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ON THE ZEROS OF FUNCTIONS IN THE SELBERG CLASS

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To Professor Swadheen Pattanayak
with deep regards

Abstract: It is proved that under some suitable conditions, the degree two functions in the Selberg class have infinitely many zeros on the critical line.
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1. Introduction

Let $\mathcal{S}$ denote the Selberg class of functions (see [12]). Let $F \in \mathcal{S}$ be a function of *degree* $d > 0$ (defined later). From the theory of integral functions of finite order, it is easy to show that F has infinitely many zeros in the critical strip $0 \leq \sigma \leq 1$. In fact, a formula analogous to the Riemann - von Mangoldt formula for the Riemann zeta-function holds for F (see [12]). It is therefore natural to ask the following

Question: *Does F admit infinitely many zeros on the critical line $\sigma = \frac{1}{2}$?*

The question has an affirmative answer for all degree 1 functions and all known functions of degree 2 in $\mathcal{S}$.

In [10],[11] Kaczorowski and Perelli show that the only functions of degree $0 < d < 5/3$ are the Riemann zeta-function $\zeta(s)$, the Dirichlet L-functions $L(s,\chi)$ associated with primitive characters χ and their imaginary shifts $L(s+i\,\omega,\chi)$ with real ω. (Recently we learnt from Professor A. Ivić in a personal communication that this result has been extended by the same authors up to $0 < d < 2$.) In the case of $\zeta(s)$, Hardy [9] showed that it has an infinity of zeros on the critical line. A similar argument works for $L(s,\chi)$. In fact, for these functions it is known that a positive proportion of zeros lie on the critical line (see [18], [13], [4], [20]).

The known examples of degree 2 functions are product of two degree 1 functions, for example $\zeta^2(s)$, $L(s,\chi_1)L(s,\chi_2)$; the L-function $L_f(s)$ associated with a suitable normalized modular form f which is either holomorphic or a Maass waveform; the Dedekind zeta-function of quadratic number field K, $\zeta_K(s)$; and the imaginary shifts of any of the above functions which are entire, for example $L_f(s)$, $L(s,\chi_1)L(s,\chi_2)$ when χ_1 and χ_2 are non-principal. In all these cases, it is

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known that they have infinitely many zeros on the critical line. In fact, it is known that a positive proportion of zeros lie on the critical line (see [7],[8]). However, since it is not yet known if this list is complete for degree 2, the question has not yet been answered in general.

In this context, Gritsenko in [6] starts with $F_1, F_2, \ldots F_N$ which are distinct primitive functions of degree 2 in $\mathcal{S}$ which satisfy the following conditions:

Condition 1. *The following asymptotic formula holds as* $x \to \infty$*:*

$$\sum_{p \le x} |a_j(p)|^2 \log p = x + O(x \log^{-1} x),$$

where $a_j(n)$ *are the Dirichlet coefficients of* $F_j(s)$, $\quad j = 1, 2, \ldots, N$.

Condition 2. *For any* j, $1 \le j \le N$, *there exists a positive constant* c_{F_j} *such that*

$$|1 + a_j(p)p^{-\frac{1}{2}-it} + a_j(p^2)p^{-1-2it} + \cdots| > c_{F_j},$$

where t *is an arbitrary number in the interval* $[T, 2T+1]$, p *is an arbitrary prime number and* $T \ge T_0 > 0$, T_0 *is a sufficiently large absolute constant.*

Condition 3. *The following asymptotic formula holds as* $x \to \infty$ *:*

$$\sum_{n \le x} |a_j(n)|^2 n^{-1} = A_{F_j} \log x + B_{F_j} + O(\log^{-4} x),$$

where $A_{F_j} > 0$ *and* B_{F_j} *are constants depending only on* $F_j(s)$, $j = 1, 2, \ldots, N$.

He then assumes the truth of the following conjecture of Selberg:

Condition 4. *The following asymptotic formula holds as* $x \to \infty$*:*

$$\sum_{p \le x} a(p)\overline{a'(p)}p^{-1} = O(1),$$

where $a(n)$ *and* $a'(n)$ *are the Dirichlet coefficients of two* ***distinct*** *primitive functions* F *and* F' *in* $\mathcal{S}$.

He then shows that *the function* $\mathcal{F}$ *given by*

$$\mathcal{F}(t) = \sum_{j=1}^{N} b_j Z_{F_j}(t),$$

has at least $T \exp(\sqrt{\log\log\log T})$ *zeros of odd order in* $[T, 2T]$, *where* $Z_{F_j}(t)$ *is defined as in Section 4 and* b_j *are real with* $b_1 \ne 0$.

