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MULTIPLE INTEGRALS AND LINEAR FORMS IN ZETA-VALUES

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Dedicated to Jean-Marc Deshouillers
on the occasion of his sixtieth birthday

Abstract: We define n-dimensional Beukers-type integrals over the unit hypercube. Using an n-dimensional birational transformation we show that such integrals are equal to suitable n-dimensional Sorokin-type integrals with linear constraints, and represent linear forms in $1, \zeta(2), \zeta(3), \dots, \zeta(n)$ with rational coefficients.
Keywords: Multiple integrals of rational functions, values of the Riemann zeta-function, birational transformations.

1. Introduction

In recent years the study of $\mathbb{Q}$-linear forms in the values of the Riemann zeta-function at positive integers aroused the interest of several authors, since Rivoal's theorem [1], [9] on the existence of infinitely many irrational values of the zeta-function at odd positive integers. Basically, two main techniques are employed in this context, often interacting with each other: namely, the arithmetical study of multiple integrals of suitable rational functions, and the study of multiple hypergeometric and polylogarithmic series. Among the many papers devoted to this or to related subjects we mention [4], [5], [6], [10], [11], [13], [16], [17], [18].

Concerning in particular the diophantine study of $\zeta(2)$ and $\zeta(3)$, successive improvements on the irrationality measures of such constants were given by several authors (see [14], pp. 562-563, for an account of this). The methods used to obtain these results are all related to the double and triple integrals introduced by Beukers in [2]. The best known irrationality measures of $\zeta(2)$ and $\zeta(3)$ were obtained by Rhin and Viola [7], [8], who studied double and triple Beukers-type integrals with unequal exponents by introducing an algebraic method based on the structure of certain permutation groups. On the other hand, Sorokin [12] employed the triple integral

$$\int_{(0,1)^3} \left(\frac{x(1-x)y(1-y)z(1-z)}{(1-xy)(1-xyz)} \right)^m \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{(1-xy)(1-xyz)}$$

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to give a further proof of Apéry's theorem on the irrationality of $\zeta(3)$. The equivalence of Beukers-type and Sorokin-type triple integrals with unequal exponents satisfying suitable conditions was shown in [4] and [15]. Specifically, in [15], Section 3, Viola introduced a family of 32 three-dimensional birational transformations which can be used as changes of variables from a Beukers-type to a Sorokin-type triple integral.

The problem of extending to higher dimensions the arithmetical equivalence of different kinds of multiple integrals was studied by several mathematicians. In particular, following Vasilyev [13] and other authors, Fischler [4] considered the generalisation of Beukers' three-dimensional measure

$$\frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1-(1-xy)z} \tag{1.1}$$

in Vasilyev's form

$$\frac{\mathrm{d}x_1\ldots\mathrm{d}x_n}{1-(1-(1-\ldots-(1-x_1)\ldots x_{n-2})x_{n-1})x_n}, \tag{1.2}$$

and gave an n-dimensional change of variables transforming a Vasilyev-type into a Sorokin-type multiple integral. However, Vasilyev's measure (1.2) is not the only reasonable generalisation of (1.1) to the higher-dimensional case, and in the present paper we analyse Beukers-type multiple integrals with the measure

$$\frac{\mathrm{d}x_1\ldots\mathrm{d}x_n}{1-(1-x_1\ldots x_{n-1})x_n}. \tag{1.3}$$

The measure (1.3) has some technical advantages over (1.2) and suggests in particular the definition of a birational transformation η_n (see (2.1) and (2.2) below) which generalises one of the above-mentioned 32 birational transformations introduced in [15]. In Theorem 2.1 we use η_n as a change of variables to transform an n-dimensional Beukers-type integral with the measure (1.3) into an n-dimensional Sorokin-type integral with linear constraints.

Since Vasilyev's multiple integrals are known to be $\mathbb{Q}$-linear forms in the values of the Riemann zeta-function at positive integers of a given parity (see [4], [5], [10], [13], [16], [17]), it is natural to investigate the arithmetical structure of Beukers-type multiple integrals with the measure (1.3). Results of this kind were obtained by Nesterenko in [6], as a consequence of more general theorems involving the arithmetic of linear forms in polylogarithms. On the other hand, in a recent paper Cresson, Fischler and Rivoal [3] make a deep study of multiple hypergeometric integrals and series related to multiple polylogarithmic functions. In particular these authors analyse the arithmetic of suitably defined multiple Sorokin-type integrals, showing that such integrals are in general $\mathbb{Q}$-linear combinations of polyzeta-values. Owing to the great arithmetical complexity of this general framework, a natural quest consists in seeking conditions for a multiple Sorokin-type integral to be a $\mathbb{Q}$-linear combination only of values of the Riemann zeta-function at positive integers, avoiding the occurrence of polyzeta-values. In Theorem 3.1, using the birational transformation η_n, we give a new proof that any convergent n-dimensional Beukers-type integral without constraints, and therefore, by Theorem 2.1, also any n-dimensional Sorokin-type integral with natural linear constraints, are indeed $\mathbb{Q}$-linear combinations of 1, $\zeta(2)$, $\zeta(3)$, $\ldots$, $\zeta(n)$.

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2. An n-dimensional birational transformation

For any $n \geqslant 2$ we introduce an n-dimensional birational transformation

$$\eta_n : (x_1, \ldots, x_n) \longmapsto (X_1, \ldots, X_n)$$

through the equations

$$\eta_n : \begin{cases} X_1 &= \dfrac{(1 - x_1 \ldots x_{n-1})x_n}{1 - x_1 \ldots x_n} \\ X_2 &= \dfrac{(1 - x_1 \ldots x_{n-2})x_{n-1}}{1 - x_1 \ldots x_{n-1}} \\ \ldots & \ldots \ldots \ldots \\ X_{n-1} &= \dfrac{(1 - x_1)x_2}{1 - x_1 x_2} \\ X_n &= 1 - x_1 \ldots x_n . \end{cases} \tag{2.1}$$

This transformation generalises to any dimension n one of the 32 birational transformations defined in [15], p. 147, for the case $n = 3$, namely the transformation $\sigma\vartheta^2\eta$ with the notation therein.

