{p} = 1$ if $x_i \ne 0$ for $i = 1, \dots, n$, Corollary 6.2 implies that f is $\mathfrak{p}$-admissible. Consequently, f is $\mathcal{V}_N$-admissible. ∎

Proposition 8.4 (Existence of admissible functions). *For each m_0 there exists a $\mathcal{V}_N$-admissible function $f \in F$ of level $m \geq m_0$ which has an M-rational zero.*

Proof. Let m_1 be an integer which is greater than m_0 and $2\,\mathrm{genus}(F/M)+2$. Let $m \geq m_1$, $\mathbf{c} \in A_m(M)$, and ε be as in Proposition 8.3. Then $\nu_{m-1} = \dim((m-1)D^*) \geq 2$ (Notation 5.4).

By (1e), the coefficients of the $\lambda_{ilr}(\mathbf{Y}_i)$ (Data 7.3(d)) are $\mathcal{T}$-units. The same holds for the polynomials Y_{ijk}. Also, by Data 8.1, $\mathcal{T}$ is a K-rational small subset of $\mathcal{V}_N$ which contains $\mathcal{S}_N$. Thus,

$$(\mathcal{T}, (\lambda_{ilr}(\mathbf{Y}_i), Y_{ijk}), \mathbf{c}, \varepsilon)_{i=1,\ldots,e;\ j,l=1,\ldots,d_i;\ k=1,\ldots,e_i}$$

is an $(S,\mathcal{V})$-Skolem density problem for M (Definition 1.10). By Proposition 1.11, there exists for each i between 1 and e a point $\mathbf{a}_i \in M^{d_i} \times M^{e_i}$ such that $|\mathbf{a}_i - \mathbf{c}_i|_{\mathcal{T}} < \frac{\varepsilon}{2}$, and $|a_{ijk}|_{\mathfrak{p}} = 1$ and $|\lambda_{ilr}(\mathbf{a}_i)|_{\mathfrak{p}} = 1$ for each $\mathfrak{p} \in \mathcal{U}$ and for $r = s, \ldots, 2s-1$, $j, l = 1, \ldots, d_i$, $k = 1, \ldots, e_i$.

The field $L = K(c_1, \ldots, c_{\nu_{m-1}})$ is a finite subextension of M/K. Since $\mathcal{T}$ is K-rational, we may apply the strong approximation theorem to L (Proposition 1.2) and find $\mathbf{c}'_0 \in L^{\nu_{m-1}}$ such that $|\mathbf{c}'_0 - \mathbf{c}_0|_{\mathcal{T}} < \frac{\varepsilon}{2}$ and $|\mathbf{c}'_0|_{\mathfrak{p}} \leq 1$ for each $\mathfrak{p} \in \mathcal{U}$. Choose $0 \neq l \in O_K$ such that $|l|_{\mathcal{T}} < \frac{\varepsilon}{2}$ (recall that $\mathcal{T}|_K$ is a finite set).

Let $g = \sum_{i=1}^{e} \sum_{j=1}^{d_i} \sum_{k=1}^{e_i} a_{ijk} u_{ijkm}$ and $f' = g + \sum_{\mu=1}^{\nu_{m-1}} c'_\mu u_\mu$. Since the u_μ and the u_{ijkm} are linearly independent over M, $u_1 = 1$, and $a_{ijk} \neq 0$, we have $f' \in F \smallsetminus M$. Let $t = -\frac{1}{l} f'$. By Proposition 8.3, f' is $\mathcal{V}_N$-admissible of level m. In particular, all of the zeros of f' (hence, also of t) are simple and in $\Gamma(N)$. By Data 2.1(c), M is weakly P$\mathcal{S}$C over $O_{M,\mathcal{V}}$. Hence, by Lemma 1.9, there exists $P \in \Gamma(M)$ which is a pole of none of the functions $t, g, u_1, \ldots, u_{\nu_{m-1}}$ such that $t(P) \in O_{M,\mathcal{V}}$. Let $a_1 = lt(P) + c'_1, a_2 = c'_2, \ldots, a_{\nu_{m-1}} = c'_{\nu_{m-1}}$, and $\mathbf{a}_0 = (a_1, \ldots, a_{\nu_{m-1}})$. Then $|a_1 - c_1|_{\mathcal{T}} = |lt(P) + c'_1 - c_1|_{\mathcal{T}} < \varepsilon$, so $|\mathbf{a}_0 - \mathbf{c}_0|_{\mathcal{T}} < \varepsilon$ and $|\mathbf{a}_0|_{\mathfrak{p}} \leq 1$ for each $\mathfrak{p} \in \mathcal{U}$. Since $u_1 = 1$, we have

