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DOMINANT RESIDUE CLASSES CONCERNING THE SUMMANDS OF PARTITIONS

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To Jean-Marc Deshouillers
on his 60 th birthday

Abstract: For $d \leqq n^{1/8-\varepsilon}$, we determine in a large range of integers $N_1,\dots,N_d$ the asymptotic number of partitions of n with exactly N_r parts congruent to r modulo d for $1 \leqslant r \leqslant d$. In the second part of the paper we derive many results on the distributions of the parts in residue classes. In particular we obtain for $1 \leqq a < b \leqq d \leqq n^{1/8-\varepsilon}$, an asymptotic formula for the number of partitions of n in which there are more parts $\equiv a (\operatorname{mod} d)$ than parts $\equiv b (\operatorname{mod} d)$.
Keywords: partitions, residue classes.

1. Introduction

Recently András Sárközy and the authors [3] proved that for almost all partitions of an integer n, the parts are well distributed in arithmetic progressions modulo d for $d < n^{1/2-\varepsilon}$. This range for d is large if we compare it with the largest parts of almost all partitions. Indeed, Erdős and Lehner [6] proved in 1941 that for almost all partitions of n (with at most $o(p(n))$ exceptions) the biggest part is $(1+o(1))\frac{\sqrt{6n}}{2\pi}\log n$. However this well distribution is limited by some phenomenon of preponderance of parts with small module. For example, it is well known that for almost all partitions the number of parts equal to 1 is $\approx \sqrt{n}$ (see [11]).

In order to some applications, the aim of this paper is to study precisely the distribution of the parts congruent to j modulo d. Let $d \geqslant 2$ and $\mathcal{R} = \{N_1, \dots, N_d\}$ a set of some positive integers.

We denote by $\Pi_d(n, \mathcal{R})$ the number of partitions of n with exactly N_r parts congruent to r mod d for $1 \leqslant r \leqslant d$.

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We immediately remark that $\Pi_d(n,\mathcal{R}) \geqslant 1$ if and only if $n \equiv R \,(\mathrm{mod}\, d)$ with

$$R := \sum_{r=1}^{d} rN_r. \tag{1.1}$$

It is the reason why we will compute $\Pi_d(n+R,\mathcal{R})$ for $n \equiv 0 \,(\mathrm{mod}\, d)$. In the following result we give an asymptotic formula for $\Pi_d(n+R,\mathcal{R})$ in a large range of $N_1,\dots,N_d$.

Theorem 1.1. *Let $0<\varepsilon<10^{-2}$. There exists n_0 such that for $n \geqslant n_0$, $d \leqslant n^{\frac{1}{8}-\varepsilon}$, $d|n$ and*

$$\Big(\frac{3}{4}+\varepsilon\Big)\frac{\sqrt{6n}}{2\pi d}\log n \leqslant N_r \leqslant \frac{n^{\frac{5}{8}}}{d} \quad (1 \leqslant r \leqslant d) \tag{1.2}$$

we have

$$\begin{aligned}\Pi_d(n+R,\mathcal{R}) &= (1+o(1))p(n)d^{\frac{2+d}{2}}\Big(\frac{1}{2\sqrt{6n}}\Big)^{\frac{d-1}{2}} \\ &\quad \times \exp\Big(-\frac{\sqrt{6n}}{\pi d}\sum_{r=1}^{d}\exp\Big(-\frac{dN_r\pi}{\sqrt{6n}}\Big)\Big).\end{aligned} \tag{1.3}$$

The condition $d \leqslant n^{\frac{1}{8}-\varepsilon}$ is a consequence of the use of saddle point method. This condition is probably not optimal. It is clear that we must have $d \ll \sqrt{n}\log n$ but perhaps another approach could give some significative result in some part of the range $[n^{\frac{1}{8}-\varepsilon}, n^{\frac{1}{2}-\varepsilon}]$.

The error term $(o(1))$ in (1.3) depends mainly on the computation of the term S_1 (see paragraphs 4 and 5). We could replace it by $O(n^{-\varepsilon/6})$. In fact if we take a smaller range for $N_1,\dots,N_d$ than the one given in (1.2), then we can obtain a more precise error term in (1.3).

The first part of the paper (the paragraphs 2,3,4,5,6,7) is devoted to the proof of this theorem by the saddle point method.

In the second part of the paper we derive many results on the distributions of the parts in residue classes. Some of these results solve problems posed in [1], [2] and [4].

We first obtain a statistical result on the size of all N_r for $1 \leqslant r \leqslant d$.

Corollary 1.2. *For $0<\varepsilon<10^{-2}$, $n \geqslant n_2(\varepsilon)$, and $d \leqslant n^{\frac{1}{8}-\varepsilon}$, in almost all partitions of n the number of summands $\equiv r \,(\mathrm{mod}\, d)$ are between $\lceil(\frac{3}{4}+\varepsilon)\frac{\sqrt{6}}{2\pi d^2}\sqrt{n}\log n\rceil d$ and $\lfloor\frac{\sqrt{6}n^{5/8}}{\pi d^2}\rfloor d-1$ simultaneously for $r=1,\dots,d$.*

It should be noted that, for $d=o(\log^2 n)$, Corollary 1.2 is implied by the Theorem 1 and Corollary 2 of the article of András Sárközy with the two authors [3]. Next we will state a corollary which shows that for almost all partitions, two given residue classes doesn't contain the same number of summands.

Corollary 1.3. *For* $0<\varepsilon<10^{-2}$, $n\geqslant n_3(\varepsilon)$, $d\leqslant n^{\frac18-\varepsilon}$, *and* $1\leqslant a<b\leqslant d$, *the number of partitions of* n *with the same number of summands in the residue classes* a *and* b *(mod* d*) is* $o(p(n))$.

In [1] and [2] Dartyge and Sárközy proved that for a positive proportion of partitions some residue classes are much more represented than others. For a given partition Π of n and for any $1\leqslant j\leqslant d$, we denote by $N_j=N_j(\Pi)$ the number of parts congruent to j modulo d. Dartyge and Sárközy [2] showed that, for d fixed, n large enough ($n\geqslant n_1(d)$) and any $1\leqslant a<b\leqslant d$, the inequality $N_a-N_b>\frac{(a+b)\sqrt{n}}{50ab}$ is satisfied for at least $p(n)/12$ partitions of n. In the introduction of [1] and in the end of [4] it is conjectured that for $1\leqslant a<b\leqslant d$ there exists $C=C(a,b,d)>1/2$ such that $N_a>N_b$ for at least $Cp(n)$ partitions of n.

In the following theorem we prove this conjecture. In fact, we obtain an asymptotic estimation of the number of such partitions.

Theorem 1.4. *For any* $0<\varepsilon<10^{-2}$, $n>n_4(\varepsilon)$, $d\leqslant n^{\frac18-\varepsilon}$ *and* $1\leqslant a<b\leqslant d$, *we have the three following properties.*

(i) *The number of partitions of* n *in which there are more parts* $\equiv a\,(\mathrm{mod}\,d)$ *than parts* $\equiv b\,(\mathrm{mod}\,d)$ *is*

$$(1+o(1))p(n)\frac{1}{\Gamma\left(\frac{a}{d}\right)\Gamma\left(\frac{b}{d}\right)}\int_0^\infty x^{\frac{a}{d}-1}\mathrm{e}^{-x}\Big(\int_x^\infty y^{\frac{b}{d}-1}\mathrm{e}^{-y}\,\mathrm{d}y\Big)\,\mathrm{d}x. \tag{1.4}$$

(ii) *The number of partitions of* n *in which there are at least as many parts* $\equiv a\,(\mathrm{mod}\,d)$ *as parts* $\equiv b\,(\mathrm{mod}\,d)$ *is*

$$(1+o(1))p(n)\frac{1}{\Gamma\left(\frac{a}{d}\right)\Gamma\left(\frac{b}{d}\right)}\int_0^\infty x^{\frac{a}{d}-1}\mathrm{e}^{-x}\Big(\int_x^\infty y^{\frac{b}{d}-1}\mathrm{e}^{-y}\,\mathrm{d}y\Big)\,\mathrm{d}x. \tag{1.5}$$

(iii) *For fixed* d, $1\leqslant a<b\leqslant d$, *and large enough* n, *the number of partitions of* n *in which there are more parts* $\equiv a\,(\mathrm{mod}\,d)$ *than parts* $\equiv b\,(\mathrm{mod}\,d)$ *is*

$$>p(n)\Big(\frac12+\frac{b-a}{12d}\Big)\geqslant p(n)\Big(\frac12+\frac{1}{12d}\Big). \tag{1.6}$$

On the other hand, this number is less than

$$p(n)2^{-\frac{a}{d}}(1+o(1)). \tag{1.7}$$

When $b=d$ in the above theorem, it is possible to compute the integrals in (1.4) or in (1.5). We obtain that for $1\leqslant a<d$, the number of partitions of n such that $N_a>N_d$ (or such that $N_a\geqslant N_d$) is $(1+o(1))2^{-a/d}p(n)$.

In [2], Dartyge and Sárközy proved by combinatorics arguments that for at least $p(n)/d$ partitions of n, we have $N_1\geqslant N_j$ for any $2\leqslant j\leqslant d$. In [4], it is conjectured that there are at least $(\frac1d+c)p(n)$ such partitions for some $c=c(d)>0$. We state this for fixed d in the following theorem.

Theorem 1.5. *For fixed $d \geqslant 2$ and $1 \leqslant a \leqslant d$, the three following assertions are satisfied.*

(i) *The number of partitions of n in which there are more parts $\equiv a \pmod d$ than parts $\equiv b \pmod d$ for all $b \in \{1, \cdots, d\} \setminus \{a\}$ is*

$$(1+o(1))p(n)\frac{1}{\Gamma\left(\frac{1}{d}\right)\cdots\Gamma\left(\frac{d}{d}\right)}\int_0^\infty x^{\frac{a}{d}-1}\mathrm{e}^{-x}\Big(\prod_{\substack{r=1\\ r\neq a}}^{d}\int_x^\infty y^{\frac{r}{d}-1}\mathrm{e}^{-y}\,\mathrm{d}y\Big)\,\mathrm{d}x.$$

(ii) *The number of partitions of n in which there are at least as many parts $\equiv a \pmod d$ as parts $\equiv b \pmod d$ for all $b \in \{1, \cdots, d\} \setminus \{a\}$ is*

$$(1+o(1))p(n)\frac{1}{\Gamma\left(\frac{1}{d}\right)\cdots\Gamma\left(\frac{d}{d}\right)}\int_0^\infty x^{\frac{a}{d}-1}\mathrm{e}^{-x}\Big(\prod_{\substack{r=1\\ r\neq a}}^{d}\int_x^\infty y^{\frac{r}{d}-1}\mathrm{e}^{-y}\,\mathrm{d}y\Big)\,\mathrm{d}x.$$

(iii) *For n large enough, the number of partitions of n in which there are more parts $\equiv 1 \pmod d$ than parts $\equiv b \pmod d$ for all $b \in \{2, \ldots, d\}$ is*

$$> p(n)\Big(\frac{1}{d}+\frac{1}{14d}\Big(1-\frac{1}{d}\Big)\Big).$$

In [2], Dartyge and Sárközy proved that for at least $\frac{p(n)}{d!}(1+O(d!d^4/\sqrt{n}))$ we have $N_1 > N_2 > \cdots > N_d$. In [4] we conjectured that this holds in fact for at least $Cp(n)$ partitions with $C > 1/d!$. In the following result we solve this conjecture for fixed d.