As is easily seen, the inverse of (2.1) is given by the equations

$$\eta_n^{-1} : \begin{cases} x_1 &= \dfrac{1 - X_n}{1 - (1 - X_1 \ldots X_{n-1})X_n} \\ x_2 &= \dfrac{1 - (1 - X_1 \ldots X_{n-1})X_n}{1 - (1 - X_1 \ldots X_{n-2})X_n} \\ \ldots & \ldots \ldots \ldots \\ x_{n-1} &= \dfrac{1 - (1 - X_1 X_2)X_n}{1 - (1 - X_1)X_n} \\ x_n &= 1 - (1 - X_1)X_n . \end{cases} \tag{2.2}$$

Using η_n as a change of variables we prove the following

Theorem 2.1. *For any non-negative integers $a_1, \ldots, a_n$; $b_1, \ldots, b_n$; $c_1, \ldots, c_{n-1}$ satisfying the linear conditions*

$$\begin{aligned} a_1 + c_2 &= a_2 + c_1 \\ a_2 + c_3 &= a_3 + c_2 \\ &\ldots \ldots \\ a_{n-2} + c_{n-1} &= a_{n-1} + c_{n-2} \end{aligned} \tag{2.3}$$

the Beukers-type integral

$$B_n = \int\limits_{(0,1)^n} \frac{X_1^{a_1}(1-X_1)^{b_1}\dots X_n^{a_n}(1-X_n)^{b_n}}{(1-(1-X_1\dots X_{n-1})X_n)^{b_n+a_1-c_1}} \frac{\mathrm{d}X_1\dots\mathrm{d}X_n}{1-(1-X_1\dots X_{n-1})X_n} \tag{2.4}$$

is equal to the Sorokin-type integral

$$S_n = \tag{2.5}$$

$$\int\limits_{(0,1)^n} \frac{x_1^{b_n}(1-x_1)^{a_{n-1}}\,x_2^{c_{n-1}}(1-x_2)^{b_{n-1}}\dots x_n^{c_1}(1-x_n)^{b_1}}{(1-x_1x_2)^{a_{n-1}+b_{n-1}-a_{n-2}}\dots(1-x_1\dots x_{n-1})^{a_2+b_2-a_1}(1-x_1\dots x_n)^{a_1+b_1-a_n}}$$

$$\times\frac{\mathrm{d}x_1\dots\mathrm{d}x_n}{(1-x_1x_2)\dots(1-x_1\dots x_n)}\,.$$

Proof. Since $c_2,\dots,c_{n-1}$ do not appear explicitly in (2.4), for B_n the conditions (2.3) do not represent constraints but should be viewed instead as definitions of $c_2,\dots,c_{n-1}$ successively. We also remark that both B_n and S_n are finite if and only if $a_1,\dots,a_n;\ b_1,\dots,b_n;\ c_1,\dots,c_{n-1}\geqslant 0$.

From (2.1) and (2.2) it is plain that η_n is a one-to-one mapping of the open unit hypercube $(0,1)^n$ onto $(0,1)^n$. Moreover η_n satisfies

$$\frac{\mathrm{d}X_1\dots\mathrm{d}X_n}{1-(1-X_1\dots X_{n-1})X_n} = (-1)^{[n/2]+1}\frac{\mathrm{d}x_1\dots\mathrm{d}x_n}{(1-x_1x_2)\dots(1-x_1\dots x_n)}\,. \tag{2.6}$$

This can be proved by factoring out in the jacobian determinant

$$\frac{\mathrm{d}(X_1,\dots,X_n)}{\mathrm{d}(x_1,\dots,x_n)}$$

the denominators $(1-x_1x_2)^2,\ \dots,\ (1-x_1\dots x_n)^2$ from the first $n-1$ rows and then subtracting from the first row the last multiplied by $1-x_n$. In the determinant thus obtained the first row has all zeros except for the last entry which is $1-x_1\dots x_n$. We expand this determinant along the first row, and in the remaining determinant of order $n-1$ we factor out x_n from the last row. Then we subtract from the first row the last multiplied by $1-x_{n-1}$, and we iterate the process. This easily yields (2.6).

By (2.6) the change of variables η_n transforms B_n into the integral S_n, since from (2.3) we have

$$a_1-c_1 = a_2-c_2 = \dots = a_{n-1}-c_{n-1}. \tag{2.7}$$

Conversely, if for any non-negative integers $a_1,\dots,a_n;\ b_1,\dots,b_n;\ c_1,\dots,c_{n-1}$ we apply to S_n the change of variables η_n^{-1} we get the above integral B_n provided the linear conditions (2.3) hold, since (2.3) are necessary and sufficient to drop from B_n the factors $1-(1-X_1)X_n, \ldots, 1-(1-X_1\ldots X_{n-2})X_n$ arising from the change of variables (2.2). ■

Theorem 2.1 shows that there is a natural equivalence, obtained through the action of the birational transformation η_n, between an integral B_n without constraints and an integral S_n with the linear constraints (2.3).

3. Linear forms in zeta-values

With the next theorem we show that any n-dimensional Beukers-type integral B_n, and hence, by Theorem 2.1, any n-dimensional Sorokin-type integral S_n with the linear constraints (2.3), are linear combinations of 1, $\zeta(2)$, $\zeta(3)$, $\ldots$, $\zeta(n)$ with rational coefficients.

Theorem 3.1. *For any $n\geqslant 2$ and any non-negative integers $a_1,\ldots,a_n$; $b_1,\ldots,b_n$; $c_1,\ldots,c_{n-1}$ satisfying the linear conditions (2.3), the integral*

$$B_n = \int\limits_{(0,1)^n} \frac{X_1^{a_1}(1-X_1)^{b_1}\ldots X_n^{a_n}(1-X_n)^{b_n}}{(1-(1-X_1\ldots X_{n-1})X_n)^{b_n+a_1-c_1}}\,\frac{\mathrm{d}X_1\ldots\mathrm{d}X_n}{1-(1-X_1\ldots X_{n-1})X_n} \tag{3.1}$$

equals

$$A_1 + A_2\zeta(2) + A_3\zeta(3) + \ldots + A_{n-1}\zeta(n-1) + A_n(n-1)\zeta(n)$$

with $A_1, A_2, \ldots, A_{n-1} \in \mathbb{Q}$, $A_n \in \mathbb{Z}$. Moreover, if $n\geqslant 3$ and if

$$a_n + b_n \leqslant b_1 + \ldots + b_{n-1} + n - 3 \tag{3.2}$$

then $A_2 = 0$.