An important result in this context is by Bombieri and Hejhal (see [2]), which deals with a class of functions $L(s)$ very close to the Selberg class. Roughly speaking, their result is as follows. *If* $L_1(s), \ldots, L_N(s)$ *satisfy the same functional equation, are orthogonal in the sense of Selberg, satisfy the Riemann Hypothesis* *and a certain weak form of the pair correlation conjecture (see* [14]*), then almost all zeros of the linear combination*

$$\sum_{j=1}^{N} b_j L_j(s),$$

where b_j are real, lie on the critical line $\sigma = 1/2$ and are simple.

Unconditionally, the question seems to be difficult for a general function in $\mathcal{S}$ of degree $d \geq 2$. An attempt has been made in this note to answer the question for functions of degree 2 under some mild conditions.

To state the main theorem we need some preparation. We recall that every $F \in \mathcal{S}$ satisfies a functional equation of the type

$$\Phi(s) = \omega \bar{\Phi}(1-s),$$

where

$$\Phi(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s) = \gamma(s) F(s), \tag{1.1}$$

say, with $Q > 0, \lambda_j > 0, \Re\mu_j \geq 0$ and $|\omega| = 1$. Here $\bar{f}(s) = \overline{f(\bar{s})}$.

The functional equation can be rewritten as

$$F(s) = \Delta_F(s) \bar{F}(1-s),$$

where

$$\Delta_F(s) = \omega \frac{\bar{\gamma}(1-s)}{\gamma(s)}\cdot$$

We also recall that $\log F(s)$ is a Dirichlet series with coefficients $b(n)$ satisfying $b(n) = 0$ unless $n = p^m$, for some prime p and integer $m \geq 1$. This implies that F has an Euler product expansion and hence the Dirichlet coefficients $a(n)$ of F are multiplicative. Multiplicativity of the coefficients is crucial in the proof of our theorem.

>From now onwards the function $F \in \mathcal{S}$ will always be denoted by $F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$, $s = \sigma + it$ with the following *data*, which are invariants, associated with it (see [12]): the degree $d = 2\sum_{j=1}^{r} \lambda_j$, the ξ-invariant $\xi = 2\sum_{j=1}^{r} (\mu_j - 1/2)$ and the conductor $q = (2\pi)^d Q^2 \prod_{j=1}^{r} \lambda_j^{2\lambda_j}$. The implied constants in the O-terms appearing at various places may depend only on the conductor q. The aim of this note is to establish the following

Theorem. *Let $F \in S$ with $d = 2$, ξ real and $\sqrt{q}$ irrational and suppose the Dirichlet coefficients $a(n)$ of F satisfy*

$$\sum_{n \leq x} |a(n)|^2 = O(x), \tag{1.2}$$

then $F(s)$ has infinitely many zeros on the critical line $\Re s = \frac{1}{2}$.

Remark 1. The condition (1.2) can be derived from Condition 3 by a straight forward application of partial summation.

Remark 2. The presence of the factor $t^{-i\Im\xi}$ in (2.1) makes it difficult to apply Lemma 1 to evaluate the integral in (4.4). Therefore, we imposed the condition that ξ is real. Moreover, this condition is benign for functions F which are regular at $s = 1$, as a suitable imaginary shift of F can be made to satisfy this and hence the conclusion of the theorem will hold for F also.

Remark 3. The conditions of the theorem are satisfied for the functions $L(s, \chi_1)L(s, \chi_2)$ where χ_1 and χ_2 are primitive Dirichlet characters to the modulus q_1 and q_2 respectively provided $q_1 q_2$ is not a perfect square. The conditions also hold for $L_f(s)$ for a cuspidal modular form f of level N if N is not a perfect square.

Remark 4. The conditions of the theorem are *not* satisfied for $\zeta^2(s)$ or $L^2(s, \chi)$. However, since our proof follows closely the Hardy - Littlewood proof (as given in [19], pp 260–262) of Hardy's theorem for $\zeta(s)$, which counts the zeros of odd order on the critical line, these exceptions are to be expected.

Remark 5. The main step in the proof of the theorem is the application of Daboussi and Delange's result (see Lemma 2 below). It must be pointed out that, although not in $\mathcal{S}$, in the case of the Epstein zeta-function associated to certain positive definite binary quadratic forms and in the case of the ideal class zeta-functions associated to certain quadratic number fields, the crucial exponential sum estimate (3.5) was derived from the arithmetic information contained in their Dirichlet coefficients and thereby the infinitude of zeros on the critical line was established (see [16] and [3] respectively).