Theorem 1.6. *For fixed $d \geqslant 2$, the number of partitions of n in which there are more parts $\equiv a \pmod d$ than parts $\equiv b \pmod d$ for any $1 \leqslant a < b \leqslant d$ is*

$$\frac{(1+o(1))p(n)}{\Gamma\left(\frac{1}{d}\right)\Gamma\left(\frac{2}{d}\right)\cdots\Gamma\left(\frac{d}{d}\right)} \times \int\cdots\int_{0<x_1<x_2<\cdots<x_d} x_1^{\frac{1}{d}-1}x_2^{\frac{2}{d}-1}\cdots x_d^{\frac{d}{d}-1}\mathrm{e}^{-(x_1+x_2+\cdots+x_d)}\,\mathrm{d}x_d\cdots\,\mathrm{d}x_1.$$

For n large enough this is

$$> \frac{p(n)}{d!}.$$

We won't give the details of the proof of this theorem because it is an adaptation of the proof of Theorem 1.5. In fact, the proof of Theorem 1.5 may be also adapted easily to obtain the more general result:

Theorem 1.7. *For fixed $d \geqslant 2$ and any permutation σ on the set $\{1,\dots,d\}$, the number of partitions of n in which there are more parts $\equiv \sigma(a) \pmod d$ than parts $\equiv \sigma(b) \pmod d$ for any $1 \leqslant a < b \leqslant d$ is*

$$\frac{(1+o(1))p(n)}{\Gamma\left(\frac{1}{d}\right)\Gamma\left(\frac{2}{d}\right)\cdots\Gamma\left(\frac{d}{d}\right)}\int_0^\infty x_1^{\frac{\sigma(1)}{d}-1}\mathrm{e}^{-x_1}\int_{x_1}^\infty x_2^{\frac{\sigma(2)}{d}-1}\mathrm{e}^{-x_2}\int_{x_2}^\infty x_3^{\frac{\sigma(3)}{d}-1}\mathrm{e}^{-x_3}$$
$$\cdots\int_{x_{d-1}}^\infty x_d^{\frac{\sigma(d)}{d}-1}\mathrm{e}^{-x_d}\prod_{r=1}^d \mathrm{d}x_r.$$

With much more computations some results could be more precise. Some estimations are obtained only for d fixed mainly because in some steps we apply many times Corollary 1.3. It is probably possible to improve this corollary by a more direct use of the saddle point method.

2. A lemma on some generating function

In order to use the saddle point method we define the generating function:

$$G(z) := \sum_{\substack{n=0\\ n\equiv R \pmod d}}^{\infty} \Pi_d(n,\mathcal{R})z^n.$$

We will prove that this function is a finite product.

Lemma 2.1. *For $z\in\mathbb{C}$ and $|z|<1$, we have*

$$G(z) = \frac{z^{1N_1+\cdots+dN_d}}{\prod_{r=1}^d \prod_{j=1}^{N_r}(1-z^{jd})}.$$

We will give two proofs of this result. The first one uses a multi-variable generating function and a formula of Euler, the second is more combinatoric.

First proof of Lemma 2.1. According to Euler's theorem, for $|t|<1$ and $|q|<1$, we have

$$1+\sum_{n=1}^\infty \frac{t^n}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=0}^\infty \frac{1}{1-tq^n}, \tag{2.1}$$

for example, see [10] Theorem 349 p. 280.

For $z, w_r \in \mathbb{C}, |z| < 1$, and $|w_r| < |z|^{-r}$, $(1 \leqslant r \leqslant d)$ we have

$$\begin{aligned}
&\prod_{r=1}^{d}\prod_{k_r=0}^{\infty}\frac{1}{1-w_r z^{r+k_r d}}\\
&=\prod_{r=1}^{d}\prod_{k_r=0}^{\infty}(1+w_r z^{r+k_r d}+w_r^2 z^{2(r+k_r d)}+\cdots)\\
&=\sum_{N_1=1}^{\infty}\cdots\sum_{N_d=1}^{\infty}\Big(\sum_{n\in\mathbb{N}}{}^{*}\,\Pi_d(n,\{N_1,\ldots,N_d\})z^n\Big)w_1^{N_1}\cdots w_d^{N_d},
\end{aligned}\tag{2.2}$$

where $*$ indicates that the sum is over the $n \in \mathbb{N}$ such that $n \equiv R \,(\mathrm{mod}\, d)$.

On the other hand, for $1 \leqslant r \leqslant d$, we write $w_r z^{r+k_r d} = (w_r z^r)(z^d)^{k_r}$ and we apply (2.1) with $t = w_r z^r$, $q = z^d$:

$$\begin{aligned}
&\prod_{r=1}^{d}\prod_{k_r=0}^{\infty}\frac{1}{1-w_r z^{r+k_r d}}\\
&=\prod_{r=1}^{d}\Big(1+\sum_{N_r=1}^{\infty}\frac{(w_r z^r)^{Nr}}{(1-z^d)(1-z^{2d})\cdots(1-z^{N_r d})}\Big)\\
&=\prod_{r=1}^{d}\sum_{N_r=0}^{\infty}\frac{w_r^{N_r}z^{rN_r}}{\prod_{j=1}^{N_r}(1-z^{jd})}\\
&=\sum_{N_1=0}^{\infty}\cdots\sum_{N_d=0}^{\infty}\Big(\frac{z^{N_1+\cdots+dN_d}}{\prod_{r=1}^{d}\prod_{j=1}^{N_r}(1-z^{jd})}\Big)w_1^{N_1}\cdots w_d^{N_d}.
\end{aligned}\tag{2.3}$$

We finish the proof by comparing the coefficient of $w_1^{N_1}\cdots w_r^{N_r}$ in (2.2) and (2.3).

Second proof of Lemma 2.1. Let Π be a partition of n counted in $\Pi_d(n,\mathcal{R})$. This partition is of the form:

$$\Pi : n = \sum_{r=1}^{d}\sum_{j=1}^{N_r}(r+\lambda_{r,j}d),$$

with

$$\lambda_{r,1} \geqslant \ldots \geqslant \lambda_{r,N_r} \geqslant 0 \quad (1 \leqslant r \leqslant d).$$

Thus we have

$$n = R + d\sum_{r=1}^{d} m_r, \text{ with } m_r = \sum_{j=1}^{N_r}\lambda_{r,j} \ (1 \leqslant r \leqslant d).$$

For each $1 \leqslant r \leqslant d$, $\lambda_{r,1}, \dots, \lambda_{r,N_r}$ is a partition of m_r in at most N_r parts. Let $p_{N_r}(m_r)$ denote the number of such partitions. We have

$$\begin{aligned} G(z) &= z^R \sum_{\substack{n=0 \\ n \equiv R \,(\mathrm{mod}\, d)}}^{\infty} \sum_{\substack{m_1+\cdots+m_d=\frac{n-R}{d} \\ m_j \in \mathbb{N}}} p_{N_1}(m_1)\cdots p_{N_d}(m_d) z^{d(m_1+\cdots+m_d)} \\ &= z^R \prod_{r=1}^{d} \Big(\sum_{m=0}^{\infty} z^{dm} p_{N_r}(m) \Big) \\ &= \frac{z^R}{\prod_{r=1}^{d} \prod_{j=1}^{N_r} (1 - z^{dj})}, \end{aligned}$$

where we have used the formula for $|x| < 1$

$$\sum_{n=0}^{\infty} p_m(n) x^n = \frac{1}{\prod_{j=1}^{m} (1 - x^j)}.$$

3. The saddle point method

For $v \in \mathbb{C}$, $|v| < 1$, it follows from Lemma 2.1 that

$$\sum_{m=0}^{\infty} \Pi_d(dm + R, \mathcal{R}) v^{dm} = \prod_{r=1}^{d} \prod_{j=1}^{N_r} (1 - v^{jd})^{-1}.$$

For $d|n$, and some $0 < \varrho < 1$, we obtain by the Cauchy formula that

$$\Pi_d(n + R, \mathcal{R}) = \frac{1}{2i\pi} \int_{|v|=\varrho} v^{-n-1} \prod_{r=1}^{d} \prod_{j=1}^{N_r} (1 - v^{jd})^{-1} \, \mathrm{d}v.$$

Let $x > 0$, $\varrho = e^{-x}$, $z = x + iy$, $v = \mathrm{e}^{-z}$. Then we have:

$$\begin{aligned} \Pi_d(n + R, \mathcal{R}) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \Big\{ \prod_{r=1}^{d} \prod_{j=1}^{N_r} \frac{1}{1 - \exp(-jd(x+iy))} \Big\} \exp(n(x+iy)) \, \mathrm{d}y \\ &= \frac{d}{2\pi} \int_{-\pi/d}^{\pi/d} \Big\{ \prod_{r=1}^{d} \prod_{j=1}^{N_r} \frac{1}{1 - \exp(-jd(x+iy))} \Big\} \exp(n(x+iy)) \, \mathrm{d}y \end{aligned}$$

since the integrand is periodic in y and has period $2\pi/d$. For $\Re w > 0$, we set

$$f(w) := \prod_{\nu=1}^{\infty} (1 - \exp(-\nu w))^{-1}$$

and

$$g_k(w) := \prod_{\nu=1}^{k}(1-\exp(-\nu w))^{-1} = f(w)\prod_{\nu=k+1}^{\infty}(1-\exp(-\nu w)).$$

With this notation,

$$\Pi_d(n+R,\mathcal{R}) = \frac{d}{2\pi}\int_{-\pi/d}^{\pi/d}\Big\{\prod_{r=1}^{d} g_{N_r}(d(x+iy))\Big\}\exp(n(x+iy))\,\mathrm{d}y.$$

For $\varepsilon>0,\ 0<\varepsilon<10^{-2},\ d\leqslant n^{\frac18-\varepsilon}$ and $n>n_0$, we consider the interval

$$I = I_{n,d,\varepsilon} := \Big[\big(\frac34+\varepsilon\big)\frac{\sqrt6}{2\pi d}\sqrt n\log n, \frac{n^{\frac58}}{d}\Big].$$

We will estimate $\Pi_d(n+R,\mathcal{R})$ for $N_1,\dots,N_d\in I$ and $d|n$. Choosing $x=x_0=\frac{\pi}{\sqrt{6n}}$, $y_1=n^{-\frac34+\frac{\varepsilon}{3}}$, $y_2=n^{-\frac58+\frac{\varepsilon}{3}}$ and $y_3=\pi x_0$, we write $\Pi_d(n+R,\mathcal{R})$ as

$$\begin{aligned}\Pi_d(n+R,\mathcal{R}) &= \frac{d}{2\pi}\Big\{\int_{|y|\leqslant y_1}+\int_{y_1\leqslant|y|\leqslant y_2}+\int_{y_2\leqslant|y|\leqslant y_3}+\int_{y_3\leqslant|y|\leqslant\pi/d}\Big\}\\ &= S_1+S_2+S_3+S_4.\end{aligned}\tag{3.1}$$

Theorem 1.1 will be derived by the following lemma:

Lemma 3.1. *Under the hypotheses of Theorem 1.1, we have*

$$S_1 = (1+o(1))p(n)d^{\frac{2+d}{2}}\Big(\frac{x_0}{2\pi}\Big)^{\frac{d-1}{2}}\exp\Big(-\frac{1}{dx_0}\sum_{r=1}^{d}\exp(-dN_rx_0)\Big);\tag{3.2}$$

$$S_i = o(S_1)\quad (i=2,3,4).\tag{3.3}$$

In the next paragraph we state some estimates of g_k and in the paragraphs 5, 6, and 7 we prove (3.2), (3.3) respectively.

4. The function g_k

By elementary arguments we will prove the following lemma which compares g_k with f.