A proof of this theorem can be obtained as a special case of some formulae given by Nesterenko [6] relating multiple integrals over the unit hypercube to suitable Mellin-Barnes-type hypergeometric integrals and to linear forms in polylogarithms. We give here a more elementary and direct proof of Theorem 3.1 using the birational transformation η_n and a reduction from S_n to S_{n-1} (see the proof of Lemma 3.6 below).

Our proof of Theorem 3.1 is by induction on the dimension n. For $n=2$ the theorem holds by [7], Theorems 2.1 or 2.2, after replacing X_1 with $1-X_1$. Let $n\geqslant 3$ and assume the theorem holds for $n-1$. We require some lemmas.

Lemma 3.2. *The value of the integral B_n in (3.1) is unchanged under the action of the permutation*

$$\boldsymbol{\nu} = (a_1\ c_1)\ \ldots\ (a_{n-1}\ c_{n-1})(a_n\ b_n)$$

which preserves the linear conditions (2.3).

Proof. We apply to B_n the change of variable

$$X_n = \frac{1-\xi_n}{1-(1-X_1\ldots X_{n-1})\xi_n}$$

which satisfies

$$\frac{\mathrm{d}X_n}{1-(1-X_1\ldots X_{n-1})X_n} = -\frac{\mathrm{d}\xi_n}{1-(1-X_1\ldots X_{n-1})\xi_n}$$

and maps the interval $0<\xi_n<1$ onto $0<X_n<1$. The lemma easily follows from (2.7). ■

Lemma 3.3. *For any $n \geqslant 3$ and any integers $a_1,\ldots,a_{n-1} \geqslant 0$ we have*

$$\int\limits_{(0,1)^n} X_1^{a_1}\ldots X_{n-1}^{a_{n-1}} \frac{\mathrm{d}X_1\ldots\mathrm{d}X_n}{1-(1-X_1\ldots X_{n-1})X_n} \tag{3.3}$$
$$= A_1 + A_3\zeta(3) + \ldots + A_{n-1}\zeta(n-1) + A_n(n-1)\zeta(n)$$

with $A_1, A_3, \ldots, A_n \in \mathbb{Q}$ such that

$$\begin{cases} A_n = 1 \quad \textit{and } A_3 = \ldots = A_{n-1} = 0, & \textit{if } a_1 = \ldots = a_{n-1}, \\ A_n = 0, \quad \textit{otherwise.} \end{cases}$$

Proof. Denote the integral (3.3) by $I_{a_1,\ldots,a_{n-1}}$. On integrating with respect to X_n we get

$$\begin{aligned} I_{a_1,\ldots,a_{n-1}} &= -\int\limits_{(0,1)^{n-1}} X_1^{a_1}\ldots X_{n-1}^{a_{n-1}} \frac{\log(X_1\ldots X_{n-1})}{1-X_1\ldots X_{n-1}}\,\mathrm{d}X_1\ldots\mathrm{d}X_{n-1} \\ &= -\sum_{j=1}^{n-1}\int\limits_{(0,1)^{n-1}} X_1^{a_1}\ldots X_{n-1}^{a_{n-1}} \frac{\log X_j}{1-X_1\ldots X_{n-1}}\,\mathrm{d}X_1\ldots\mathrm{d}X_{n-1}. \end{aligned}$$

Since

$$\begin{aligned} -&\int\limits_{(0,1)^{n-1}} X_1^{a_1}\ldots X_{n-1}^{a_{n-1}} \frac{\log X_1}{1-X_1\ldots X_{n-1}}\,\mathrm{d}X_1\ldots\mathrm{d}X_{n-1} \\ &= -\sum_{k=0}^{\infty}\int\limits_{(0,1)^{n-1}} X_1^{a_1}\ldots X_{n-1}^{a_{n-1}}(X_1\ldots X_{n-1})^k \log X_1\,\mathrm{d}X_1\ldots\mathrm{d}X_{n-1} \\ &= -\sum_{k=0}^{\infty}\int_0^1 X_1^{a_1+k}\log X_1\,\mathrm{d}X_1 \int_0^1 X_2^{a_2+k}\mathrm{d}X_2 \ldots \int_0^1 X_{n-1}^{a_{n-1}+k}\mathrm{d}X_{n-1} \\ &= \sum_{h=1}^{\infty}\frac{1}{(a_1+h)^2\,(a_2+h)\ldots(a_{n-1}+h)}\,, \end{aligned}$$

we obtain

$$I_{a_1,\dots,a_{n-1}} = \sum_{h=1}^{\infty} \frac{1}{(a_1+h)\dots(a_{n-1}+h)} \left(\frac{1}{a_1+h} + \dots + \frac{1}{a_{n-1}+h}\right). \tag{3.4}$$

Up to applying a permutation of the variables $X_1, \dots, X_{n-1}$ we may assume $a_1 \leqslant \dots \leqslant a_{n-1}$. If $a_1 = \dots = a_{n-1}$ we have

$$I_{a_1,\dots,a_1} = (n-1)\sum_{h=1}^{\infty} \frac{1}{(a_1+h)^n} = (n-1)\Big(\zeta(n) - \sum_{j=1}^{a_1} \frac{1}{j^n}\Big),$$

which proves the lemma in this case. For $n=3$ and $a_1 < a_2$ we have by (3.4)

$$\begin{aligned} I_{a_1,a_2} &= \frac{1}{a_2-a_1} \sum_{h=1}^{\infty} \left(\frac{1}{a_1+h} - \frac{1}{a_2+h}\right)\left(\frac{1}{a_1+h} + \frac{1}{a_2+h}\right) \\ &= \frac{1}{a_2-a_1} \sum_{h=1}^{\infty} \left(\frac{1}{(a_1+h)^2} - \frac{1}{(a_2+h)^2}\right) = \frac{1}{a_2-a_1} \sum_{j=a_1+1}^{a_2} \frac{1}{j^2}, \end{aligned}$$

whence the lemma holds for $n=3$.