$$g + (lt + c'_1) + c'_2 u_2 + \cdots + c'_{\nu_{m-1}} u_{\nu_{m-1}} = lt + f' = 0.$$

Hence, P is a zero of the function

$$f = lt(P) + f' = g + \sum_{\mu=1}^{\nu_{m-1}} a_\mu u_\mu.$$

Thus $\mathbf{a} = (\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_e) \in A_m(M)$ satisfies (2) and f has the form (3). By Proposition 8.3, f is $\mathcal{V}_N$-admissible of level m. ■

Proposition 8.4 is a reformulation of Theorem 3.5. The latter implies Theorem 3.4, which is a reformulation of Theorem 2.2(c) for curves. We state the latter for the record.

Proposition 8.5 (Strong approximation theorem for integral points on curves). *Let C be an absolutely irreducible affine curve defined over K. Suppose that $C(D_{N,\mathfrak{p}}) \neq \emptyset$ for each $\mathfrak{p} \in \mathcal{V}_N \smallsetminus \mathcal{W}_N$. Consider $(\mathbf{z}_\mathfrak{q})_{\mathfrak{q} \in \mathcal{W}_N} \in C_{D,\mathcal{S},\mathcal{W}}$ and $\varepsilon > 0$. Then there exists $\mathbf{z} \in C(D_{M,\mathcal{V}})$ such that $|\mathbf{z} - \mathbf{z}_\mathfrak{q}|_\mathfrak{q} < \varepsilon$ for each $\mathfrak{q} \in \mathcal{W}_N$.*

9. The approximation theorems and the local global principle for arbitrary affine varieties

In this section we use the strong approximation theorem for integral points on curves (Proposition 8.5) to prove the approximation theorem for integral points on arbitrary varieties. Then we prove all other theorems of Section 2.

Lemma 9.1. *Let V be an absolutely irreducible variety defined over K. Let $\mathcal{R}_1$ be a finite subset of $\mathcal{V}_N$ whose elements are mutually nonconjugate over K. For each $\mathfrak{p} \in \mathcal{R}_1$ let $\mathbf{z}_\mathfrak{p} \in V(N_\mathfrak{p})$. Let $\mathcal{R} = \{\mathfrak{p}^\sigma \mid \mathfrak{p} \in \mathcal{R}_1,\ \sigma \in \mathrm{Gal}(N/K)\}$. Then we can find a finite extension L of M/K and extend the point $(\mathbf{z}_\mathfrak{p})_{\mathfrak{p}\in\mathcal{R}_1}$ into a point $(\mathbf{z}_\mathfrak{q})_{\mathfrak{q}\in\mathcal{R}}$ such that $\mathbf{z}_\mathfrak{q} \in V(L_\mathfrak{q})$ and $\mathbf{z}_{\mathfrak{q}^\sigma} = \mathbf{z}_\mathfrak{q}^\sigma$ for each $\mathfrak{q} \in \mathcal{R}$ and each $\sigma \in \mathrm{Gal}(N/L)$.*

Proof. We first prove that M/K has a finite subextension L such that $\mathbf{z}_\mathfrak{p}^\sigma \in V(L_{\mathfrak{p}^\sigma})$ for each $\mathfrak{p} \in \mathcal{R}_1$ and each $\sigma \in \mathrm{Gal}(N/K)$. It suffices to do it in the case that $\mathcal{R}_1$ consists of one prime $\mathfrak{p}$.

If $\mathfrak{p}$ is an infinite prime, then there exists a finite subextension L of M/K such that $L_\mathfrak{p} = M_\mathfrak{p} = N_\mathfrak{p}$ is either real closed or algebraically closed. Then, $L_{\mathfrak{p}^\sigma} = N_{\mathfrak{p}^\sigma}$ for each $\sigma \in \mathrm{Gal}(N/K)$.