The plan of this note is as follows. In section 2 we shall derive an asymptotic expression for $\Delta_F(s)$ and state some growth estimates. Section 3 will deal with some exponential lemmas and a lower bound estimate for the first power mean of a Dirichlet series on the critical line. In section 4, we shall prove the theorem.

2. Asymptotic expansion and order estimates

The well-known Stirling's formula for the Γ-function states that in any fixed vertical strip $-\infty < \alpha \le \sigma \le \beta < \infty$,

$$\Gamma(\sigma + it) = (2\pi)^{1/2} t^{\sigma+it-1/2} e^{-\frac{\pi}{2}t+\frac{\pi}{2}i(\sigma-1/2)-it}(1 + O(1/t)) \quad \text{as } t \to \infty.$$

Using this formula along with the functional equation for $F(s)$, we obtain

$$\Delta_F(s) = \omega_1 (Q_1 t^{d/2})^{1-2\sigma-2it} t^{-i\Im\xi} e^{idt}(1 + O(1/t)), \tag{2.1}$$

where $\omega_1 = \omega e^{-\frac{\pi}{2}i(d/2+\Re\xi)+i\Im\xi} \prod_{j=1}^{r} \lambda_j^{-2i\Im\mu_j}$, with $|\omega_1| = 1$, and $Q_1 = Q \prod_{j=1}^{r} \lambda_j^{\lambda_j}$. Note that $Q_1 = \sqrt{q}\,(2\pi)^{-d/2}$.

As $d = 2$ and $\Im\xi = 0$ in the present case, we get

$$\Delta_F(s) = \omega_1'(Q_1 t)^{1-2\sigma-2it} e^{2it}(1 + O(1/t)). \tag{2.2}$$

where $\omega_1' = \omega e^{-\frac{\pi}{2}i(1+\xi)} \prod_{j=1}^{r} \lambda_j^{-2i\Im\mu_j}$.

Thus, we have

$$\Delta_F^{-1/2}(s) = \omega_2(Q_1 t)^{\sigma-1/2+it} e^{-it}(1 + O(1/t)), \tag{2.3}$$

where $\omega_2 = (\omega_1')^{-1/2}$ with $|\omega_2| = 1$.

Next, we use the following uniform convexity estimates which are easy to verify.

$$F(s) = \begin{cases} O(Q_1^{1-\sigma} t^{1-\sigma+\epsilon}) & 0 \leq \sigma \leq 1 \\ O(t^{\epsilon}), \ \sigma > 1. \end{cases}$$

From (2.3) we get

$$\Delta_F^{-1/2}(s)F(s) = \begin{cases} O(Q_1^{1/2} t^{1/2+\epsilon}) & 0 \leq \sigma \leq 1 \\ O(Q_1^{1/2} t^{\sigma-1/2+\epsilon}) & 1 < \sigma \leq 1 + \delta \end{cases} \tag{2.4}$$

3. Estimates on oscillatory integrals

We use the following result from Potter and Titchmarsh [16] on exponential integrals.

Lemma 1. *Let*

$$J = \int_T^{T'} t^{\alpha} \left(\frac{t}{e\beta}\right)^{it} dt,$$

where $\alpha, \beta > 0$, *and* $0 < T \leq T' \leq 2T$. *Then*

$$J = O(T^{\alpha}/\log(T/\beta)) \quad \textit{if} \quad \beta < T, \tag{3.1}$$

$$J = (2\pi)^{1/2}\beta^{\alpha+1/2} e^{i\pi/4-i\beta} + O(T^{\alpha+2/5}) + O(T^{\alpha}/\log(\beta/T)) + + O(T^{\alpha}/\log(T'/\beta)) \quad \textit{if} \quad T < \beta < T', \tag{3.2}$$

$$J = O(T^{\alpha}/\log(\beta/T')) \quad \textit{if} \quad T' < \beta, \tag{3.3}$$

$$J = O(T^{\alpha+1/2}) \quad \textit{in any case.} \tag{3.4}$$

We will also use the following result on exponential sums from Daboussi and Delange [5].