Lemma 4.1. (i) *For $k\in I$ and $|y|\leqslant\pi/d$ we have*

$$\begin{aligned}g_k(d(x_0+iy)) &= f(d(x_0+iy))\exp\Big\{-\frac{\exp(-dk(x_0+iy))}{d(x_0+iy)}\Big\}\\ &\times\exp\Big\{O(\exp(-dkx_0))+O\Big(\sqrt{\frac{n}{d}}\exp(-2dkx_0)\Big)\Big\},\end{aligned}\tag{4.1}$$

and

$$g_k(d(x_0+iy)) = f(d(x_0+iy))\exp\Big(-\frac{\exp(-dkx_0)}{dx_0}\Big) \times \exp\Big\{O(1)\exp(-dkx_0)(\sqrt{n}k|y|+1+\frac{\sqrt{n}}{d}\exp(-dkx_0))\Big\}. \tag{4.2}$$

(ii) *For* $k\in I$ *and* $|y|\leqslant y_1$ *we have*

$$\begin{aligned} g_k(d(x_0+iy)) &= f(d(x_0+iy))\exp\Big(\frac{-\exp(-dkx_0)}{dx_0}\Big)\exp\Big(O\big(\frac{n^{\frac{3}{8}+\frac{\varepsilon}{3}}}{d}\exp(-dkx_0)\big)\Big)\\ &= (1+o(d^{-1}))f(d(x_0+iy))\exp\Big(-\frac{1}{dx_0}\exp(-dkx_0)\Big). \end{aligned} \tag{4.3}$$

Proof. Consider $g_k(dz)$ for $k\in I$ and $|y|\leqslant \pi/d$. If $\nu\geqslant k+1$ then

$$|\exp(-\nu d(x_0+iy))| = \exp(-\nu dx_0) < \exp(-kdx_0) \leqslant n^{-\frac{3}{8}-\frac{\varepsilon}{2}}.$$

Therefore (here log denotes the principal determination of logarithm defined on $\mathbb{C}\setminus\mathbb{R}^-$),

$$\begin{aligned} &g_k(d(x_0+iy))\\ &= f(d(x_0+iy))\exp\Big\{\sum_{\nu=k+1}^{\infty}\log(1-\exp(-\nu d(x_0+iy)))\Big\}\\ &= f(d(x_0+iy))\exp\Big\{-\sum_{\nu=k+1}^{\infty}\big(\exp(-\nu d(x_0+iy))+O(\exp(-2\nu dx_0))\big)\Big\}\\ &= f(d(x_0+iy))\exp\Big\{-\frac{\exp(-dk(x_0+iy))}{\exp(d(x_0+iy))-1}+O\Big(\frac{\exp(-2dkx_0)}{\exp(2dx_0)-1}\Big)\Big\}. \end{aligned}$$

Here, $|d(x_0+iy)|\leqslant dx_0+\pi<6$. Thus

$$\frac{1}{\exp(d(x_0+iy))-1} = \frac{1}{d(x_0+iy)}+O(1).$$

This yields that

$$\begin{aligned} g_k(d(x_0+iy)) &= f(d(x_0+iy))\\ &\times\exp\Big\{-\frac{\exp(-dk(x_0+iy))}{d(x_0+iy)}+O(\exp(-dkx_0))+O\big(\frac{\sqrt{n}}{d}\exp(-2dkx_0)\big)\Big\}, \end{aligned}$$

this ends the proof of (4.1).

To prove (4.2) we remark that

$$\begin{aligned}
&\Big|\frac{\exp(-dk(x_0+iy))}{d(x_0+iy)}-\frac{\exp(-dkx_0)}{dx_0}\Big|\\
&=\frac{\exp(-dkx_0)}{d}\Big|\frac{\exp(-dkiy)-1-iyx_0^{-1}}{x_0+iy}\Big|\\
&\leqslant\frac{\exp(-dkx_0)}{d}\frac{(dk|y|+|y|x_0^{-1})}{x_0}\\
&=O(\sqrt{n}k|y|\exp(-dkx_0)),
\end{aligned}\tag{4.4}$$

since $x_0^{-1}=O(dk)$. It remains to insert (4.4) in (4.1) to obtain (4.2).

Now we prove (4.3). For $k\in I$ and $|y|\leqslant y_1=n^{-\frac{3}{4}+\varepsilon}$, the different factors in the error term of (4.2) become:

$$\sqrt{n}k|y|+1+\frac{\sqrt{n}}{d}\exp(-dkx_0)\leqslant\sqrt{n}\frac{n^{5/8}}{d}n^{-\frac{3}{4}+\frac{\varepsilon}{3}}+\frac{d}{d}+\frac{\sqrt{n}}{d}n^{-\frac{3}{8}-\frac{\varepsilon}{2}}=O\big(\frac{n^{\frac{3}{8}+\frac{\varepsilon}{3}}}{d}\big),$$

and

$$\frac{n^{\frac{3}{8}+\frac{\varepsilon}{3}}}{d}\exp(-dkx_0)\leqslant\frac{n^{-\frac{\varepsilon}{6}}}{d}=o\big(\frac{1}{d}\big).\tag{4.5}$$

Consequently, for $k\in I$ and $|y|\leqslant y_1$,

$$\begin{aligned}
g_k(d(x_0+iy))&=f(d(x_0+iy))\exp\Big\{-\frac{\exp(-dkx_0)}{dx_0}+O\Big(\frac{n^{\frac{3}{8}+\frac{\varepsilon}{3}}}{d}\exp(-dkx_0)\Big)\Big\}\\
&=(1+o(d^{-1}))f(d(x_0+iy))\exp\Big\{-\frac{\exp(-dkx_0)}{dx_0}\Big\},
\end{aligned}$$

this ends the proof of (4.3).

5. The main term S_1

By (3.1) and Lemma 4.1 we have

$$\begin{aligned}
S_1&=\frac{d}{2\pi}\int_{-y_1}^{y_1}\Big\{\prod_{r=1}^{d}g_{N_r}(d(x_0+iy))\Big\}\exp(n(x_0+iy))\,\mathrm{d}y\\
&=\frac{d}{2\pi}\exp\Big(-\frac{1}{dx_0}\sum_{r=1}^{d}\exp(-dN_rx_0)\Big)\\
&\quad\times\int_{-y_1}^{y_1}f^d(d(x_0+iy))\exp\Big\{n(x_0+iy)+O\Big(\frac{n^{\frac{3}{8}+\frac{\varepsilon}{3}}}{d}\sum_{r=1}^{d}\exp(-dN_rx_0)\Big)\Big\}\,\mathrm{d}y\\
&=d\exp\Big(-\frac{1}{dx_0}\sum_{r=1}^{d}\exp(-dN_rx_0)\Big)\\
&\quad\times\frac{1}{2\pi}\int_{-y_1}^{y_1}f^d(d(x_0+iy))\exp\big(n(x_0+iy)+O(n^{-\frac{\varepsilon}{6}})\big)\,\mathrm{d}y.
\end{aligned}$$

Next we use the well-known formula (see for example [7] or [8])

$$f(w) = \exp\Big(\frac{\pi^2}{6w} + \frac{1}{2}\log\frac{w}{2\pi} + O(|w|)\Big)$$

for $w \to 0$ in $|\arg w| \leqslant \kappa < \pi/2$ and $\Re w > 0$.

For $|y| \leqslant y_3 = \pi x_0$,

$$\begin{aligned} f(d(x_0+iy)) &= \exp\Big(\frac{\pi^2}{6d(x_0+iy)} + \frac{1}{2}\log\big(\frac{d(x_0+iy)}{2\pi}\big) + O(dx_0)\Big),\\ f^d(d(x_0+iy)) &= \exp\Big(\frac{\pi^2}{6(x_0+iy)} + \frac{d}{2}\log\big(\frac{d(x_0+iy)}{2\pi}\big) + O(d^2x_0)\Big)\\ &= f(x_0+iy)\exp\Big(\frac{d}{2}\log d + \frac{d-1}{2}\log\frac{x_0+iy}{2\pi} + O(d^2x_0)\Big). \end{aligned}$$

For $|y| \leqslant y_2 = n^{-\frac{5}{8}+\frac{\varepsilon}{3}}$,

$$\begin{aligned} f^d(d(x_0+iy)) &= f(x_0+iy)\exp\Big(\frac{d}{2}\log d + \frac{d-1}{2}\log\frac{x_0}{2\pi} + O(d)\big(\frac{|y|}{x_0} + dx_0\big)\Big)\\ &= f(x_0+iy)d^{d/2}\big(\frac{x_0}{2\pi}\big)^{\frac{d-1}{2}}\exp(O(dn^{-\frac{1}{8}+\frac{\varepsilon}{3}})). \end{aligned} \tag{5.1}$$

Finally by (5.1) and (4.5),

$$\begin{aligned} S_1 = d^{1+\frac{d}{2}}\Big(\frac{x_0}{2\pi}\Big)^{\frac{d-1}{2}} &\exp\Big(-\frac{1}{dx_0}\sum_{r=1}^{d}\exp(-dN_rx_0)\Big)\\ \times\Big\{&\frac{1}{2\pi}\int_{-y_1}^{y_1} f(x_0+iy)\exp(n(x_0+iy))\,\mathrm{d}y\\ &+ o(1)\int_{-y_1}^{y_1}|f(x_0+iy)\exp(n(x_0+iy))|\,\mathrm{d}y\Big\}. \end{aligned}$$

For $|y| \leqslant y_1$, - - - as it is well known - - -

$$\begin{aligned} &f(x_0+iy)\exp(n(x_0+iy))\\ &= \exp\Big(\frac{\pi^2}{6(x_0+iy)} + \frac{1}{2}\log\big(\frac{x_0+iy}{2\pi}\big) + o(1) + nx_0 + iny\Big)\\ &= \exp\Big(\frac{\pi^2}{6x_0}\big(1 - \frac{iy}{x_0} - \frac{y^2}{x_0^2} + O\big(\frac{y_1^3}{x_0^3}\big)\big)\\ &\quad + \frac{1}{2}\log\big(\frac{x_0}{2\pi}\big) + O\big(\frac{y_1}{x_0}\big) + o(1) + nx_0 + iny\Big)\\ &= \exp\Big(\frac{\pi^2}{6x_0} - \frac{\pi^2y^2}{6x_0^3} + \frac{1}{2}\log\big(\frac{x_0}{2\pi}\big) + o(1) + nx_0\Big)\\ &= (1+o(1))|f(x_0+iy)\exp(n(x_0+iy))|, \end{aligned}$$

and

$$\frac{1}{2\pi}\int_{-y_1}^{y_1} f(x_0+iy)\exp(n(x_0+iy))\,\mathrm{d}y = (1-o(1))p(n).$$

This ends the proof of (3.2).

6. The term S_2

We write

$$S_2 = \int_{y_1}^{y_2} + \int_{-y_2}^{-y_1} = S_2^+ + S_2^-.$$

Thus we have

$$S_2^+ = \frac{d}{2\pi}\int_{y_1}^{y_2}\Big\{\prod_{r=1}^{d} g_{N_r}(d(x_0+iy))\Big\}\exp(n(x_0+iy))\,\mathrm{d}y.$$

From Lemma 4.1 we have for $k \in I$ and $|y| \leqslant \pi/d$

$$\begin{aligned}|g_k(d(x_0+iy))| = |f(d(x_0+iy))|\exp\Big\{&-\Re\frac{\exp(-dk(x_0+iy))}{d(x_0+iy)}\\ &+O(\exp((-dkx_0))+O\big(\frac{\sqrt{n}}{d}\exp(-2dkx_0)\big)\Big\}.\end{aligned}$$

If $k \in I$ and $y_1 \leqslant y \leqslant y_2 = n^{-\frac{5}{8}+\frac{\varepsilon}{3}}$ then

$$\begin{aligned}&|g_k(d(x_0+iy))| \qquad (6.1)\\ &= |f(d(x_0+iy))|\exp\Big\{-\frac{\exp(-dkx_0)}{dx_0}\Re\frac{\exp(-dkiy)}{1+i\frac{y}{x_0}} + O(n^{-\frac{3}{8}-\frac{\varepsilon}{2}}) + O\big(\frac{n^{-\frac{1}{4}-\varepsilon}}{d}\big)\Big\}\\ &= |f(d(x_0+iy))|\exp\Big\{-\frac{\exp(-dkx_0)}{dx_0}\Re\big(\exp(-dkiy)(1+O(\frac{y_2}{x_0}))\big) + o(d^{-1})\Big\}\\ &= |f(d(x_0+iy))|\exp\Big\{-\frac{\exp(-dkx_0)}{dx_0}\big(\cos(dky) + O(n^{-\frac{1}{8}+\frac{\varepsilon}{3}})\big) + o(d^{-1})\Big\}\\ &= |f(d(x_0+iy))|\exp\Big\{-\frac{\exp(-dkx_0)}{dx_0}\Big(1-2\sin^2\big(\frac{dky}{2}\big)\Big) + o(d^{-1})\Big\}.\end{aligned}$$