Let now $n > 3$ and assume the lemma holds for $n-1$. If $a_1 < a_{n-1}$ we get by (3.4)

$$\begin{aligned} I_{a_1,\dots,a_{n-1}} = &\frac{1}{a_{n-1}-a_1} \sum_{h=1}^{\infty} \frac{1}{(a_2+h)\dots(a_{n-2}+h)} \left(\frac{1}{a_1+h} - \frac{1}{a_{n-1}+h}\right) \\ &\times \left(\frac{1}{a_1+h} + \dots + \frac{1}{a_{n-1}+h}\right) \\ = &\frac{1}{a_{n-1}-a_1} \sum_{h=1}^{\infty} \frac{1}{(a_1+h)\dots(a_{n-2}+h)} \left(\frac{1}{a_1+h} + \dots + \frac{1}{a_{n-2}+h}\right) \\ &- \frac{1}{a_{n-1}-a_1} \sum_{h=1}^{\infty} \frac{1}{(a_2+h)\dots(a_{n-1}+h)} \left(\frac{1}{a_2+h} + \dots + \frac{1}{a_{n-1}+h}\right) \\ = &\frac{1}{a_{n-1}-a_1} I_{a_1,\dots,a_{n-2}} - \frac{1}{a_{n-1}-a_1} I_{a_2,\dots,a_{n-1}}. \end{aligned}$$

By the inductive assumption, both $I_{a_1,\dots,a_{n-2}}$ and $I_{a_2,\dots,a_{n-1}}$ are linear combinations of 1, $\zeta(3)$, $\dots$, $\zeta(n-1)$ with rational coefficients, and therefore so is $I_{a_1,\dots,a_{n-1}}$. ■

Lemma 3.4. *For any $n \geqslant 3$ and any non-negative integers $a_1, \dots, a_{n-1}$; $b_1, \dots, b_{n-1}$ we have*

$$\int_{(0,1)^n} X_1^{a_1}(1-X_1)^{b_1} \dots X_{n-1}^{a_{n-1}}(1-X_{n-1})^{b_{n-1}} \frac{\mathrm{d}X_1 \dots \mathrm{d}X_n}{1-(1-X_1\dots X_{n-1})X_n} \tag{3.5}$$

$$= A_1 + A_3\zeta(3) + \dots + A_{n-1}\zeta(n-1) + A_n(n-1)\zeta(n)$$

with $A_1, A_3, \dots, A_{n-1} \in \mathbb{Q}$, $A_n \in \mathbb{Z}$.

Proof. We expand $(1-X_1)^{b_1}, \ldots, (1-X_{n-1})^{b_{n-1}}$ using the binomial theorem. Thus we express the integral (3.5) as a linear combination with integer coefficients of integrals of type (3.3), and the conclusion follows from Lemma 3.3. ∎

Lemma 3.5. *For any non-negative integers* $a_1,\ldots,a_{n-1}\,;\, b_1,\ldots,b_{n-1}\,;\, c_1,\ldots,c_{n-1}$ *satisfying (2.3) and such that* $a_1 \neq c_1$ *we have*

$$\int\limits_{(0,1)^n} \frac{X_1^{a_1}(1-X_1)^{b_1}\ldots X_{n-1}^{a_{n-1}}(1-X_{n-1})^{b_{n-1}}}{(1-(1-X_1\ldots X_{n-1})X_n)^{a_1-c_1}} \times \frac{\mathrm{d}X_1\ldots\mathrm{d}X_n}{1-(1-X_1\ldots X_{n-1})X_n} \in \mathbb{Q}. \tag{3.6}$$

Proof. If $a_1 < c_1$, (3.6) is the integral of a polynomial with integer coefficients:

$$\int\limits_{(0,1)^n} X_1^{a_1}(1-X_1)^{b_1}\ldots X_{n-1}^{a_{n-1}}(1-X_{n-1})^{b_{n-1}} \times (1-(1-X_1\ldots X_{n-1})X_n)^{c_1-a_1-1}\,\mathrm{d}X_1\ldots\mathrm{d}X_n,$$

and hence is a rational number. If $a_1 > c_1$ we apply the permutation $\boldsymbol{\nu}$. By Lemma 3.2 the integral (3.6) becomes

$$\int\limits_{(0,1)^n} X_1^{c_1}(1-X_1)^{b_1}\ldots X_{n-1}^{c_{n-1}}(1-X_{n-1})^{b_{n-1}} \times (1-(1-X_1\ldots X_{n-1})X_n)^{a_1-c_1-1}\,\mathrm{d}X_1\ldots\mathrm{d}X_n,$$

and hence is a rational number. ∎

Lemma 3.6. *For any non-negative integers* $a_1,\ldots,a_n\,;\, b_1,\ldots,b_{n-1}\,;\, c_1,\ldots,c_{n-1}$ *satisfying (2.3) and such that*

$$a_n > \min(a_1+b_1, \ldots, a_{n-1}+b_{n-1}) \tag{3.7}$$

we have

$$\int\limits_{(0,1)^n} \frac{X_1^{a_1}(1-X_1)^{b_1}\ldots X_{n-1}^{a_{n-1}}(1-X_{n-1})^{b_{n-1}}X_n^{a_n}}{(1-(1-X_1\ldots X_{n-1})X_n)^{a_1-c_1}} \times \frac{\mathrm{d}X_1\ldots\mathrm{d}X_n}{1-(1-X_1\ldots X_{n-1})X_n} = A_1 + A_2\zeta(2) + \ldots + A_{n-1}\zeta(n-1) \tag{3.8}$$

with $A_1, A_2, \ldots, A_{n-1} \in \mathbb{Q}$. *Moreover, if*

$$a_n \leqslant b_1 + \ldots + b_{n-1} + n - 3 \tag{3.9}$$

then $A_2 = 0$.