Lemma 2. *Let f be a multiplicative arithmetical function satisfying the condition $\sum_{n \leq x} |f(n)|^2 = O(x)$. Then, for every irrational α, we have*

$$\sum_{n \leq x} f(n) e^{2\pi i n\alpha} = o(x). \tag{3.5}$$

Remark 6. We can show that the method of Daboussi and Delange in [5] can be worked out more carefully to replace $o(x)$ in (3.5) with $O(x(\log\log x)^{-1/2})$. In fact, under the additional condition that $|f(p)| \leq A$ for all primes p, Montgomery and Vaughan [15] improved the O-estimate to $O(x\log^{-1} x)$.

Remark 7. For α rational, the estimate (3.5) was established by Daboussi and Delange [5] under the restriction that $|f(n)| \leq 1$ and some other conditions. However, in the general case, the Dirichlet coefficients of functions in $\mathcal{S}$ do not satisfy this restriction.

Now we state Theorem 3 of [1] as:

Lemma 3. *Let $B(s) = \sum_{n=1}^{\infty} b_n n^{-s}$ be any Dirichlet series satisfying the following conditions:*

(i) not all b_n's are zero;
(ii) the function can be continued analytically in $\sigma \geq a$, $|t| \geq t_0$, and in this region $B(s) = O((|t| + 10)^A)$.

Then for every $\epsilon > 0$, we have

$$\int_T^{T+H} |B(\sigma + it)| dt \gg H$$

for all $H \geq (\log T)^{\epsilon}$, $T \geq T_0(\epsilon)$, and $\sigma > a$.

Remark 8. Ramachandra showed (see [17], Chapter II) that the first power mean of a generalized Dirichlet series satisfying certain conditions can not be too small. Lemma 3 is a particular case of this general theorem, which is quite useful in obtaining lower bounds of the type (4.1), even in short-intervals.

4. Proof of the main theorem

We define the function

$$Z_F(t) = \Delta_F(1/2 + it)^{-1/2} F(1/2 + it),$$

where $Z_F(t)$ is the analogue of Hardy's function $Z(t)$ in the theory of Riemann's zeta-function. The functional equation implies that $Z_F(t)$ is real for real t. Thus the zeros of $F(s)$ on the critical line correspond to the real zeros of $Z_F(t)$.

Suppose now that $Z_F(t)$ has no zeros in the interval $[T, 2T]$ where $T \geq T_1 > 0$, T_1 is a sufficiently large absolute constant.

Consider the integral

$$I = \int_T^{2T} Z_F(t) dt.$$

We are going to estimate $|I|$ from below and above to derive eventually a contradiction. Since the integrand is of constant sign by our assumption, we have

$$|I| = \int_T^{2T} |Z_F(t)| dt.$$

The lower bound

$$|I| \gg T \tag{4.1}$$

follows on taking $H = T$ in Lemma 3.

As for the upper bound estimation, we first write the integral I as

$$I = -i \int_{1/2+iT}^{1/2+i2T} \Delta_F^{-1/2}(s)F(s)ds. \tag{4.2}$$

Next, we move the line of integration to $\sigma = 1+\delta, (\delta > 0$ is a small positive constant less than $1/10)$ and apply Cauchy's theorem to the integral

$$\int \Delta_F^{-1/2}(s)F(s)ds,$$

along the rectangle with sides $\sigma = 1/2$, $\sigma = 1+\delta$, $t = T$ and $t = 2T$.

By Cauchy's theorem and the estimate (2.4), the integral (4.2) reduces to

$$\int_T^{2T} \Delta_F^{-1/2}(1+\delta+it)F(1+\delta+it)dt, \tag{4.3}$$

with an error $O(T^{1/2+\delta+\epsilon}) = o(T)$ coming from the horizontal lines.

Substituting the asymptotic formula (2.3) in (4.3), we see that the expression (4.3) is a constant multiple of

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^{1+\delta}} \int_T^{2T} (Q_1 t)^{1/2+\delta} e^{it\log(\frac{Q_1 t}{en})}\big(1 + O(1/t)\big)dt. \tag{4.4}$$

The $O-$term in (4.4) is trivially $O(T^{1/2+\delta}) = o(T)$.

The problem, therefore, reduces to the estimation of the sum

$$Q_1^{1/2+\delta} \sum_{n=1}^{\infty} \frac{a(n)}{n^{1+\delta}} \int_T^{2T} t^{1/2+\delta} \left(\frac{Q_1 t}{en}\right)^{it} dt. \tag{4.5}$$

The integral in (4.5) is evaluated by exponential integral techniques (Lemma 1). First we subdivide the sum in (4.5) into sub-intervals

$$1 \le n \le Q_1T - 1, \qquad Q_1T + 1 \le n \le 2Q_1T - 1, \qquad 2Q_1T + 1 \le n,$$

and denote the corresponding sums over these ranges as $\Sigma_1, \Sigma_2, \Sigma_3$ respectively.