If $k \leqslant \frac{\sqrt{6}}{\pi d}\sqrt{n}\log n$ then

$$\begin{aligned}\frac{\exp(-dkx_0)}{dx_0}2\sin^2\big(\frac{dky}{2}\big) &= O\big(\frac{\sqrt{n}}{d}\big)(dky_2)^2\exp(-dkx_0)\\ &= O\big(\frac{\sqrt{n}}{d}\big)n^{-\frac{3}{8}-\frac{\varepsilon}{2}}(\log^2 n)n^{-\frac{1}{4}+\frac{2\varepsilon}{3}} = o(d^{-1}).\end{aligned} \qquad (6.2)$$

If $k \geqslant \frac{\sqrt{6n}}{\pi d} \log n$ then

$$\frac{\exp(-dkx_0)}{dx_0} 2\sin^2\big(\frac{dky}{2}\big) = O\big(\frac{\sqrt{n}}{d}\big)\exp(-dkx_0) = O\big(\frac{\sqrt{n}}{d}\big)n^{-1} = o(d^{-1}). \quad (6.3)$$

By (6.1), (6.2), (6.3), and (5.1) we have

$$|S_2^+| \leqslant \frac{d}{2\pi}\exp\Big(-\sum_{r=1}^{d}\frac{\exp(-dN_r x_0)}{dx_0}\Big)\int_{y_1}^{y_2}|f^d(d(x_0+iy))|\exp(nx_0+o(1))\,\mathrm{d}y$$
$$= O(d)\exp\Big(-\sum_{r=1}^{d}\frac{\exp(-dN_r x_0)}{dx_0}\Big)d^{\frac{d}{2}}\big(\frac{x_0}{2\pi}\big)^{\frac{d-1}{2}}\int_{y_1}^{y_2}|f(x_0+iy)|\exp(nx_0)\,\mathrm{d}y.$$

Here the usual estimation:

$$|f(x_0+iy)| = \exp\Big\{\Re\frac{\pi^2}{6(x_0+iy)} + O(\log n)\Big\}$$
$$\leqslant \exp\Big\{\frac{\pi^2}{6x_0}.\frac{x_0^2}{x_0^2+y_1^2} + O(\log n)\Big\}$$

yields that $S_2^+ = o(S_1)$ and the same goes for S_2^-.

7. The terms S_3 and S_4

Like in the previous paragraph we write

$$S_3 = \int_{y_2\leqslant y\leqslant y_3} + \int_{-y_3\leqslant y\leqslant -y_2} = S_3^+ + S_3^-$$

and in the same way we write $S_4 = S_4^+ + S_4^-$. Similarly, for $y_2 \leqslant |y| \leqslant y_3 = \pi x_0$,

$$|f^d(d(x_0+iy)) = |f(x_0+iy)|d^{\frac{d}{2}}\big(\frac{x_0}{2\pi}\big)^{\frac{d-1}{2}}\exp(O(d\log n))$$

and

$$|g_k(d(x_0+iy))| = |f(d(x_0+iy))|\exp\Big\{-\Re\frac{\exp(-dk(x_0+iy))}{d(x_0+iy)} + o(d^{-1})\Big\}$$
$$\leqslant |f(d(x_0+iy))|\exp\Big\{+\frac{\exp(-dkx_0)}{dx_0} + o(d^{-1})\Big\}$$
$$\leqslant |f(d(x_0+iy))|\exp\Big\{-\frac{\exp(-dkx_0)}{dx_0} + O\big(\frac{n^{\frac{1}{8}-\frac{\varepsilon}{2}}}{d}\big)\Big\}$$

yield that $|S_3| = o(S_1)$ since

$$\frac{\pi^2}{6x_0}.\frac{x_0^2}{x_0^2+y_2^2} \leqslant \frac{\pi^2}{6x_0}\big(1-\frac{y_2^2}{2x_0^2}\big) \leqslant \frac{\pi^2}{6x_0} - n^{\frac{1}{4}}.$$

Finally, for $y_3 \leqslant |y| \leqslant \pi/d$, we obtain again that

$$|g_k(d(x_0+iy))| \leqslant |f(d(x_0+iy))| \exp\Big\{ -\frac{\exp(-dkx_0)}{dx_0} + O\big(\frac{n^{\frac{1}{8}-\frac{\varepsilon}{2}}}{d}\big)\Big\}.$$

Since

$$f(w) = \exp\Big(\sum_{m=1}^{\infty} \frac{1}{m(\exp(mw)-1)}\Big)$$

for $\Re w > 0$, we have

$$\begin{aligned}|f(w)| &\leqslant \exp\Big(\Re\frac{1}{e^w-1} + \sum_{m=2}^{\infty}\frac{1}{m|e^{mw}-1|}\Big)\\ &\leqslant \exp\Big(\frac{1}{|e^w-1|} + \frac{1}{\Re w}\big(\frac{\pi^2}{6}-1\big)\Big)\\ &\leqslant \exp\Big(\frac{1}{\frac{2}{\pi}|\mathrm{Im} w|} + \frac{1}{\Re w}\big(\frac{\pi^2}{6}-1\big)\Big)\end{aligned}$$

if $|\mathrm{Im} w| \leqslant \pi$. Thus

$$|f(d(x_0+iy))| \leqslant \exp\Big(\frac{\pi}{2d|y|} + \frac{1}{dx_0}\big(\frac{\pi^2}{6}-1\big)\Big),$$

$$|f^d(d(x_0+iy))| \leqslant \exp\Big(\frac{\pi}{2|y|} + \frac{1}{x_0}\big(\frac{\pi^2}{6}-1\big)\Big) \leqslant \exp\big(\frac{\pi^2}{6x_0} - \frac{1}{2x_0}\big).$$

Observing that

$$d^{-\frac{d}{2}}\Big(\frac{2\pi}{x_0}\Big)^{\frac{d-1}{2}} = \exp(O(d\log n))$$

we see that $S_4 = o(S_1)$, this ends the proof of Lemma 3.1 and Theorem 1.1 is proved.

8. When $n \equiv R\,(\mathrm{mod}\, d)$

We are going to apply Theorem 1.1 for $n-R$ instead of n when $n \equiv R\,(\mathrm{mod}\, d)$. In this section we will derive from Theorem 1.1 the following result:

Corollary 8.1. *For* $0 < \varepsilon < 10^{-2}$, $n \geqslant n_1$, $d \leqslant (n-n^{3/4})^{\frac{1}{8}-\varepsilon}$, $n \equiv R\,(\mathrm{mod}\, d)$, *and*

$$\big(\frac{3}{4}+\varepsilon\big)\frac{\sqrt{6n}}{2\pi d}\log n \leqslant N_r \leqslant \frac{\sqrt{6}}{\pi}\frac{n^{5/8}}{d} \ (r=1,\dots,d) \tag{8.1}$$

we have

$$\Pi_d(n,\mathfrak{R}) = (1+o(1))p(n)d^{\frac{2+d}{2}}\Big(\frac{1}{2\sqrt{6n}}\Big)^{\frac{d-1}{2}}\exp\Big\{-\frac{\pi R}{\sqrt{6n}} - \frac{\sqrt{6n}}{\pi d}\sum_{r=1}^{d}\exp\Big(-\frac{dN_r\pi}{\sqrt{6n}}\Big)\Big\}.$$

Proof. Under the hypotheses of Corollary 8.1, we have

$$R < \frac{n^{\frac{5}{8}}}{d}.\frac{d(d+1)}{2} \leqslant dn^{\frac{5}{8}} < n^{\frac{3}{4}-\varepsilon} < n^{\frac{3}{4}},$$

thus $n - R > n - n^{3/4}$, $\frac{\sqrt{6}}{\pi} n^{5/8} < (n-R)^{5/8}$, and

$$\begin{aligned}
n - R &= n(1 + O(n^{-1/4})) = n\exp(O(n^{-1/4})) \\
\sqrt{n-R} &= \sqrt{n}\exp(O(n^{-1/4})) = \sqrt{n} + O(n^{1/4}) \\
\frac{1}{\sqrt{n-R}} &= \frac{1}{\sqrt{n}}\exp(O(n^{-1/4})) = \frac{1}{\sqrt{n}} + O(n^{-3/4}) \\
\Big(\frac{1}{\sqrt{n-R}}\Big)^{\frac{d-1}{2}} &= \Big(\frac{1}{\sqrt{n}}\Big)^{\frac{d-1}{2}}\exp(O(dn^{-1/4})) = \Big(\frac{1}{\sqrt{n}}\Big)^{\frac{d-1}{2}}(1 + o(1)).
\end{aligned}$$

Next we compute the argument of the exponential in Theorem 1.1:

$$\begin{aligned}
\Big(\frac{\sqrt{6n}}{\pi d} - \frac{\sqrt{6(n-R)}}{\pi d}\Big)\sum_{r=1}^{d}\exp\Big(-\frac{dN_r\pi}{\sqrt{6(n-R)}}\Big) &= O\big(\frac{n^{1/4}}{d}\big)\sum_{r=1}^{d}\exp\big(-\frac{dN_r\pi}{\sqrt{6n}}\big) \\
&= O\big(\frac{n^{1/4}}{d}\big)dn^{-\frac{3}{8}-\frac{\varepsilon}{2}} \\
&= O(n^{-\frac{1}{8}-\frac{\varepsilon}{2}}) = o(1).
\end{aligned}$$

In the same way we have for $1 \leqslant r \leqslant d$:

$$\begin{aligned}
\exp\Big(-\frac{dN_r\pi}{\sqrt{6(n-R)}}\Big) &= \exp\Big(-dN_r\pi\big(\frac{1}{\sqrt{6n}} + O(n^{-3/4})\big)\Big) \\
&= \exp\Big(-\frac{dN_r\pi}{\sqrt{6n}} + O\big(d\frac{n^{5/8}}{d}n^{-3/4}\big)\Big) \\
&= (1 + O(n^{-1/8}))\exp\big(-\frac{dN_r\pi}{\sqrt{6n}}\big).
\end{aligned}$$

It remains to sum this equality over $1 \leqslant r \leqslant d$:

$$\begin{aligned}
\frac{\sqrt{6n}}{\pi d}\sum_{r=1}^{d}\exp\Big(-\frac{dN_r\pi}{\sqrt{6(n-R)}}\Big) &= \frac{\sqrt{6n}}{\pi d}(1 + O(n^{-1/8}))\sum_{r=1}^{d}\exp\Big(-\frac{dN_r\pi}{\sqrt{6n}}\Big) \\
&= \frac{\sqrt{6n}}{\pi d}\sum_{r=1}^{d}\exp\Big(-\frac{dN_r\pi}{\sqrt{6n}}\Big) + O\big(\frac{n^{\frac{1}{2}-\frac{1}{8}}}{d}dn^{-\frac{3}{8}-\frac{\varepsilon}{2}}\big) \\
&= \frac{\sqrt{6n}}{\pi d}\sum_{r=1}^{d}\exp\Big(-\frac{dN_r\pi}{\sqrt{6n}}\Big) + o(1).
\end{aligned}$$

We apply Theorem 1.1:

$$\Pi_d(n,\mathcal{R}) \tag{8.2}$$
$$= (1+o(1))p(n-R)d^{\frac{2+d}{2}}\Big(\frac{1}{2\sqrt{6n}}\Big)^{\frac{d-1}{2}}\exp\Big(-\frac{\sqrt{6n}}{\pi d}\sum_{r=1}^{d}\exp\Big(-\frac{dN_r\pi}{\sqrt{6n}}\Big)\Big).$$

From the asymptotic formula

$$p(n) = (1+o(1))\frac{1}{4n\sqrt{3}}\exp\Big(\frac{2\pi\sqrt{n}}{\sqrt{6}}\Big)$$

of Hardy and Ramanujan [9] we obtain for $1 \leqslant t \leqslant n^{\frac{3}{4}-\varepsilon}$, that

$$\begin{aligned}\frac{p(n-t)}{p(n)} &= (1+o(1))\exp\Big(-\frac{2\pi}{\sqrt{6}}(\sqrt{n}-\sqrt{n-t})\Big) \qquad (8.3)\\ &= (1+o(1))\exp\Big(-\frac{\pi t}{\sqrt{6n}}\Big)\exp\Big(-\frac{2\pi}{\sqrt{6}}\big(\frac{t}{\sqrt{n}+\sqrt{n-t}}-\frac{t}{2\sqrt{n}}\big)\Big)\\ &= (1+o(1))\exp\Big(-\frac{\pi t}{\sqrt{6n}}\Big)\exp\Big(-\frac{2\pi t}{\sqrt{6}}\frac{\sqrt{n}-\sqrt{n-t}}{2\sqrt{n}(\sqrt{n}+\sqrt{n-t})}\Big)\\ &= (1+o(1))\exp\Big(-\frac{\pi t}{\sqrt{6n}}\Big)\exp\Big(O(t^2n^{-3/2})\Big) = (1+o(1))\exp\Big(-\frac{\pi t}{\sqrt{6n}}\Big).\end{aligned}$$

The equalities (8.3) and (8.2) give Corollary 8.1.