Now suppose $\mathfrak{p}$ is a finite prime. Then $N_\mathfrak{p}$ could be an infinite extension of $K_\mathfrak{p}$, so there might be no field L as in the preceding paragraph. However, we may choose a finite Galois subextension E of N/K with $\mathbf{z}_\mathfrak{p} \in V(E_\mathfrak{p})$. Let y be a primitive element of E/K. Let $\varepsilon > 0$ be a real number which is smaller than $|y - y'|_\mathfrak{p}$ for all conjugates y' of y over K with $y' \neq y$. By Lemma 1.7 applied to all conjugates of y instead of to x, there exists a finite subset B of M with the following property: For each $\mathfrak{q} \in \mathcal{V}_N$ which lies over $\mathfrak{p}|_K$ and each conjugate y' of y over K there exists $b \in B$ such that $|b - y'|_\mathfrak{q} < \varepsilon$. Then $L = K(B)$ is a finite subextension of M/K.

Consider a $\sigma \in \mathrm{Gal}(N/K)$ and let $\mathfrak{q} = \mathfrak{p}^\sigma$, $y' = y^\sigma$. Choose $b \in B$ such that $|b - y'|_\mathfrak{q} < \varepsilon$. By Krasner's lemma [15, p. 43], $K_\mathfrak{q}(y^\sigma) \subseteq K_\mathfrak{q}(b) \subseteq LK_\mathfrak{q} = L_\mathfrak{q}$. Hence $\mathbf{z}_\mathfrak{p}^\sigma \in V(K_\mathfrak{q}(y^\sigma)) \subseteq V(L_{\mathfrak{p}^\sigma})$.

Now choose a finite subset $\mathcal{R}_2$ of $\mathcal{R}$ that contains $\mathcal{R}_1$ and represents $\mathcal{R}|_L$ (Definition 3.8). For each $\mathfrak{q} \in \mathcal{R}_2 \smallsetminus \mathcal{R}_1$ there exists a unique $\mathfrak{p} \in \mathcal{R}_1$ such that $\mathfrak{q}|_K = \mathfrak{p}|_K$. Choose $\lambda \in \mathrm{Gal}(N/K)$ such that $\mathfrak{q} = \mathfrak{p}^\lambda$ and set $\mathbf{z}_\mathfrak{q} = \mathbf{z}_\mathfrak{p}^\lambda$. Then $\mathbf{z}_\mathfrak{q} \in V(L_\mathfrak{q})$.

If $\sigma \in \mathrm{Gal}(N/L)$ satisfies $\mathfrak{q}^\sigma = \mathfrak{q}$, then $\sigma \in \mathrm{Gal}(N/N \cap L_\mathfrak{q})$. Hence, the extension of σ to $N_\mathfrak{q}$ (Data 2.1(a)) fixes the elements of $L_\mathfrak{q}$. In particular $\mathbf{z}_\mathfrak{q}^\sigma = \mathbf{z}_\mathfrak{q}$. It follows that if for arbitrary $\mathfrak{q} \in \mathcal{R}_2$ and $\tau \in \mathrm{Gal}(N/L)$ we define $\mathbf{z}_{\mathfrak{q}^\tau} = \mathbf{z}_\mathfrak{q}^\tau$, then $\mathbf{z}_\mathfrak{p}$ is well defined for each $\mathfrak{p} \in \mathcal{R}$, it coincides with the original $\mathbf{z}_\mathfrak{p}$ if $\mathfrak{p} \in \mathcal{R}_1$, and satisfies $\mathbf{z}_{\mathfrak{p}^\sigma} = \mathbf{z}_\mathfrak{p}^\sigma$ for each $\mathfrak{p} \in \mathcal{R}$ and $\sigma \in \mathrm{Gal}(N/L)$. ■

We return now to the notation of Data 2.1, copy over Theorem 2.2, and prove it.