Proof. Up to applying a permutation of $X_1, \ldots, X_{n-1}$ we may assume

$$a_1 + b_1 = \min(a_1 + b_1, \ldots, a_{n-1} + b_{n-1}). \tag{3.10}$$

The integral (3.8) is equal to (2.4) with $b_n = 0$. Therefore, by Theorem 2.1, (3.8) is equal to the Sorokin-type integral

$$\int\limits_{(0,1)^n} \frac{(1-x_1)^{a_{n-1}}\, x_2^{c_{n-1}}(1-x_2)^{b_{n-1}} \ldots\, x_n^{c_1}(1-x_n)^{b_1}}{(1-x_1x_2)^{a_{n-1}+b_{n-1}-a_{n-2}} \ldots (1-x_1\ldots x_n)^{a_1+b_1-a_n}} \tag{3.11}$$
$$\times \frac{\mathrm{d}x_1 \ldots \mathrm{d}x_n}{(1-x_1x_2)\ldots(1-x_1\ldots x_n)}\,.$$

By (3.7) and (3.10) we have $a_1+b_1-a_n < 0$. Hence we can expand both $(1-x_n)^{b_1}$ and $(1-x_1\ldots x_n)^{a_n-a_1-b_1-1}$ using the binomial theorem and then integrate with respect to x_n. Thus the integral (3.11) becomes

$$\sum_{h=0}^{b_1} \sum_{k=0}^{a_n-a_1-b_1-1} \frac{(-1)^{h+k}}{c_1+h+k+1} \binom{b_1}{h} \binom{a_n-a_1-b_1-1}{k} \tag{3.12}$$
$$\times \int\limits_{(0,1)^{n-1}} \frac{x_1^k\,(1-x_1)^{a_{n-1}}\, x_2^{c_{n-1}+k}(1-x_2)^{b_{n-1}} \ldots\, x_{n-1}^{c_2+k}(1-x_{n-1})^{b_2}}{(1-x_1x_2)^{a_{n-1}+b_{n-1}-a_{n-2}} \ldots (1-x_1\ldots x_{n-1})^{a_2+b_2-a_1}}$$
$$\times \frac{\mathrm{d}x_1 \ldots \mathrm{d}x_{n-1}}{(1-x_1x_2)\ldots(1-x_1\ldots x_{n-1})}.$$

For each k, the integral in (3.12) is an $(n-1)$-dimensional Sorokin-type integral S_{n-1} of type (2.5) with exponents, say, $\alpha_1, \ldots, \alpha_{n-1}$; $\beta_1, \ldots, \beta_{n-1}$; $\gamma_1, \ldots, \gamma_{n-2}$ given by

$$\alpha_1 = a_2,\ \ldots,\ \alpha_{n-2} = a_{n-1},\ \alpha_{n-1} = a_1,$$
$$\beta_1 = b_2,\ \ldots,\ \beta_{n-2} = b_{n-1},\ \beta_{n-1} = k,$$
$$\gamma_1 = c_2 + k,\ \ldots,\ \gamma_{n-2} = c_{n-1} + k.$$

Since

$$\gamma_i - \alpha_i \;=\; c_{i+1} - a_{i+1} + k \qquad (i = 1, \ldots, n-2),$$

by (2.7) the linear conditions (2.3) hold for S_{n-1}. Again by Theorem 2.1, S_{n-1} is equal to the Beukers-type integral

$$\int\limits_{(0,1)^{n-1}} \frac{X_1^{a_2}(1-X_1)^{b_2} \ldots X_{n-2}^{a_{n-1}}(1-X_{n-2})^{b_{n-1}} X_{n-1}^{a_1}(1-X_{n-1})^k}{(1-(1-X_1\ldots X_{n-2})X_{n-1})^{k+a_2-(c_2+k)}} \tag{3.13}$$
$$\times \frac{\mathrm{d}X_1 \ldots \mathrm{d}X_{n-1}}{1-(1-X_1\ldots X_{n-2})X_{n-1}}\,.$$

Since we have assumed Theorem 3.1 to be true for the dimension $n-1$, the integral (3.13) is a $\mathbb{Q}$-linear combination of $1,\ \zeta(2),\ \ldots,\ \zeta(n-1)$, and therefore so is (3.11) and hence (3.8).

If $n=3$ and if (3.2) holds, the coefficient A_2 of $\zeta(2)$ is zero by [8], Theorem 2.1 and Remark 2.2. If $n \geqslant 4$, our argument to prove $A_2=0$ under the condition (3.9) is again by induction on the dimension. We assume that for the dimension $n-1$ the condition (3.2) implies the vanishing of the coefficient of $\zeta(2)$. Since in (3.12) we have $k \leqslant a_n - a_1 - b_1 - 1$, assuming (3.9) we get

$$a_1 + k \leqslant a_n - b_1 - 1 \leqslant b_2 + \ldots + b_{n-1} + (n-1) - 3.$$

Thus by the inductive assumption on the dimension and by (3.2), the coefficient of $\zeta(2)$ for the integral (3.13) vanishes. Hence in each term of the sum (3.12) the coefficient of $\zeta(2)$ vanishes, and we conclude that $A_2=0$. ∎

Lemma 3.7. *For any non-negative integers* $a_1,\ldots,a_n;\ b_1,\ldots,b_{n-1}$ *such that*

$$0 < a_n \leqslant \min(a_1+b_1,\ \ldots,\ a_{n-1}+b_{n-1})$$

we have

$$\int\limits_{(0,1)^n} X_1^{a_1}(1-X_1)^{b_1}\ldots X_{n-1}^{a_{n-1}}(1-X_{n-1})^{b_{n-1}}X_n^{a_n}\,\frac{\mathrm{d}X_1\ldots\mathrm{d}X_n}{1-(1-X_1\ldots X_{n-1})X_n}$$
$$= A_1 + A_2\zeta(2) + \ldots + A_{n-1}\zeta(n-1) + A_n(n-1)\zeta(n) \tag{3.14}$$

with $A_1,\ A_2,\ \ldots,\ A_{n-1} \in \mathbb{Q},\ \ A_n \in \mathbb{Z}$. *Moreover if (3.9) holds then* $A_2=0$.

Proof. We distinguish three cases.

(i) $a_n \leqslant n-3$.