9. Local stability of $\Pi_d(n,\mathcal{R})$

The next corollary says that if we take two sets $\mathcal{R} = \{N_1,\dots,N_d\} \subset \mathbb{Z}^d$ verifying (8.1) and $\mathcal{R}^* = \{N_1^*,\dots,N_d^*\} \subset \mathbb{R}^d$ such that the N_r^* are near the N_r on average, then in the estimation of $\Pi_d(n,\mathcal{R})$ we may replace the N_r by the N_r^* in cost of an admissible error term. This will be very useful for the proofs of the different results announced in the introduction.

Corollary 9.1. *For $0<\varepsilon<10^{-2}$, $n \geqslant n_1$, $d \leqslant (n-n^{3/4})^{\frac{1}{8}-\varepsilon}$, $n \equiv R\,(\mathrm{mod}\,d)$, and two sets $\mathcal{R} = \{N_1,\dots,N_d\} \subset \mathbb{Z}^d$, $\mathcal{R}^* = \{N_1^*,\dots,N_d^*\} \subset \mathbb{R}^d$ such that:*

(i) *$\mathcal{R}$ satisfies (8.1);*

(ii) *$\mathcal{R}$ and $\mathcal{R}^*$ verify*

$$\sum_{r=1}^{d}|N_r-N_r^*| \leqslant d^3, \tag{9.1}$$

we have

$$\Pi_d(n,\mathcal{R}) = (1+o(1))p(n)d^{\frac{2+d}{2}}\Big(\frac{1}{2\sqrt{6n}}\Big)^{\frac{d-1}{2}}\exp\Big\{-\frac{\pi R^*}{\sqrt{6n}}-\frac{\sqrt{6n}}{\pi d}\sum_{r=1}^{d}\exp\Big(-\frac{dN_r^*\pi}{\sqrt{6n}}\Big)\Big\}.$$

Proof. Let F be the function defined by:

$$F(N_1,\dots,N_d)=\exp\Big\{-\frac{\pi R}{\sqrt{6n}}-\frac{\sqrt{6n}}{\pi d}\sum_{r=1}^{d}\exp\Big(-\frac{dN_r\pi}{\sqrt{6n}}\Big)\Big\}. \tag{9.2}$$

If $\mathcal{R}^*$ satisfy (9.1), then in Corollary 8.1, $F(N_1,\dots,N_r)\sim F(N_1^*,\dots,N_d^*)$ since

$$\Big|\frac{1}{\sqrt{n}}\sum_{r=1}^{d} r(N_r-N_r^*)\Big|\leqslant\frac{1}{\sqrt{n}}\sum_{r=1}^{d} d|N_r-N_r^*|\leqslant\frac{d^4}{\sqrt{n}}=o(1),$$

and

$$\begin{aligned}\Big|\frac{\sqrt{n}}{d}\sum_{r=1}^{d}\Big(\exp\big(-\frac{dN_r\pi}{\sqrt{6n}}\big)-\exp\big(-\frac{dN_r^*\pi}{\sqrt{6n}}\big)\Big)\Big| &\leqslant\frac{\sqrt{n}}{d}\sum_{r=1}^{d}\exp\big(-\frac{dN_r\pi}{\sqrt{6n}}\big)\\ &\quad\times\Big|1-\exp\big(-\frac{d(N_r^*-N_r)\pi}{\sqrt{6n}}\big)\Big|\\ &\leqslant\frac{\sqrt{n}}{d}\sum_{r=1}^{d} n^{-\frac{3}{8}-\frac{\varepsilon}{2}}O\Big(\frac{d}{\sqrt{n}}|N_r^*-N_r|\Big)\\ &=O(d^3n^{-\frac{3}{2}-\frac{\varepsilon}{2}})=o(1).\end{aligned}$$

This ends the proof of Corollary 9.1.

10. Partitions without abnormally represented residue classes; proof of Corollary 1.2

If we shall sum over certain choices of $N_1,\dots,N_d$ then the product in

$$F(N_1,\dots,N_d)=\prod_{r=1}^{d}\exp\Big\{-\frac{\pi rN_r}{\sqrt{6n}}-\frac{\sqrt{6n}}{\pi d}\exp\Big(-\frac{dN_r\pi}{\sqrt{6n}}\Big)\Big\}$$

would be useful for an "independent" computation but we have the condition

$$N_1\equiv n-\sum_{r=2}^{d} rN_r \pmod d. \tag{10.1}$$

For $N_1^*=\lfloor\frac{N_1}{d}\rfloor d$ (or $\lceil\frac{N_1}{d}\rceil d$) and $N_r^*=N_r$ $(r=2,\dots,d)$, Corollary 9.1 implies that in an asymptotic sense, we can substitute the condition (10.1) by the condition $d|N_1$. Let $A:=\lceil\big(\frac{3}{4}+\varepsilon\big)\frac{\sqrt{6}}{2\pi d^2}\sqrt{n}\log n\rceil d$ and $B:=\lfloor\frac{\sqrt{6}n^{5/8}}{\pi d^2}\rfloor d$.

Thus $d|A$, $d|B$, and

$$\big(\frac{3}{4}+\varepsilon\big)\frac{\sqrt{6n}}{2\pi d}\log n\leqslant A<B\leqslant\frac{\sqrt{6}n^{5/8}}{\pi d}.$$

In the following lines, for each $A \leqslant N_1, \dots, N_d < B$, $\mathcal{R}$ is the associated set $\mathcal{R} = \{N_1, \dots, N_d\}$ and the integer R is $\sum_{r=1}^{d} rN_r$. By Corollary 9.1,

$$\sum_{\substack{A\leqslant N_1,\dots,N_d<B\\ R\equiv n \,(\mathrm{mod}\, d)}} \Pi_d(n,\mathcal{R}) = (1+o(1))p(n)d^{\frac{2+d}{2}}\Big(\frac{1}{2\sqrt{6n}}\Big)^{\frac{d-1}{2}} \sum_{\substack{A\leqslant N_1,\dots,N_d<B\\ d|N_1}} F(N_1,\dots,N_d).$$

Here the sum is

$$\begin{aligned} S &:= \sum_{\substack{A/d\leqslant N_1'<B/d\\ A\leqslant N_2,\dots,N_d<B}} F(dN_1',N_2,\dots,N_d) \\ &= \sum_{\substack{A/d\leqslant N_1'<B/d\\ A\leqslant N_2,\dots,N_d<B}} \int_{N_1'}^{N_1'+1}\int_{N_2}^{N_2+1}\cdots\int_{N_d}^{N_d+1} F(dN_1',N_2,\dots,N_d)\,\mathrm{d}t_1'\,\mathrm{d}t_2\cdots\mathrm{d}t_d. \end{aligned}$$

Next we apply Corollary 9.1

$$S = \sum_{\substack{A/d\leqslant N_1'<B/d\\ A\leqslant N_2,\dots,N_d<B}} \int_{N_1'}^{N_1'+1}\int_{N_2}^{N_2+1}\cdots\int_{N_d}^{N_d+1} (1+o(1))F(dt_1',t_2,\dots,t_d)\,\mathrm{d}t_1'\cdots\mathrm{d}t_d,$$

since $(dt_1' - dN_1') + (t_2 - N_2) + \cdots + (t_d - N_d) \leqslant d + d - 1 \leqslant d^3$.

By $dt_1' = t_1$, it is

$$\begin{aligned} S &= (1+o(1))\frac{1}{d}\int_A^B\int_A^B\cdots\int_A^B F(t_1,\dots,t_d)\,\mathrm{d}t_1\cdots\mathrm{d}t_d \\ &= (1+o(1))\frac{1}{d}\prod_{r=1}^{d}\int_A^B \exp\Big(-\frac{\pi rt}{\sqrt{6n}} - \frac{\sqrt{6n}}{\pi d}\exp\Big(-\frac{dt\pi}{\sqrt{6n}}\Big)\Big)\,\mathrm{d}t. \end{aligned}$$

We set $t = u\sqrt{6n}/\pi d$ in the integral:

$$S = (1+o(1))\frac{1}{d}\Big(\frac{\sqrt{6n}}{\pi d}\Big)^d \prod_{r=1}^{d}\int_{A\pi d/\sqrt{6n}}^{B\pi d/\sqrt{6n}} \mathrm{e}^{-\frac{ur}{d}}\mathrm{e}^{-\frac{\mathrm{e}^{-u}\sqrt{6n}}{\pi d}}\,\mathrm{d}u.$$

Next we write $x = \frac{\sqrt{6n}}{\pi d}\mathrm{e}^{-u}$

$$\begin{aligned} S &= (1+o(1))\frac{1}{d}\Big(\frac{\sqrt{6n}}{\pi d}\Big)^{d-\sum_{r=1}^{d}\frac{r}{d}} \prod_{r=1}^{d}\int_{\frac{\sqrt{6n}}{\pi d}\exp\left(-\frac{B\pi d}{\sqrt{6n}}\right)}^{\frac{\sqrt{6n}}{\pi d}\exp\left(-\frac{A\pi d}{\sqrt{6n}}\right)} x^{\frac{r}{d}-1}\mathrm{e}^{-x}\,\mathrm{d}x \\ &= (1+o(1))\frac{1}{d}\Big(\frac{\sqrt{6n}}{\pi d}\Big)^{\frac{d-1}{2}} \prod_{r=1}^{d}\int_{\frac{\sqrt{6n}}{\pi d}\exp\left(-\frac{B\pi d}{\sqrt{6n}}\right)}^{\frac{\sqrt{6n}}{\pi d}\exp\left(-\frac{A\pi d}{\sqrt{6n}}\right)} x^{\frac{r}{d}-1}\mathrm{e}^{-x}\,\mathrm{d}x. \end{aligned}$$

We shall estimate the complementary integrals:

$$\int_0^{\frac{\sqrt{6n}}{\pi d}\exp\left(-\frac{B\pi d}{\sqrt{6n}}\right)} x^{\frac{r}{d}-1}\mathrm{e}^{-x}\,\mathrm{d}x = \int_0^{\exp(-n^{1/8}+o(1))\sqrt{6n}/(\pi d)} x^{\frac{r}{d}-1}\mathrm{e}^{-x}\,\mathrm{d}x.$$

$$< \int_0^{\sqrt{n}d^{-1}\exp(-n^{\frac{1}{8}})} x^{\frac{r}{d}-1}\,\mathrm{d}x = \frac{d}{r}\Big(\frac{\sqrt{n}}{d}\exp(-n^{1/8})\Big)^{\frac{r}{d}}$$

$$\leqslant \frac{d}{r}\Big(\exp(\frac{\log n}{2} - n^{1/8})\Big)^{r/d} \leqslant \frac{d}{r}\exp(-\frac{n^{1/8}}{2d})$$

$$\leqslant \frac{d}{r}\exp\big(-\frac{n^{\varepsilon}}{2}\big) = O(\Gamma(\frac{r}{d}))\exp(-\frac{n^{\varepsilon}}{2})$$

$$= o\Big(\frac{1}{d}\Big)\Gamma\Big(\frac{r}{d}\Big),$$

by

$$\Gamma(x) = \frac{1}{x\mathrm{e}^{\gamma x}}\prod_{\nu=1}^{\infty}\frac{\mathrm{e}^{x/\nu}}{1+\frac{x}{\nu}} > \frac{1}{x\mathrm{e}^{\gamma x}},$$

where γ is the Euler constant.