Theorem 9.2 (Strong approximation theorem). *Let V be an absolutely irreducible affine variety defined over K. Consider $(\mathbf{z}_\mathfrak{q})_{\mathfrak{q}\in\mathcal{W}_N} \in V_{K,\mathcal{S},\mathcal{W}}$ and $\varepsilon > 0$.*

(a) There exists $\mathbf{z} \in V(M)$ such that $|\mathbf{z} - \mathbf{z}_\mathfrak{q}|_\mathfrak{q} < \varepsilon$ for each $\mathfrak{q} \in \mathcal{W}_N$.

(b) If $V(D_{N,\mathfrak{p}}) \neq \emptyset$ for each $\mathfrak{p} \in \mathcal{V}_N \smallsetminus \mathcal{W}_N$, then there exists $\mathbf{z} \in V(M)$ such that $|\mathbf{z} - \mathbf{z}_\mathfrak{q}|_\mathfrak{q} < \varepsilon$ for each $\mathfrak{q} \in \mathcal{W}_N$ and $\mathbf{z} \in D_{N,\mathfrak{p}}^n$ for each $\mathfrak{p} \in \mathcal{V}_N \smallsetminus \mathcal{W}_N$.

(c) If $V(D_{N,\mathfrak{p}}) \neq \emptyset$ for each $\mathfrak{p} \in \mathcal{V}_N \smallsetminus \mathcal{W}_N$, and $\mathbf{z}_\mathfrak{q} \in D_{N,\mathfrak{q}}^n$ for each $\mathfrak{q} \in \mathcal{W}_N$, then there exists $\mathbf{z} \in V(D_{M,\mathcal{V}})$ such that $|\mathbf{z} - \mathbf{z}_\mathfrak{q}|_\mathfrak{q} < \varepsilon$ for each $\mathfrak{q} \in \mathcal{W}_N$.

Proof. By assumption $\mathbf{z}_\mathfrak{q} \in V_{\mathrm{simp}}(N_\mathfrak{q})$ for each $\mathfrak{q} \in \mathcal{W}_N$. Also, there exists a finite subextension L of M/K such that $\mathbf{z}_\mathfrak{q} \in V_{\mathrm{simp}}(L_\mathfrak{q})$ and $\mathbf{z}_\mathfrak{q}^\sigma = \mathbf{z}_{\mathfrak{q}^\sigma}$ for each $\mathfrak{q} \in \mathcal{W}_N$ and $\sigma \in \mathrm{Gal}(N/L)$. Our primary goal is to find a point $\mathbf{z} \in V(M)$ such that $|\mathbf{z} - \mathbf{z}_\mathfrak{q}|_\mathfrak{q} < \varepsilon$ for each $\mathfrak{q} \in \mathcal{W}_N$.

Proof of (c). Here we assume in addition that $V(D_{N,\mathfrak{p}}) \neq \emptyset$ for each $\mathfrak{p} \in \mathcal{V}_N \smallsetminus \mathcal{W}_N$ and $\mathbf{z}_\mathfrak{q} \in V_{\mathrm{simp}}(D_{N,\mathfrak{q}})$ for each $\mathfrak{q} \in \mathcal{W}_N$. We have to approximate the points $\mathbf{z}_\mathfrak{q}$ with $\mathbf{z} \in V(D_{M,\mathcal{V}})$.

Choose a point $\mathbf{z}_0 \in V(K_s)$ and recall that $N_\mathfrak{p} = K_s$ for each $\mathfrak{p} \in \mathcal{V}_N \smallsetminus \mathcal{W}_N$ (Remark 2.3(c)). Let

$$\mathcal{U} = \{\mathfrak{p} \in \mathcal{V}_{\mathrm{fin},N} \smallsetminus \mathcal{W}_N \mid |\mathbf{z}_0^\sigma|_\mathfrak{p} \leq 1 \text{ for each } \sigma \in \mathrm{Gal}(L)\}, \qquad \mathcal{T} = \mathcal{V}_N \smallsetminus \mathcal{U}.$$

Then $\mathcal{T}$ is an L-rational small set which contains $\mathcal{W}_N$. Choose a finite subset $\mathcal{W}_1$ of $\mathcal{W}_N$ which represents $\mathcal{W}_N|_L$ and a finite subset $\mathcal{R}_1$ of $\mathcal{R} = \mathcal{T} \smallsetminus \mathcal{W}_N$ which represents $\mathcal{R}|_L$ (Definition 3.8). Set $\mathcal{T}_1 = \mathcal{W}_1 \cup \mathcal{R}_1$.