In this case (3.9) holds for any values $b_1,\ldots,b_{n-1} \geqslant 0$. If $b_1>0$ we decompose the integral (3.14) as

$$\begin{aligned}&\int\limits_{(0,1)^n} X_1^{a_1}(1-X_1)^{b_1-1}\ldots X_{n-1}^{a_{n-1}}(1-X_{n-1})^{b_{n-1}}X_n^{a_n}\\&\qquad\times\frac{\mathrm{d}X_1\ldots\mathrm{d}X_n}{1-(1-X_1\ldots X_{n-1})X_n}\\&\quad-\int\limits_{(0,1)^n} X_1^{a_1+1}(1-X_1)^{b_1-1}\ldots X_{n-1}^{a_{n-1}}(1-X_{n-1})^{b_{n-1}}X_n^{a_n}\\&\qquad\times\frac{\mathrm{d}X_1\ldots\mathrm{d}X_n}{1-(1-X_1\ldots X_{n-1})X_n}\,,\end{aligned} \tag{3.15}$$

and for each of these two integrals we iterate the same decomposition until the exponent of $1-X_1$ is pared down to zero. In each integral thus obtained we repeat the same process for every positive b_i $(2 \leqslant i \leqslant n-1)$. Hence we express (3.14) as a linear combination with integer coefficients of integrals of the type

$$J_{a'_1,\dots,a'_{n-1},a_n} := \int\limits_{(0,1)^n} X_1^{a'_1} \dots X_{n-1}^{a'_{n-1}} X_n^{a_n} \frac{\mathrm{d}X_1 \dots \mathrm{d}X_n}{1-(1-X_1\dots X_{n-1})X_n}$$

where $a_i \leqslant a'_i \leqslant a_i + b_i$ $(i = 1, \dots, n-1)$. If one of the a'_i is zero, $J_{a'_1,\dots,a'_{n-1},a_n}$ satisfies the assumptions (3.7) and (3.9) and therefore, by Lemma 3.6, it is a $\mathbb{Q}$-linear combination of 1, $\zeta(3)$, $\dots$, $\zeta(n-1)$. If $a'_1 \dots a'_{n-1} > 0$, using the identity

$$X_1 \dots X_n = (1-(1-X_1\dots X_{n-1})X_n) + X_n - 1 \tag{3.16}$$

we get

$$\begin{aligned} J_{a'_1,\dots,a'_{n-1},a_n} &= \int\limits_{(0,1)^n} X_1^{a'_1-1} \dots X_{n-1}^{a'_{n-1}-1} X_n^{a_n-1} \,\mathrm{d}X_1 \dots \mathrm{d}X_n \\ &\quad + \int\limits_{(0,1)^n} X_1^{a'_1-1} \dots X_{n-1}^{a'_{n-1}-1} X_n^{a_n} \frac{\mathrm{d}X_1 \dots \mathrm{d}X_n}{1-(1-X_1\dots X_{n-1})X_n} \\ &\quad - \int\limits_{(0,1)^n} X_1^{a'_1-1} \dots X_{n-1}^{a'_{n-1}-1} X_n^{a_n-1} \frac{\mathrm{d}X_1 \dots \mathrm{d}X_n}{1-(1-X_1\dots X_{n-1})X_n} \\ &= \frac{1}{a'_1 \dots a'_{n-1}\, a_n} + J_{a'_1-1,\dots,a'_{n-1}-1,a_n} - J_{a'_1-1,\dots,a'_{n-1}-1,a_n-1}. \end{aligned}$$

By iterating the same decomposition for the last two integrals, in finitely many steps we express $J_{a'_1,\dots,a'_{n-1},a_n}$ as a sum of a rational number and of a linear combination with integer coefficients of integrals of the same type, where we have either $a_n = 0$, so that for such integrals we can apply Lemma 3.3, or $a_n > \min(a'_1,\dots,a'_{n-1})$ so that (3.7) and (3.9) hold, and we can apply Lemma 3.6. This proves the lemma in the present case.

(ii) $a_n > b_1 + \dots + b_{n-1} + n - 3$.
Here we argue as in case (i) except that we ignore (3.9).

(iii) $n - 3 < a_n \leqslant b_1 + \dots + b_{n-1} + n - 3$.
We now use a method of descent, based on an iterated use of (3.15) and (3.16) successively. With the integral (3.14) we associate the weight

$$\sum_{i=1}^{n} a_i + 2\sum_{i=1}^{n-1} b_i = a_n + \sum_{i=1}^{n-1}(a_i+b_i) + \sum_{i=1}^{n-1} b_i. \tag{3.17}$$

We repeatedly apply the decomposition (3.15) for suitable positive b_i's until we pare down the sum $b_1 + \dots + b_{n-1} + n - 3$ to a_n. In other words, using (3.15) we express (3.14) as a linear combination with integer coefficients of integrals of the same type for each of which

$$0 < a_n = b_1 + \ldots + b_{n-1} + n - 3. \tag{3.18}$$

If $a_n > \min(a_1 + b_1, \ldots, a_{n-1} + b_{n-1})$ we have (3.7) and (3.9), and we can apply Lemma 3.6. Hence for any such integral we may assume

$$0 < a_n \leqslant \min(a_1 + b_1, \ldots, a_{n-1} + b_{n-1}). \tag{3.19}$$

Let

$$J^{b_1, \ldots, b_{n-1}}_{a_1, \ldots, a_n} \tag{3.20}$$

be any integral of type (3.14) satisfying (3.18) and (3.19). If $n = 3$ we have $J^{b_1,b_2}_{a_1,a_2,a_3} = A_1 + A_3\, 2\zeta(3)$ with $A_1 \in \mathbb{Q}$ and $A_3 \in \mathbb{Z}$ by virtue of [8], Theorem 2.1. Thus we may assume $n \geqslant 4$. Since $n - 3 > 0$, (3.18) implies $b_i < a_n$ $(i = 1, \ldots, n-1)$ whence, by (3.19), $a_i \geqslant a_n - b_i > 0$. Therefore we can repeatedly apply the identity (3.16) to (3.20). Since through the application of (3.16) the exponents $b_1, \ldots, b_{n-1}$ do not change and a_n does not increase, in finitely many steps we express (3.20) as a sum of a rational number and of a linear combination with integer coefficients of integrals again of type (3.14) and satisfying (3.9), in each of which either $a_n = 0$ (Lemma 3.4), or $a_i = 0$ for at least one i $(1 \leqslant i \leqslant n - 1)$. In the latter case we iterate the above process by applying again the decomposition (3.15) to the integral considered if some b_i is positive, until (3.18) holds again for the integrals obtained. Plainly the weight (3.17) strictly decreases at each application of (3.15) or (3.16), so that the algorithm terminates after finitely many iterations, yielding a rational number plus a linear combination with integer coefficients of finitely many integrals, each of which satisfies the assumptions of one of Lemmas 3.3, or 3.4, or 3.6 with the assumption (3.9). This suffices for the proof. ∎