For the other side, we have:

$$\int_{\frac{\sqrt{6n}}{\pi d}\exp\left(-\frac{A\pi d}{\sqrt{6n}}\right)}^{\infty} x^{\frac{r}{d}-1}\mathrm{e}^{-x}\,\mathrm{d}x = \int_{\exp\left(-\left(\frac{3}{8}+\frac{\varepsilon}{2}\right)\log n+o(1)\right)\frac{\sqrt{6n}}{\pi d}}^{\infty} x^{\frac{r}{d}-1}\mathrm{e}^{-x}\,\mathrm{d}x$$

$$= \int_{(1+o(1))\frac{\sqrt{6}}{\pi}\frac{n^{\frac{1}{8}-\frac{\varepsilon}{2}}}{d}}^{\infty} x^{\frac{r}{d}-1}\mathrm{e}^{-x}\,\mathrm{d}x$$

$$\leqslant \int_{\frac{n^{\frac{\varepsilon}{2}}}{2}}^{\infty} x^{\frac{r}{d}-1}\mathrm{e}^{-x}\,\mathrm{d}x \leqslant \int_{\frac{n^{\frac{\varepsilon}{2}}}{2}}^{\infty}\mathrm{e}^{-x}\,\mathrm{d}x$$

$$\leqslant \exp\big(-\frac{n^{\frac{\varepsilon}{2}}}{2}\big) = o\big(\frac{1}{d}\big) = o\big(\frac{1}{d}\big)\Gamma\big(\frac{r}{d}\big),$$

since $\Gamma\left(\frac{r}{d}\right) \geqslant 1$.

Finally we obtain that

$$\sum_{\substack{A\leqslant N_1,\dots,N_d<B\\ R\equiv n\,(\mathrm{mod}\,d)}} \Pi_d(n,\mathcal{R}) = (1+o(1))p(n)d^{\frac{2+d}{2}}\Big(\frac{1}{2\sqrt{6n}}\Big)^{\frac{d-1}{2}}\frac{1}{d}\Big(\frac{\sqrt{6n}}{\pi d}\Big)^{\frac{d-1}{2}}$$

$$\times\prod_{r=1}^{d}\Big\{\Gamma\big(\frac{r}{d}\big)+o\big(\frac{1}{d}\big)\Gamma\big(\frac{r}{d}\big)\Big\}$$

$$= (1+o(1))p(n)\sqrt{d}\Big(\frac{1}{2\pi}\Big)^{\frac{d-1}{2}}(1+o(d^{-1}))^d\prod_{r=1}^{d}\Gamma\big(\frac{r}{d}\big)$$

$$= (1+o(1))p(n)\frac{\Gamma\left(\frac{1}{d}\right)\cdots\Gamma\left(\frac{d-1}{d}\right)}{\frac{(2\pi)^{\frac{d-1}{2}}}{\sqrt{d}}}$$

$$= (1+o(1))p(n).$$

11. Partitions with equilibrated residue classes: proof of Corollary 1.3

For $1 \leqslant a < b \leqslant d$, we can estimate the number of partitions of n with the property that the residue classes a and b (mod d) contain the same number of summands. Let $E(a,b)$ denote the set of such partitions. By Corollary 1.2, apart from $o(p(n))$ partitions of n we may assume that $A \leqslant N_1, \ldots, N_d < B$. Thus we have:

$$E(a,b) = \sum_{\substack{A\leqslant N_1,\ldots,N_d\leqslant B\\ n\equiv R\,(\mathrm{mod}\,d)\\ N_a=N_b}} \Pi_d(n,\mathcal{R}) + o(p(n)).$$

We can follow the proof of to make the $N_1, \ldots, N_d$ independent.

There is a technical difficulty when d is small (when $\varphi(d) < 3$). We would like to replace for some convenient $j \in \{1,\ldots,d\} \setminus \{a,b\}$ the condition

$$jN_j \equiv n - \sum_{\substack{1\leqslant r\leqslant d\\ r\neq j}} rN_r \,(\mathrm{mod}\, d)$$

by $d|N_j^*$. But in this way, when d is small we are not sure that the correspondence between the corresponding sets $\mathcal{R}$ and $\mathcal{R}^*$ is one-to-one.

We will choose our set $\mathcal{R}^*$ in the following way. If $a \neq 1$ then we take $N_1^* = d\lfloor \frac{N_1}{d} \rfloor$.

If $a = 1$, $b \neq d-1$ and $d \geqslant 3$ then we use $j = d-1$, $N_{d-1}^* = d\lfloor \frac{N_{d-1}}{d} \rfloor$.

If $a = 1$, $b = d-1$ and $d \notin \{2,3,4,6\}$ we use $j = c$, $N_c^* = d\lfloor \frac{N_c}{d} \rfloor$ with c minimal satisfying $1 < c < d-1$ and $(c,d) = 1$.

If $(a,b,d) = (1,5,6)$, we use $N_2^* = 3\lfloor \frac{N_2}{3} \rfloor$, $N_3^* = 2\lfloor \frac{N_3}{2} \rfloor$ (thus in this case we have $\mathcal{R}^* = \{N_1, N_2^*, N_3^*, N_4, N_5, N_6\}$).

The cases $(a,b,d) \in \{(1,2,2),(1,2,3),(1,3,4)\}$ are to be investigated separately. Later we have to substitute

$$\int_A^B \exp\Big(-\frac{\pi}{\sqrt{6n}} a t_a - \frac{\sqrt{6n}}{\pi d}\exp\Big(-\frac{d t_a \pi}{\sqrt{6n}}\Big)\Big)\,\mathrm{d}t_a \\ \times \int_A^B \exp\Big(-\frac{\pi}{\sqrt{6n}} b t_b - \frac{\sqrt{6n}}{\pi d}\exp\Big(-\frac{d t_b \pi}{\sqrt{6n}}\Big)\Big)\,\mathrm{d}t_b$$

by

$$\int_A^B \exp\Big(-\frac{\pi}{\sqrt{6n}}(a+b)t - 2\frac{\sqrt{6n}}{\pi d}\exp\Big(-\frac{d t \pi}{\sqrt{6n}}\Big)\Big)\,\mathrm{d}t;$$

moreover, $\Gamma\big(\frac{a}{d}\big)\Gamma\big(\frac{b}{d}\big)$ by

$$\frac{\pi d}{\sqrt{6n}} \int_0^\infty x^{\frac{a+b}{d}-1} \mathrm{e}^{-2x}\,\mathrm{d}x = \frac{\pi d}{\sqrt{6n}} \frac{\Gamma\big(\frac{a+b}{d}\big)}{2^{\frac{a+b}{d}}}.$$

The complementary integrals change unessentially.

Thus the final result is

$$o(p(n)) + (1+o(1))p(n)\frac{\pi d}{\sqrt{6n}} 2^{-\frac{a+b}{d}} \frac{\Gamma\left(\frac{a+b}{d}\right)}{\Gamma\left(\frac{a}{b}\right)\Gamma\left(\frac{b}{d}\right)}$$
$$= o(p(n)) + O\Big(p(n)\frac{d^2}{\sqrt{n}}\Big) = o(p(n)),$$

we have used the facts that $\Gamma\left(\frac{a}{d}\right)\Gamma\left(\frac{b}{d}\right) \geqslant 1$, $\Gamma\left(\frac{a+b}{d}\right) \leqslant \Gamma\left(\frac{1}{d}\right) = d\Gamma\left(\frac{1}{d}+1\right) \leqslant d$.

This result is valid for $(a,b,d) = (1,2,2)$ too. For $(a,b,d) \in \{(1,2,3),(1,3,4)\}$ we can obtain similar expressions weighted by constants depending on the residue of n mod d: $0,0,3$; $0,2,0,2$.

12. Comparison between the number of summands in two residue classes: proof of Theorem 1.4

12.1. Proof of the propositions (i) and (ii) of Theorem 1.4. In this section, for $1 \leqslant a < b \leqslant d$, we investigate the number of partitions of n in which there are more parts $\equiv a \pmod d$ than parts $\equiv b \pmod d$, briefly the case $N_a > N_b$. We shall consider the cases $N_a > N_b$ resp. $N_a \geqslant N_b$ together as $N_a \geqslant N_b + \Delta$ with $\Delta = 1$ resp. $\Delta = 0$.

By Corollary 1.2 the N_r belong to $[A,B]$ for almost partitions:

$$\sum_{\substack{N_1,\dots,N_d \\ R\equiv n \,(\mathrm{mod}\, d) \\ N_a \geqslant N_b+\Delta}} \Pi_d(n,\mathcal{R}) = o(p(n)) + \sum_{\substack{A\leqslant N_1,\dots,N_d<B \\ R\equiv n \,(\mathrm{mod}\, d) \\ N_a \geqslant N_b+\Delta}} \Pi_d(n,\mathcal{R}).$$

Apart from $(a,b,d) \in \{(1,2,2),(1,2,3),(1,3,4)\}$ - as in the proof of Corollary 1.3 — we can suppose that $1 < a$ and follow the proof of Corollary 1.2.

We have to substitute:

$$\sum_{A\leqslant N_a<B}\ \sum_{A\leqslant N_b<B} \int_{N_a}^{N_a+1}\int_{N_b}^{N_b+1} F(\dots,t_a,\dots,t_b,\dots)\,\mathrm{d}t_a\,\mathrm{d}t_b$$

by

$$T_{a,b} := \sum_{A+\Delta\leqslant N_a<B}\ \sum_{A\leqslant N_b\leqslant N_a-\Delta} \int_{N_a}^{N_a+1}\int_{N_b}^{N_b+1} F(\dots,t_a,\dots,t_b,\dots)\,\mathrm{d}t_a\,\mathrm{d}t_b.$$

We have

$$T_{a,b} = \sum_{\Delta+A\leqslant N_a<B} \int_{N_a}^{N_a+1}\int_{A}^{N_a+1-\Delta} F(\dots,t_a,\dots,t_b,\dots)\,\mathrm{d}t_a\,\mathrm{d}t_b.$$

When $\Delta = 1$ we have the upper bound

$$T_{a,b} \leqslant \int_A^B \int_A^{t_a} F(\dots, t_a, \dots, t_b, \dots)\, \mathrm{d}t_a\, \mathrm{d}t_b.$$

If $\Delta = 0$, then it is a lower bound:

$$T_{a,b} \geqslant \int_A^B \int_A^{t_a} F(\dots, t_a, \dots, t_b, \dots)\, \mathrm{d}t_a\, \mathrm{d}t_b.$$

Taking into account Corollary 1.3, apart from $o(p(n))$ partitions of n, we can compute both cases substituting $\int_{A \leqslant t_a \leqslant B} \int_{A \leqslant t_b \leqslant B}$ by $\int_{A \leqslant t_a \leqslant B} \int_{A \leqslant t_b \leqslant t_a}$. Later, considering also the complementary integrals, we have to substitute

$$(1 + o(d^{-1}))\Gamma\Big(\frac{a}{d}\Big)(1 + o(d^{-1}))\Gamma\Big(\frac{b}{d}\Big)$$

by

$$\int_0^\infty x^{\frac{a}{d}-1} \mathrm{e}^{-x} \Big(\int_x^\infty y^{\frac{b}{d}-1} \mathrm{e}^{-y}\, \mathrm{d}y \Big)\, \mathrm{d}x + o(d^{-1})\Gamma\Big(\frac{a}{d}\Big)\Gamma\Big(\frac{b}{d}\Big).$$

For $(a, b, d) \in \{(1,2,2), (1,2,3), (1,3,4)\}$ we use both $N_1^* = d\lfloor \frac{N_1}{d} \rfloor$, $N_1^{**} = \lceil \frac{N_1}{d} \rceil d$.