For each $\mathfrak{p} \in \mathcal{R}_1$ choose $\mathbf{z}_\mathfrak{p} \in V(D_{N,\mathfrak{p}})$. Now apply Lemma 9.1 to L and $\mathcal{R}_1$, extend L (hence, also $\mathcal{W}_1$, $\mathcal{R}_1$, and $\mathcal{T}_1$), if necessary, and extend the point $(\mathbf{z}_\mathfrak{p})_{\mathfrak{p} \in \mathcal{T}_1}$ to a point $(\mathbf{z}_\mathfrak{p})_{\mathfrak{p} \in \mathcal{T}}$ such that $\mathbf{z}_\mathfrak{p} \in V(L_\mathfrak{p})$ and $\mathbf{z}_{\mathfrak{p}^\sigma} = \mathbf{z}_\mathfrak{p}^\sigma$ for all $\mathfrak{p} \in \mathcal{T}$ and $\sigma \in \mathrm{Gal}(N/L)$. In particular, each $\mathbf{z}_\mathfrak{p}$ belongs to $V(D_{L,\mathfrak{p}})$, hence to $V(D_{N,\mathfrak{p}})$. Finally, if $\mathfrak{p} \in \mathcal{U}$, then $N_\mathfrak{p} = K_s$. Thus, let $\mathbf{z}_\mathfrak{p} = \mathbf{z}_0$.

In an appendix to [12] we show that there exists an affine absolutely irreducible curve C which is defined over L, hence also over M, which lies on V and passes through $\mathbf{z}_0$ and through $\mathbf{z}_\mathfrak{p}$ for each $\mathfrak{p} \in \mathcal{T}_1$. Moreover, $\mathbf{z}_\mathfrak{p}$ is simple on C for each $\mathfrak{p} \in \mathcal{W}_1$. For an arbitrary $\mathfrak{p}' \in \mathcal{V}_N$ the point $\mathbf{z}_{\mathfrak{p}'}$ is conjugate over L to a point $\mathbf{z}_\mathfrak{p}$ for some $\mathfrak{p} \in \mathcal{T}_1 \cup \mathcal{U}$. Hence $\mathbf{z}_{\mathfrak{p}'}$ belongs to $C(D_{N,\mathfrak{p}'})$ and is simple if $\mathfrak{p}' \in \mathcal{W}_N$, so $(\mathbf{z}_\mathfrak{q})_{\mathfrak{q} \in \mathcal{W}_N} \in C_{D,\mathcal{S},\mathcal{W}}$.

By Proposition 8.5 and Remark 2.3(a), there exists $\mathbf{z} \in C(D_{M,\mathcal{V}})$ which satisfies $|\mathbf{z} - \mathbf{z}_\mathfrak{q}|_\mathfrak{q} < \varepsilon$ for each $\mathfrak{q} \in \mathcal{W}_N$. Then $\mathbf{z} \in V(D_{M,\mathcal{V}})$, as desired.

Proof of (b). Here we only assume that $V(D_{N,\mathfrak{p}}) \neq \emptyset$ for each $\mathfrak{p} \in \mathcal{V}_N \smallsetminus \mathcal{W}_N$. We have to approximate the points $\mathbf{z}_\mathfrak{q}$ by a point $\mathbf{z} \in V(M)$ such that $|\mathbf{z}|_\mathfrak{p} \leq 1$ for each $\mathfrak{p} \in \mathcal{V}_{\mathrm{fin},N} \smallsetminus \mathcal{W}_N$ and $|\mathbf{z}|_\mathfrak{p} < 1$ for each $\mathfrak{p} \in \mathcal{V}_{\mathrm{inf},N} \smallsetminus \mathcal{W}_N$.