Lemma 3.8. *For any non-negative integers $a_1, \ldots, a_n$; $b_1, \ldots, b_{n-1}$; $c_1, \ldots, c_{n-1}$ satisfying (2.3) and*

$$0 < a_n \leqslant \min(a_1 + b_1, \ldots, a_{n-1} + b_{n-1}) \tag{3.21}$$

we have

$$\int\limits_{(0,1)^n} \frac{X_1^{a_1}(1 - X_1)^{b_1} \ldots X_{n-1}^{a_{n-1}}(1 - X_{n-1})^{b_{n-1}} X_n^{a_n}}{(1 - (1 - X_1 \ldots X_{n-1})X_n)^{a_1 - c_1}} \, \frac{\mathrm{d}X_1 \ldots \mathrm{d}X_n}{1 - (1 - X_1 \ldots X_{n-1})X_n}$$
$$= A_1 + A_2\zeta(2) + \ldots + A_{n-1}\zeta(n-1) + A_n(n-1)\zeta(n) \tag{3.22}$$

with $A_1, A_2, \ldots, A_{n-1} \in \mathbb{Q}$, $A_n \in \mathbb{Z}$. Moreover if (3.9) holds then $A_2 = 0$.

Proof. If $a_1 = c_1$ the conclusion holds by Lemma 3.7. If $a_1 < c_1$ the integral (3.22) is a rational number, as in the proof of Lemma 3.5. If $a_1 > c_1$ and

$0 < a_n < a_1 - c_1$, we apply the permutation $\boldsymbol{\nu}$ as in Lemma 3.5. By Lemma 3.2 the integral (3.22) equals

$$\int\limits_{(0,1)^n} \frac{X_1^{c_1}(1-X_1)^{b_1}\dots X_{n-1}^{c_{n-1}}(1-X_{n-1})^{b_{n-1}}(1-X_n)^{a_n}}{(1-(1-X_1\dots X_{n-1})X_n)^{a_n+c_1-a_1}} \times \frac{\mathrm{d}X_1\dots\mathrm{d}X_n}{1-(1-X_1\dots X_{n-1})X_n},$$

and again this is a rational number since we are assuming $a_n + c_1 - a_1 < 0$. Thus, by (3.21), it remains to consider the case

$$0 < a_1 - c_1 \leqslant a_n \leqslant \min(a_1+b_1, \dots, a_{n-1}+b_{n-1}).$$

We transform the integral

$$\int_0^1 X_1^{a_1}(1-X_1)^{b_1} \frac{\mathrm{d}X_1}{(1-(1-X_1\dots X_{n-1})X_n)^{a_1-c_1+1}} \tag{3.23}$$

by (a_1-c_1)-fold partial integration. If $b_1 \geqslant a_1 - c_1$, (3.23) becomes

$$\begin{aligned}&\frac{1}{(a_1-c_1)X_2\dots X_n}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}X_1}\bigl(X_1^{a_1}(1-X_1)^{b_1}\bigr)\frac{\mathrm{d}X_1}{(1-(1-X_1\dots X_{n-1})X_n)^{a_1-c_1}}\\ &= \dots = \frac{1}{(a_1-c_1)!\,X_2^{a_1-c_1}\dots X_n^{a_1-c_1}}\\ &\quad\times\int_0^1\Bigl(\frac{\mathrm{d}}{\mathrm{d}X_1}\Bigr)^{a_1-c_1}\bigl(X_1^{a_1}(1-X_1)^{b_1}\bigr)\frac{\mathrm{d}X_1}{1-(1-X_1\dots X_{n-1})X_n}.\end{aligned}$$

By Leibniz' rule this is easily seen to be

$$\frac{1}{X_2^{a_1-c_1}\dots X_n^{a_1-c_1}}\sum_{k=0}^{a_1-c_1}(-1)^k\binom{a_1}{c_1+k}\binom{b_1}{k} \times\int_0^1 X_1^{c_1+k}(1-X_1)^{b_1-k}\frac{\mathrm{d}X_1}{1-(1-X_1\dots X_{n-1})X_n}.$$

Hence, by (2.7), the integral (3.22) equals

$$\begin{aligned}&\sum_{k=0}^{a_1-c_1}(-1)^k\binom{a_1}{c_1+k}\binom{b_1}{k} \\ &\times\int\limits_{(0,1)^n} X_1^{c_1+k}(1-X_1)^{b_1-k}X_2^{c_2}(1-X_2)^{b_2}\dots X_{n-1}^{c_{n-1}}(1-X_{n-1})^{b_{n-1}}X_n^{a_n-(a_1-c_1)}\\ &\times\frac{\mathrm{d}X_1\dots\mathrm{d}X_n}{1-(1-X_1\dots X_{n-1})X_n}.\end{aligned} \tag{3.24}$$

Each integral in the sum (3.24) satisfies the assumptions of Lemma 3.4 if $a_n = a_1 - c_1$, or of Lemma 3.7 if $a_n > a_1 - c_1$. Moreover if (3.9) holds then

$$a_n - (a_1 - c_1 - k) \leqslant a_n \leqslant b_1 + \ldots + b_{n-1} + n - 3,$$

so that the integral in (3.24) satisfies

$$a_n - (a_1 - c_1) \leqslant (b_1 - k) + b_2 + \ldots + b_{n-1} + n - 3$$

and therefore, by Lemmas 3.4 and 3.7, the coefficient of $\zeta(2)$ is zero. This proves the lemma if $b_1 \geqslant a_1 - c_1$.