Thus the final result is

$$\begin{aligned} &\sum_{\substack{N_1, \dots, N_d \\ R \equiv n \,(\mathrm{mod}\, d) \\ N_a \geqslant N_b + \Delta}} \Pi_d(n, \mathcal{R}) \\ &= o(p(n)) + \frac{(1 + o(1))}{\Gamma\big(\frac{a}{d}\big)\Gamma\big(\frac{b}{d}\big)} p(n) \int_0^\infty x^{\frac{a}{d}-1} \mathrm{e}^{-x} \int_x^\infty y^{\frac{b}{d}-1} \mathrm{e}^{-y}\, \mathrm{d}y\, \mathrm{d}x. \end{aligned} \tag{12.1}$$

This ends the proofs of (i) and (ii) of Theorem 1.4.

12.2. Proof of the lower bound (1.6). For the special case $1 \leqslant a < b = d$, (12.1) becomes

$$\begin{aligned} &o(p(n)) + \frac{(1 + o(1))p(n)}{\Gamma\big(\frac{a}{d}\big)\Gamma\big(1\big)} \int_0^\infty x^{\frac{a}{d}-1} \mathrm{e}^{-2x}\, \mathrm{d}x = o(p(n)) + \frac{(1 + o(1))p(n)}{2^{\frac{a}{d}}} \\ &= (1 + o(1)) \frac{p(n)}{2^{\frac{a}{d}}}, \end{aligned}$$

since $1 < 2^{\frac{a}{d}} < 2$.

Moreover,

$$\frac{1}{2^{\frac{a}{d}}} \geqslant \frac{1}{2^{\frac{(d-1)}{d}}} = \frac{1}{2} \exp\Big(\frac{\log 2}{d}\Big) > \frac{1}{2} + \frac{\log 2}{2d}.$$

For the general case $1 \leqslant a < b \leqslant d$ let us consider the integrals

$$I_1 = \int_0^\infty x^{\frac{a}{d}-1} e^{-x} \Big(\int_x^\infty y^{\frac{b}{d}-1} e^{-y} \, dy \Big) \, dx$$

and

$$I_2 = \int_0^\infty x^{\frac{b}{d}-1} e^{-x} \Big(\int_x^\infty y^{\frac{a}{d}-1} e^{-y} \, dy \Big) \, dx.$$

Then we have $I_1 + I_2 = \Gamma\Big(\frac{a}{d}\Big)\Gamma\Big(\frac{b}{d}\Big)$ and

$$I_1 - I_2 = \int_0^\infty \int_x^\infty e^{-x-y} (xy)^{\frac{a}{d}-1} \big(y^{\frac{b-a}{d}} - x^{\frac{b-a}{d}} \big) \, dy \, dx > 0.$$

Therefore, $I_1 > \frac{1}{2}\Gamma\Big(\frac{a}{d}\Big)\Gamma\Big(\frac{b}{d}\Big)$ and

$$o(p(n)) + (1+o(1))p(n) \frac{I_1}{\Gamma\Big(\frac{a}{d}\Big)\Gamma\Big(\frac{b}{d}\Big)} \sim p(n) \frac{I_1}{\Gamma\Big(\frac{a}{d}\Big)\Gamma\Big(\frac{b}{d}\Big)}.$$

We can estimate

$$\frac{I_1}{\Gamma\Big(\frac{a}{d}\Big)\Gamma\Big(\frac{b}{d}\Big)} - \frac{1}{2} = \frac{I_1 - I_2}{2\Gamma\Big(\frac{a}{d}\Big)\Gamma\Big(\frac{b}{d}\Big)}$$

from below in the following way. For any $\delta > 0$,

$$\begin{aligned}
I_1 - I_2 &> \int_0^\infty \int_{x(1+\delta)}^\infty e^{-x-y} (xy)^{\frac{a}{d}-1} \big(y^{\frac{b-a}{d}} - x^{\frac{b-a}{d}} \big) \, dy \, dx \\
&\geqslant \int_0^\infty \int_{x(1+\delta)}^\infty e^{-x-y} (xy)^{\frac{a}{d}-1} \Big(y^{\frac{b-a}{d}} - \big(\frac{y}{1+\delta}\big)^{\frac{b-a}{d}} \Big) \, dy \, dx \\
&= \Big(1 - \frac{1}{(1+\delta)^{\frac{b-a}{d}}} \Big) \int_0^\infty \int_{x(1+\delta)}^\infty x^{\frac{a}{d}-1} e^{-x} y^{\frac{b}{d}-1} e^{-y} \, dy \, dx \\
&= \Big(1 - \frac{1}{(1+\delta)^{\frac{b-a}{d}}} \Big) \Big\{ I_1 - \int_0^\infty \int_x^{x(1+\delta)} x^{\frac{a}{d}-1} e^{-x} y^{\frac{b}{d}-1} e^{-y} \, dy \, dx \Big\} \\
&> \Big(1 - \frac{1}{(1+\delta)^{\frac{b-a}{d}}} \Big) \Big\{ \frac{1}{2}\Gamma\Big(\frac{a}{d}\Big)\Gamma\Big(\frac{b}{d}\Big) - \int_0^\infty x^{\frac{a}{d}-1} e^{-x} \int_x^{x(1+\delta)} y^{\frac{b}{d}-1} e^{-y} \, dy \, dx \Big\} \\
&\geqslant \Big(1 - \frac{1}{(1+\delta)^{\frac{b-a}{d}}} \Big) \Big\{ \frac{1}{2}\Gamma\Big(\frac{a}{d}\Big)\Gamma\Big(\frac{b}{d}\Big) - \int_0^\infty x^{\frac{a}{d}-1} e^{-x} (x^{\frac{b}{d}-1} e^{-x} \delta x) \, dx \Big\} \\
&= \Big(1 - \frac{1}{(1+\delta)^{\frac{b-a}{d}}} \Big) \Big\{ \frac{1}{2}\Gamma\Big(\frac{a}{d}\Big)\Gamma\Big(\frac{b}{d}\Big) - \delta\Gamma\Big(\frac{a+b}{d}\Big) 2^{-\frac{a+b}{d}} \Big\},
\end{aligned}$$

We obtain

$$\frac{I_1 - I_2}{2\Gamma\Big(\frac{a}{d}\Big)\Gamma\Big(\frac{b}{d}\Big)} > \Big(1 - \frac{1}{(1+\delta)^{\frac{b-a}{d}}} \Big) \Big\{ \frac{1}{4} - \frac{\delta\Gamma\Big(\frac{a+b}{d}\Big)}{2^{1+\frac{a+b}{d}}\Gamma\Big(\frac{a}{d}\Big)\Gamma\Big(\frac{b}{d}\Big)} \Big\}.$$

For $x, y > 0$,

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t.$$

For $0 < x \leqslant y \leqslant 1$, we get $B(x,y) \geqslant \int_0^1 t^{x-1}\,\mathrm{d}t = \frac{1}{x}$ and $\dfrac{\Gamma(x+y)}{\Gamma(x)\Gamma(y)} \leqslant x$. Further, $x4^{-x} \leqslant \frac{1}{\log 4} 4^{\frac{-1}{\log 4}} = \frac{1}{2\mathrm{e}\log 2}$.

Therefore,

$$\frac{\delta\Gamma\Big(\frac{a+b}{d}\Big)}{2^{1+\frac{a+b}{d}}\Gamma\Big(\frac{a}{d}\Big)\Gamma\Big(\frac{b}{d}\Big)} \leqslant \frac{\delta\frac{a}{d}}{2^{1+\frac{2a}{d}+\frac{b-a}{d}}} \leqslant \frac{\delta}{2^{\frac{b-a}{d}}}\frac{1}{4\mathrm{e}\log 2}.$$

Let $\alpha := 0.59$ and

$$\delta := \Big(\frac{1}{1-\alpha\frac{b-a}{d}}\Big)^{\frac{d}{b-a}} - 1.$$

Then

$$\begin{aligned}
\frac{\delta}{2^{\frac{b-a}{d}}} &= 2^{-\frac{b-a}{d}}\exp\Big(\frac{d}{b-a}\log\frac{1}{1-\alpha\frac{b-a}{d}}\Big) - 2^{-\frac{b-a}{d}}\\
&= 2^{-\frac{b-a}{d}}\exp\Big(\alpha + \sum_{m=2}^{\infty}\frac{1}{m}\alpha^m\big(\frac{b-a}{d}\big)^{m-1}\Big) - 2^{-\frac{b-a}{d}}\\
&\leqslant 2^{-\frac{b-a}{d}}\exp\Big(\alpha + \big(\log\frac{1}{1-\alpha} - \alpha\big)\frac{b-a}{d}\Big) - 2^{-\frac{b-a}{d}}\\
&= \exp\Big(\alpha - \big(\log 2 + \alpha - \log\frac{1}{1-\alpha}\big)\frac{b-a}{d}\Big) - \exp\big(-(\log 2)\frac{b-a}{d}\big)
\end{aligned}$$

which is monotonically decreasing in $\frac{b-a}{d}$ (for $\alpha = 0.59$). Therefore

$$\frac{\delta}{2^{\frac{b-a}{d}}} \leqslant \mathrm{e}^{\alpha} - 1.$$

Finally,

$$\Big(1 - \frac{1}{(1+\delta)^{\frac{b-a}{d}}}\Big)\Big\{\frac{1}{4} - \frac{\delta}{2^{\frac{b-a}{d}}}\frac{1}{4\mathrm{e}\log 2}\Big\} \geqslant \alpha\frac{b-a}{d}\frac{1}{4}\Big(1 - \frac{\mathrm{e}^{\alpha}-1}{\mathrm{e}\log 2}\Big) > \frac{1}{12}\frac{b-a}{d}.$$

We remind the reader of the fact that we considered the cases $N_a > N_b$ resp. $N_a \geqslant N_b$ together. Increasing ε, we can use $d \leqslant n^{\frac{1}{8}-\varepsilon}$. Thus (1.6) is proved.

12.3. Proof of the upper bound (1.7). For $1 \leqslant a, b \leqslant d$, we denote by $S_{a,b}$ the set of the partitions of n satisfying $N_a \geqslant N_b$.

As it is said in the introduction, when $b = d$, we can compute $|S_{a,d}|$ by (1.5), $|S_{a,d}| = p(n)(2^{-\frac{a}{d}} + o(1))$. The upper bound (1.7) in Theorem 1.4 is a consequence of the following lemma:

Lemma 12.1. *For* $1 \leqslant a < b < d$, *we have* $|S_{a,b}| \leqslant |S_{a,d}| + o(p(n))$.

Proof. For any $1 \leqslant c_1, c_2, c_3 \leqslant d$, let $S(c_1, c_2, c_3)$ denote the set of the partitions of n such that $N_{c_1} \geqslant N_{c_2} \geqslant N_{c_3}$ (here as before, N_{c_i} is the number of parts $\equiv c_i \pmod d$).

We have the two equalities:

$$S_{a,b} = S(a,b,d) \cup S(a,d,b) \cup S(d,a,b),$$

and

$$S_{a,d} = S(a,b,d) \cup S(a,d,b) \cup S(b,a,d).$$

By Corollary 1.3, $|S(c_1,c_2,c_3) \cap S(c_{\sigma(1)}, c_{\sigma(2)}, c_{\sigma(3)})| = o(p(n))$ for any non trivial permutation σ on the set $\{1,2,3\}$. Thus we have:

$$\begin{aligned} |S_{a,b}| &= |S(a,b,d)| + |S(a,d,b)| + |S(d,a,b)| + o(p(n)),\\ |S_{a,d}| &= |S(a,b,d)| + |S(a,d,b)| + |S(b,a,d)| + o(p(n)). \end{aligned}$$

To prove Lemma 12.1, it is sufficient to show that

$$|S(d,a,b)| \leqslant |S(b,a,d)| + o(p(n)). \tag{12.2}$$

To prove this inequality, we will show that there exists an injective map Ψ defined on $S(d,a,b)$ such that for almost all partitions $\Pi \in S(d,a,b)$, $\Psi(\Pi) \in S(b,a,d)$. This map consists in exchanging the parts $\equiv b \pmod d$ with the parts $\equiv d \pmod d$ and to put some appropriate parts to compensate the quantity $(d-b)(N_d - N_b)$ arising from this exchange. Such sort of idea was already used in some proofs of [2].