The set $\mathcal{R} = \mathcal{V}_{\mathrm{inf},N} \smallsetminus \mathcal{W}_N$ is K-rational and small. Choose a finite subset $\mathcal{R}_1$ of $\mathcal{R}$ which represents $\mathcal{R}|_L$. For each $\mathfrak{p} \in \mathcal{R}_1$, let $\mathbf{z}_\mathfrak{p} \in V(D_{N,\mathfrak{p}})$. Since $N_\mathfrak{p} = K_s = \tilde{K}$ (if $\mathcal{R} \neq \emptyset$, then $\mathrm{char}(K) = 0$) and $V_{\mathrm{simp}}(K_s)$ is Zariski open in $V(K_s)$ [5, Lemma 2.2], $V_{\mathrm{simp}}(N_\mathfrak{p})$ is $\mathfrak{p}$-dense in $V(N_\mathfrak{p})$. Hence, we may assume that $\mathbf{z}_\mathfrak{p} \in V_{\mathrm{simp}}(N_\mathfrak{p})$ and $|\mathbf{z}_\mathfrak{p}|_\mathfrak{p} < 1$. By Lemma 9.1, we may extend L, if necessary, and extend the point $(\mathbf{z}_\mathfrak{p})_{\mathfrak{p} \in \mathcal{R}_1}$ to a point $(\mathbf{z}_\mathfrak{q})_{\mathfrak{q} \in \mathcal{R}}$ such that $\mathbf{z}_\mathfrak{q} \in V_{\mathrm{simp}}(L_\mathfrak{q})$, $|\mathbf{z}_\mathfrak{q}|_\mathfrak{q} < 1$, and $\mathbf{z}_{\mathfrak{q}^\sigma} = \mathbf{z}_\mathfrak{q}^\sigma$ for each $\mathfrak{q} \in \mathcal{R}$ and each $\sigma \in \mathrm{Gal}(N/L)$.

Let $\mathcal{T} = \mathcal{R} \cup \mathcal{W}_N$. Since $(\mathbf{z}_\mathfrak{q})_{\mathfrak{q} \in \mathcal{T}}$ is L-rational, the set $\{|\mathbf{z}_\mathfrak{q}|_\mathfrak{q} \mid \mathfrak{q} \in \mathcal{T}\}$ is finite. Hence, $\alpha = \min(1, 1/|\mathbf{z}_\mathfrak{q}|_\mathfrak{q})_{\mathfrak{q} \in \mathcal{T}}$ is a well defined positive real number ≤ 1. Also, we may assume that $\varepsilon < 1 - |\mathbf{z}_\mathfrak{q}|_\mathfrak{q}$ for each $\mathfrak{q} \in \mathcal{R}$. By Proposition 1.11 applied to X instead of to f_i, there exists $a \in M$ such that $|a|_\mathfrak{q} < \alpha$ for each $\mathfrak{q} \in \mathcal{T}$ and $|a|_\mathfrak{p} = 1$ for each $\mathfrak{p} \in \mathcal{V}_{\mathrm{fin},N} \smallsetminus \mathcal{W}_N$.

Consider the automorphism λ of $\mathbb{A}^n$ defined by $\lambda(\mathbf{x}) = a\mathbf{x}$. It maps V onto an affine absolutely irreducible variety V' which is defined over $K(a)$. For each $\mathfrak{p} \in \mathcal{V}_{\mathrm{fin},N} \smallsetminus \mathcal{W}_N$ we have $V'(D_{N,\mathfrak{p}}) \neq \emptyset$. If $\mathfrak{q} \in \mathcal{T}$, then $\mathbf{z}'_\mathfrak{q} = a\mathbf{z}_\mathfrak{q} \in V'_{\mathrm{simp}}(D_{N,\mathfrak{q}})$. Moreover, if $\sigma \in \mathrm{Gal}(L(a))$, then $\mathbf{z}'_{\mathfrak{q}^\sigma} = (\mathbf{z}'_\mathfrak{q})^\sigma$.