Notice that there are at most 4 integers n from the range of (4.5) which have not been included in the above ranges. As $n \asymp Q_1T$ (by which we mean $Q_1T \ll n \ll Q_1T$) for these integers and $a(n) = O(n^\epsilon)$, their contribution to (4.5) is

$$O\Big(\frac{n^\epsilon}{n^{1+\delta}} T^{1/2+\delta} T\Big) = O\big(T^{1/2+\epsilon}\big) = o(T).$$

The sum Σ_1 is estimated by using (3.1) (with $T' = 2T$) to be

$$\Sigma_1 = O\left(T^{1/2+\delta} \sum \frac{|a(n)|}{n^{1+\delta} \log(Q_1 T/n)}\right). \tag{4.6}$$

We then further sub-divide the range of (4.6) into two sub-sums as follows

$$1 \le n \le Q_1 T/2, \qquad Q_1 T/2 < n \le Q_1 T - 1,$$

and denote the corresponding sums by Σ_{11} and Σ_{12} respectively.

For estimating Σ_{11}, we note that $\log(Q_1 T/n) \ge \log 2$ in this range and hence we get

$$\Sigma_{11} = O\left(T^{1/2+\delta} \sum \frac{|a(n)|}{n^{1+\delta}}\right) = O\left(T^{1/2+\delta}\right) = o(T).$$

For estimating Σ_{12}, we use the inequality $\log(Q_1 T/n) \ge (Q_1 T - n)/Q_1 T$, $n \asymp Q_1 T$ and $a(n) = O(n^\epsilon)$ and obtain

$$\Sigma_{12} = O\left(T^{1/2+\delta} \sum \frac{|a(n)|}{n^{1+\delta}} \frac{Q_1 T}{Q_1 T - n}\right) = O\left(T^{1/2+\epsilon} \sum \frac{1}{Q_1 T - n}\right). \tag{4.7}$$

Observe that the last sum in (4.7) is

$$\sum (Q_1 T - n)^{-1} = \sum_{1 \le k < Q_1 T/2} (k + f)^{-1} = O(\log T),$$

where $f = Q_1 T - \lfloor Q_1 T \rfloor \ge 0$ is the fractional part of $Q_1 T$. Using this in (4.7) gives $\Sigma_{12} = O(T^{1/2+\epsilon} \log T) = o(T)$.

Estimating Σ_3 is similar to that of Σ_1 except that we use (3.3) instead of (3.1). Hence, we find in a similar manner that $\Sigma_3 = o(T)$.

The sum Σ_2 is estimated by using (3.2) (with $T' = 2T$) to be

$$\Sigma_2 = C \sum a(n) e^{-in/Q_1} + O\left(\sum \frac{|a(n)|}{n^{1+\delta}} T^{9/10+\delta}\right) + \tag{4.8}$$
$$+ O\left(T^{1/2+\delta} \sum \frac{|a(n)|}{n^{1+\delta}} E_n\right),$$

where $C = (2\pi/Q_1)^{\frac{1}{2}} e^{i\frac{\pi}{4}}$ is a constant and $E_n = \log^{-1}(n/(Q_1 T)) + \log^{-1}(2Q_1 T/n)$.

Using the fact that $Q_1 = \sqrt{q}(2\pi)^{-1}$, we get that the main exponential sum in (4.8) is a constant multiple of

$$\sum_{Q_1 T + 1 \le n \le 2Q_1 T - 1} a(n) e^{-2\pi i n/\sqrt{q}}.$$

By the assumption (1.2) in Theorem 1, the irrationality of $\sqrt{q}$ and Lemma 2, we get this sum to be $o(T)$.

The first O-term in (4.8) is estimated by using $n \asymp Q_1 T$ and $a(n) = O(n^\epsilon)$ to be $O(T^{9/10+\epsilon}) = o(T)$.

The estimation of the second O-term in (4.8) is done in the same way as was done for Σ_{12} to conclude that it is $O((T)^{1/2+\epsilon}\log T) = o(T)$.

Collecting all the estimates, we have

$$\int_T^{2T} Z_F(t)dt = o(T). \tag{4.9}$$

Thus from (4.1) and (4.9), we derive a contradiction. This completes the proof of the theorem.

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