If $b_1 < a_1 - c_1$, again by $(a_1 - c_1)$-fold partial integration one easily finds that the integral (3.23) is equal to

$$\frac{1}{X_2^{a_1-c_1} \ldots X_n^{a_1-c_1}} \sum_{k=0}^{b_1} (-1)^k \binom{a_1}{c_1+k} \binom{b_1}{k} \tag{3.25}$$

$$\times \int_0^1 X_1^{c_1+k} (1 - X_1)^{b_1-k} \frac{\mathrm{d}X_1}{1 - (1 - X_1 \ldots X_{n-1}) X_n}$$

$$+ (-1)^{b_1+1} \sum_{k=b_1+1}^{a_1-c_1} \frac{\binom{a_1}{k-b_1-1}}{k \binom{a_1-c_1}{k} X_2^k \ldots X_n^k (1 - (1 - X_2 \ldots X_{n-1}) X_n)^{a_1-c_1-k+1}}.$$

The contribution to the integral (3.22) given by the first sum in (3.25) is treated as in the previous case. By (2.7), the contribution to (3.22) given by the second sum in (3.25) is a linear combination with rational coefficients of $(n-1)$-dimensional integrals of the type

$$\int_{(0,1)^{n-1}} \frac{X_2^{a_2-k} (1 - X_2)^{b_2} \ldots X_{n-1}^{a_{n-1}-k} (1 - X_{n-1})^{b_{n-1}} X_n^{a_n-k}}{(1 - (1 - X_2 \ldots X_{n-1}) X_n)^{(a_2-k)-c_2}} \tag{3.26}$$

$$\times \frac{\mathrm{d}X_2 \ldots \mathrm{d}X_n}{1 - (1 - X_2 \ldots X_{n-1}) X_n}.$$

Since we assume Theorem 3.1 to hold for the dimension $n-1$, (3.26) is a $\mathbb{Q}$-linear combination of $1, \zeta(2), \ldots, \zeta(n-1)$. Also, for the second sum in (3.25) we have $k \geqslant b_1 + 1$. Thus if (3.9) holds we get

$$a_n - k \leqslant a_n - b_1 - 1 \leqslant b_2 + \ldots + b_{n-1} + (n-1) - 3,$$

whence, by the inductive assumption on the dimension together with [8], Theorem 2.1 and Remark 2.2 for the dimension 3, the coefficient of $\zeta(2)$ for the integral (3.26) vanishes. This completes the proof of the lemma. ∎

Proof of Theorem 3.1. If $b_n = 0$ the conclusion holds by Lemmas 3.4, 3.5, 3.6 or 3.8. If $b_n > 0$ we use the identity

$$1 - X_n = (1 - (1 - X_1 \ldots X_{n-1}) X_n) - X_1 \ldots X_n.$$

The integral (3.1) becomes

$$\int\limits_{(0,1)^n} \frac{X_1^{a_1}(1-X_1)^{b_1}\dots X_n^{a_n}(1-X_n)^{b_n-1}}{(1-(1-X_1\dots X_{n-1})X_n)^{(b_n-1)+a_1-c_1}} \frac{\mathrm{d}X_1\dots\mathrm{d}X_n}{1-(1-X_1\dots X_{n-1})X_n} \tag{3.27}$$
$$- \int\limits_{(0,1)^n} \frac{X_1^{a_1+1}(1-X_1)^{b_1}\dots X_n^{a_n+1}(1-X_n)^{b_n-1}}{(1-(1-X_1\dots X_{n-1})X_n)^{(b_n-1)+(a_1+1)-c_1}} \frac{\mathrm{d}X_1\dots\mathrm{d}X_n}{1-(1-X_1\dots X_{n-1})X_n}.$$

By iterating this decomposition we express the integral (3.1) as a sum of a rational number (whenever some exponent in the denominator becomes negative) plus a linear combination with integer coefficients of integrals where the exponent of $1-X_n$ is zero, and hence satisfying the assumptions of one of the previous lemmas. If the condition (3.2) holds for the integral (3.1), it also holds for the two integrals in (3.27). This suffices for the proof of Theorem 3.1. ∎

References

[1] K. Ball and T. Rivoal, *Irrationalité d'une infinité de valeurs de la fonction zêta aux entiers impairs*, Invent. Math. **146** (2001), 193–207.

[2] F. Beukers, *A note on the irrationality of* $\zeta(2)$ *and* $\zeta(3)$, Bull. London Math. Soc. **11** (1979), 268–272.

[3] J. Cresson, S. Fischler and T. Rivoal, *Séries hypergéométriques multiples et polyzêtas*, Bull. Soc. Math. France, to appear.

[4] S. Fischler, *Groupes de Rhin-Viola et intégrales multiples*, J. Théor. Nombres Bordeaux **15** (2003), 479–534.

[5] C. Krattenthaler and T. Rivoal, *Hypergéométrie et fonction zêta de Riemann*, Mem. Amer. Math. Soc., vol. 186 no. 875, March 2007.

[6] Yu. V. Nesterenko, *Integral identities and constructions of approximations to zeta-values*, J. Théor. Nombres Bordeaux **15** (2003), 535–550.

[7] G. Rhin and C. Viola, *On a permutation group related to* $\zeta(2)$, Acta Arith. **77** (1996), 23–56.

[8] G. Rhin and C. Viola, *The group structure for* $\zeta(3)$, Acta Arith. **97** (2001), 269–293.

[9] T. Rivoal, *La fonction zêta de Riemann prend une infinité de valeurs irrationnelles aux entiers impairs*, C. R. Acad. Sci. Paris, Sér. I Math., **331** (2000), 267–270.

[10] V. Kh. Salikhov and A. I. Frolovichev, *On multiple integrals represented as a linear form in* 1, $\zeta(3)$, $\zeta(5)$, …, $\zeta(2k-1)$, Fundam. Prikl. Mat. **11** no. 6 (2005), 143–178 (russian).

[11] V. N. Sorokin, *On the measure of transcendency of the number* π^2, Sb. Math. **187** no. 12 (1996), 1819–1852.

[12] V. N. Sorokin, *Apéry's theorem*, Moscow Univ. Math. Bull. **53** no. 3 (1998), 48–52.

[13] D. V. Vasilyev, *Approximations of zero by linear forms in values of the Riemann zeta-function*, Dokl. Belarus Acad. Sci. **45** no. 5 (2001), 36–40 (russian).

[14] C. Viola, *Birational transformations and values of the Riemann zeta-function*, J. Théor. Nombres Bordeaux **15** (2003), 561–592.

[15] C. Viola, *The arithmetic of Euler's integrals*, Riv. Mat. Univ. Parma (7) **3*** (2004), 119–149.

[16] S. A. Zlobin, *Properties of the coefficients of some linear forms of generalized polylogarithms*, Fundam. Prikl. Mat. **11** no. 6 (2005), 41–58 (russian).

[17] W. Zudilin, *Well-poised hypergeometric service for diophantine problems of zeta values*, J. Théor. Nombres Bordeaux **15** (2003), 593–626.

[18] W. Zudilin, *Arithmetic of linear forms involving odd zeta values*, J. Théor. Nombres Bordeaux **16** (2004), 251–291.

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