Since $\mathcal{T}$ is small, the set $\{|a|_\mathfrak{q} \mid \mathfrak{q} \in \mathcal{T}\}$ is finite. Hence, by (c), there exists $\mathbf{z}' \in V'(D_{M,\mathcal{V}})$ such that $|\mathbf{z}' - \mathbf{z}'_\mathfrak{q}|_\mathfrak{q} < \varepsilon|a|_\mathfrak{q}$ for each $\mathfrak{q} \in \mathcal{T}$. It follows that $\mathbf{z} = a^{-1}\mathbf{z}' \in V(M)$ and $|\mathbf{z} - \mathbf{z}_\mathfrak{q}|_\mathfrak{q} < \varepsilon$ for each $\mathfrak{q} \in \mathcal{T}$. Hence, $|\mathbf{z} - \mathbf{z}_\mathfrak{q}|_\mathfrak{q} < \varepsilon$ for each $\mathfrak{q} \in \mathcal{W}_N$ and $|\mathbf{z}|_\mathfrak{q} \le |\mathbf{z} - \mathbf{z}_\mathfrak{q}|_\mathfrak{q} + |\mathbf{z}_\mathfrak{q}|_\mathfrak{q} < 1$ for each $\mathfrak{q} \in \mathcal{R}$. Finally, since a is a $\mathfrak{p}$-unit for each finite prime outside $\mathcal{W}_N$, we have $|\mathbf{z}|_\mathfrak{p} \le 1$ for each $\mathfrak{p} \in \mathcal{V}_{\mathrm{fin},N} \smallsetminus \mathcal{W}_N$.

Proof of (a). Choose $\mathbf{z}_0 \in V(K_s)$ and recall that $N_\mathfrak{p} = K_s$ for each $\mathfrak{p} \in \mathcal{V}_N \smallsetminus \mathcal{W}_N$. Then $\mathcal{U} = \{\mathfrak{p} \in \mathcal{V}_{\mathrm{fin},N} \smallsetminus \mathcal{W}_N \mid |\mathbf{z}_0^\sigma|_\mathfrak{p} \le 1 \text{ for all } \sigma \in \mathrm{Gal}(K)\}$ is a well defined K-rational big subset of $\mathcal{V}_N$. Hence, $\mathcal{T} = \mathcal{V}_N \smallsetminus \mathcal{U}$ and $\mathcal{R} = \mathcal{T} \smallsetminus \mathcal{W}_N$ are K-rational small subsets of $\mathcal{V}_N$.

As in the proof of (b), $\alpha = \min(1, 1/|\mathbf{z}_0|_\mathfrak{p})_{\mathfrak{p}\in\mathcal{R}}$ is a well defined positive real number ≤ 1. By Proposition 1.11, there exists $a \in M$ such that $|a|_\mathfrak{p} < \alpha$ for each $\mathfrak{p} \in \mathcal{R}$ and $|a|_\mathfrak{p} = 1$ for each $\mathfrak{p} \in \mathcal{V}_{\mathrm{fin},N} \smallsetminus \mathcal{R}$. Consider the automorphism λ of $\mathbb{A}^n$ defined by $\lambda(\mathbf{x}) = a\mathbf{x}$ over $K(a)$. It maps V onto an absolutely irreducible variety V' which is defined over $K(a)$. If $\mathfrak{q} \in \mathcal{W}_N$, then $\mathbf{z}'_\mathfrak{q} = a\mathbf{z}_\mathfrak{q} \in V'_{\mathrm{simp}}(N_\mathfrak{q})$. Moreover, if $\sigma \in \mathrm{Gal}(L(a))$, then $\mathbf{z}'_{\mathfrak{q}^\sigma} = (\mathbf{z}'_\mathfrak{q})^\sigma$. If $\mathfrak{p} \in \mathcal{R}$, then $N_\mathfrak{p} = K_s$ and hence, $\mathbf{z}'_\mathfrak{p} = a\mathbf{z}_0 \in V'(N_\mathfrak{p})$ and satisfies $|\mathbf{z}'_\mathfrak{p}|_\mathfrak{p} = |a\mathbf{z}_\mathfrak{p}|_\mathfrak{p} < \alpha|\mathbf{z}_\mathfrak{p}|_\mathfrak{p} \le 1$. Similarly, if $\mathfrak{p} \in \mathcal{U}$, then $\mathbf{z}'_\mathfrak{p} = a\mathbf{z}_0 \in V'(N_\mathfrak{p})$ and $|\mathbf{z}'_\mathfrak{p}|_\mathfrak{p} \le 1$.

By (b), there exists $\mathbf{z}' \in V'(M)$ such that $|\mathbf{z}' - \mathbf{z}'_\mathfrak{q}|_\mathfrak{q} < \varepsilon|a|_\mathfrak{q}$ for each $\mathfrak{q} \in \mathcal{W}_N$. Hence, $\mathbf{z} = a^{-1}\mathbf{z}'$ belongs to $V(M)$ and satisfies $|\mathbf{z} - \mathbf{z}_\mathfrak{q}|_\mathfrak{q} < \varepsilon$ for each $\mathfrak{q} \in \mathcal{W}_N$.

This concludes the proof of the theorem. ∎

Next we show how to deduce the weak approximation theorem from the strong approximation theorem.

Proof of Theorem 2.4(a). There exists a finite subextension K' of M/K over which V is defined and such that the map res: $\mathcal{T} \to \mathcal{T}|_{K'}$ is injective. Assume without loss that $K' = K$. Extend each $\mathfrak{p} \in \mathcal{T}$ to a prime of N, if necessary, to assume that $\mathcal{T} \subset \mathcal{V}_N$. Recall that $D_{M,\mathfrak{p}} = D_{N,\mathfrak{p}}$ for each $\mathfrak{p} \in \mathcal{V}_N$ (Remark 2.3(c)).

For each $\mathfrak{p} \in \mathcal{T} \cap \mathcal{S}_N$ let $\mathbf{z}_\mathfrak{p} \in V_{\mathrm{simp}}(D_{N,\mathfrak{p}})$ and for each $\mathfrak{p} \in \mathcal{T} \smallsetminus \mathcal{S}_N$ let $\mathbf{z}_\mathfrak{p} \in V(D_{N,\mathfrak{p}})$. Also, let ε be a positive real number. We have to find $\mathbf{z} \in V(D_{M,\mathcal{V}})$ such that $|\mathbf{z} - \mathbf{z}_\mathfrak{p}|_\mathfrak{p} < \varepsilon$ for each $\mathfrak{p} \in \mathcal{T}$.

Let $\mathcal{T}' = \{\mathfrak{p}^\sigma \mid \mathfrak{p} \in \mathcal{T},\ \sigma \in \mathrm{Gal}(N/K)\}$. Then $\mathcal{W}' = \mathcal{S}_N \cup \mathcal{T}'$ and $\mathcal{R} = \mathcal{S}_N \smallsetminus \mathcal{T}'$ are K-rational small sets. Choose a finite subset $\mathcal{R}_1$ of $\mathcal{R}$ that represents $\mathcal{R}|_K$. Then $\mathcal{W}_1 = \mathcal{R}_1 \cupdot (\mathcal{T} \cap \mathcal{S}_N) \cupdot (\mathcal{T} \smallsetminus \mathcal{S}_N)$ represents $\mathcal{W} = \mathcal{W}'|_K$.

If $\mathfrak{p} \in \mathcal{T} \smallsetminus \mathcal{S}_N$, then $N_\mathfrak{p} = K_s$ (Remark 2.3(c)). Since $V_{\mathrm{simp}}(K_s)$ is Zariski open in $V(K_s)$, it is $\mathfrak{p}$-dense in $V(K_s)$ [5, Lemma 2.2]. Hence, we can assume without loss that $\mathbf{z}_\mathfrak{p}$ is simple. Finally, for each $\mathfrak{p} \in \mathcal{R}_1$ we choose $\mathbf{z}_\mathfrak{p} \in V_{\mathrm{simp}}(D_{N,\mathfrak{p}})$.

By Lemma 9.1, the point $(\mathbf{z}_\mathfrak{q})_{\mathfrak{q}\in\mathcal{W}_1}$ extends to a point $(\mathbf{z}_\mathfrak{q})_{\mathfrak{q}\in\mathcal{W}'}$ of $V_{K,\mathcal{S},\mathcal{W}}$. Theorem 9.2(c) gives a point $\mathbf{z} \in V(D_{M,\mathcal{V}})$ such that $|\mathbf{z}-\mathbf{z}_\mathfrak{q}|_\mathfrak{q} < \varepsilon$ for each $\mathfrak{q} \in \mathcal{W}'$ and in particular for each $\mathfrak{q} \in \mathcal{T}$.

Proof of Theorem 2.4(b). Replace the use of Theorem 9.2(c) in the proof of (a) by a use of Theorem 9.2(a). ■

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