• We suppose that $a \neq 1$. Let Π be a generic partition of n in $S(d,a,b)$. We write Π in the following way:

$$\Pi : n = \sum_{r=1}^{d} \sum_{j=1}^{N_r} (r + \lambda_{j,r} d) \quad \text{with} \quad \lambda_{j,r} \geqslant 0, \ for\ 1 \leqslant r \leqslant d,\ 1 \leqslant j \leqslant N_r,$$

so that for $1 \leqslant r \leqslant d$, $r + \lambda_{1,r} d, \dots, r + \lambda_{N_r,r} d$ are the parts $\equiv r \pmod d$. To this partition Π we assign the following partition $\Psi(\Pi)$

$$\Psi(\Pi) : n = \sum_{r=1}^{d} \sum_{j=1}^{M_r} (r + \mu_{j,r} d) \quad \text{with} \quad \mu_{j,r} \geqslant 0,\ (1 \leqslant r \leqslant d,\ 1 \leqslant j \leqslant M_r),$$

with

$$M_r = \begin{cases} N_r & \text{if } r \notin \{1,b,d\} \\ N_d & \text{if } r = b \\ N_b & \text{if } r = d \\ N_1 + (d-b)(N_d - N_b) & \text{if } r = 1, \end{cases}$$

and the integers $\mu_{j,r}$ are defined by:

$$\mu_{j,r} = \lambda_{j,r} \quad \text{for} \quad r \notin \{1,b,d\},\ 1 \leqslant j \leqslant M_r,$$

$$\mu_{j,b} = \lambda_{j,d} \ (1 \leqslant j \leqslant M_b), \ \mu_{j,d} = \lambda_{j,b} \ (1 \leqslant j \leqslant M_d),$$

$$\mu_{j,1} = \begin{cases} \lambda_{j,1} & \text{if } 1 \leqslant j \leqslant N_1 \\ 0 & \text{if } N_1 + 1 \leqslant j \leqslant M_1. \end{cases}$$

We check easily that this application Ψ is injective, and that we have $M_b \geqslant M_a \geqslant M_d$, $\Psi(\Pi) \in S(b, a, d)$.

• Case $a = 1$. If $a = 1$, the above application is not good because it may happen that $M_a = M_1 = N_1 + (d - b)(N_d - N_b) > M_b$, $\Psi(\Pi) \notin S(b, a, d)$.

In the case $a = 1$, we transform the quantity $(d - b)(N_d - N_b)$ in parts equal to 2 and eventually add a part equal to 1. We set $Z = \lfloor \frac{(N_d - N_b)(d-b)}{2} \rfloor$. The partition $\Psi(\Pi)$ is defined by:

$$\text{for } r \notin \{1, 2, b, d\}, \ M_r = N_r \ \text{ and } \ \mu_{j,r} = \lambda_{j,r} \ \text{ for } \ 1 \leqslant j \leqslant M_r,$$

$$M_d = N_b \text{ and } \mu_{j,d} = \lambda_{j,b} \text{ for } 1 \leqslant j \leqslant M_d,$$

$$M_1 = \begin{cases} N_1 & \text{if } (N_d - N_b)(d - b) \equiv 0 \pmod 2 \\ N_1 + 1 & \text{if } (N_d - N_b)(d - b) \equiv 1 \pmod 2 \end{cases}, \ \mu_{j,1} = \lambda_{j,1} \ \text{ for } \ 1 \leqslant j \leqslant N_1,$$

and if $(N_d - N_b)(d - b) \equiv 1 \pmod 2$, $\mu_{N_1+1,1} = 0$.

If $b \neq 2$, then we take

$$M_b = N_d \ \text{ and } \ \mu_{j,b} = \lambda_{j,d} \ \text{ for } \ 1 \leqslant j \leqslant M_b,$$

$$M_2 = N_2 + Z \text{ and } \mu_{j,2} = \begin{cases} \lambda_{j,2} & \text{if } 1 \leqslant j \leqslant N_2 \\ 0 & \text{if } N_2 + 1 \leqslant j \leqslant M_2. \end{cases}$$

If $b = 2$, then we take

$$M_2 = N_d + Z \ \text{ and } \ \mu_{j,2} = \begin{cases} \lambda_{j,d} & \text{if } 1 \leqslant j \leqslant N_d \\ 0 & \text{if } N_d + 1 \leqslant j \leqslant M_2. \end{cases}$$

In all cases we have $M_b \geqslant M_d$, and $M_a \geqslant M_d$. Furthermore, we have $M_1 \leqslant N_1 + 1 \leqslant N_d + 1$ thus the situation $M_1 > M_b$ can happen only if $N_d = N_1$. By Corollary 1.3, this can arrive for at most $o(p(n))$ partitions of n. Thus $\Psi(\Pi) \in S(b, a, d)$ for almost all $\Pi \in S(d, a, b)$. This ends the proof of Lemma 12.1.

Thus Theorem 1.4 is proved.

13. Dominant residue class

We investigate the number of partitions of n in which there are more parts $\equiv a \pmod d$ than parts $\equiv b \pmod d$ for all $b \in \{1, \ldots, d\} \setminus \{a\}$, briefly the case $N_a > N_b$ for $1 \leqslant b \leqslant d$, $b \neq a$. We shall consider the cases $N_a > N_b$ $(b \neq a)$ resp. $N_a \geqslant N_b$ $(b \neq a)$ together as $N_a \geqslant N_b + \Delta$ $(b \neq a)$ with $\Delta = 1$ resp. $\Delta = 0$.

We have to estimate

$$M_a := \sum_{\substack{N_1,\dots,N_d \\ R\equiv n \,(\mathrm{mod}\, d) \\ N_a \geqslant \Delta + \max_{b\neq a} N_b}} \Pi_d(n,R).$$

Like in the proof of Corollary 1.3 or Theorem 1.4 we apply Corollary 1.2 to avoid the abnormally small or big N_r and Corollary 9.1 to make the N_r independent.

Lemma 13.1. *We have the equality:*

$$M_a = o(p(n)) + (1+o(1))p(n)d^{\frac{2+d}{2}}\left(\frac{1}{2\sqrt{6n}}\right)^{\frac{d-1}{2}} \sum_{\substack{A\leqslant N_1,\dots,N_d<B \\ d|N_1 \\ N_a\geqslant \Delta+\max_{b\neq a} N_b}} F(N_1,\dots,N_d). \quad (13.1)$$

We use both $N_1^* = \lfloor \frac{N_1}{d} \rfloor d$ and $N_1^{**} = \lceil \frac{N_1}{d} \rceil d$.

We first state the case $a = 1$, next we will quote the modifications to handle the case $a \geqslant 2$.

By Corollary 9.1 and Corollary 1.2 we have

$$\begin{aligned} M_1 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (13.2) \\ &= o(p(n)) + (1+o(1))p(n)d^{\frac{2+d}{2}}\left(\frac{1}{2\sqrt{6n}}\right)^{\frac{d-1}{2}} \sum_{\substack{A\leqslant N_1,\dots,N_d<B \\ R\equiv n\,(\mathrm{mod}\, d) \\ N_1\geqslant \Delta+\max_{b\neq 1} N_b}} F(N_1^*,\dots,N_d) \\ &= o(p(n)) + (1+o(1))p(n)d^{\frac{2+d}{2}}\left(\frac{1}{2\sqrt{6n}}\right)^{\frac{d-1}{2}} \sum_{\substack{A\leqslant N_1,\dots,N_d\leqslant B \\ R\equiv n\,(\mathrm{mod}\, d) \\ N_1\geqslant \Delta+\max_{b\neq 1} N_b}} F(N_1^{**},\dots,N_d). \end{aligned}$$

We have

$$\begin{aligned} &\sum_{\substack{A\leqslant N_1,\dots,N_d<B \\ N_1\equiv n-\sum_{r=2}^d rN_r\,(\mathrm{mod}\, d) \\ N_1\geqslant \Delta+\max_{2\leqslant b\leqslant d} N_b}} F(N_1^*,\dots,N_d) \\ \geqslant &\sum_{\substack{A\leqslant N_1,\dots,N_d<B \\ N_1\equiv n-\sum_{r=2}^d rN_r\,(\mathrm{mod}\, d) \\ N_1^*\geqslant \Delta+\max_{2\leqslant b\leqslant d} N_b}} F(N_1^*,\dots,N_d) \qquad\qquad (13.3) \\ = &\sum_{\substack{A\leqslant N_1^*,\dots,N_d<B \\ N_1^*\geqslant \Delta+\max_{2\leqslant b\leqslant d} N_b}} F(N_1^*,\dots,N_d) \end{aligned}$$

and

$$\begin{aligned}
&\sum_{\substack{A\leqslant N_1,\dots,N_d\leqslant B\\ N_1\equiv n-\sum_{r=2}^d rN_r \ (\mathrm{mod}\, d)\\ N_1\geqslant \Delta+\max_{2\leqslant b\leqslant d} N_b}} F(N_1^{**},\dots,N_d)\\
&\leqslant \sum_{\substack{A\leqslant N_1,\dots,N_d\leqslant B\\ N_1\equiv n-\sum_{r=2}^d rN_r \ (\mathrm{mod}\, d)\\ N_1^{**}\geqslant \Delta+\max_{2\leqslant b\leqslant d} N_b}} F(N_1^{**},\dots,N_d)\\
&\leqslant \sum_{\substack{A\leqslant N_1^{**},\dots,N_d\leqslant B\\ N_1^{**}\geqslant \Delta+\max_{2\leqslant b\leqslant d} N_b}} F(N_1^{**},\dots,N_d)\\
&\leqslant \sum_{\substack{A\leqslant N_1^{**},\dots,N_d< B\\ N_1^{**}\geqslant \Delta+\max_{2\leqslant b\leqslant d} N_b}} F(N_1^{**},\dots,N_d)+E,
\end{aligned} \tag{13.4}$$

where E is an error term collecting the $(N_1^{**},\dots,N_d)$ with $N_1^{**}=B$. This term is small enough by Corollary 1.2. Therefore

$$M_1 = o(p(n)) + (1+o(1))p(n)d^{\frac{2+d}{2}}\Big(\frac{1}{2\sqrt{6n}}\Big)^{\frac{d-1}{2}} \sum_{\substack{A\leqslant N_1,\dots,N_d<B\\ d\mid N_1\\ N_1\geqslant\Delta+\max_{b\neq 1} N_b}} F(N_1,\dots,N_d).$$

This proves (13.1) for $a=1$. For $a\neq 1$ we replace in (13.2) the conditions $N_1\geqslant \Delta+\max_{2\leqslant b\leqslant d} N_b$ by the conditions $N_a\geqslant\Delta+\max_{b\neq a} N_b$. When we replace in these conditions N_1 by N_1^* and change $\leqslant B$ to $<B$, the corresponding (13.3) becomes an upper bound and when we replace N_1 by N_1^{**}, (13.4) becomes a lower bound. (The inequalities are permuted). This ends the proof of the lemma.

Proof of (i) and (ii) of Theorem 1.5 for $a=1$. It remains to compute the summations of

$$T_1 := \sum_{\substack{A\leqslant dN_1',N_2,\dots,N_d<B\\ dN_1'\geqslant N_b+\Delta\\ b=2,\dots,d}} F(dN_1',N_2,\dots,N_d).$$

We have:

$$T_1 = \sum_{\substack{A\leqslant dN_1',N_2,\dots,N_d<B\\ dN_1'\geqslant N_b+\Delta\\ b=2,\dots,d}} \int_{N_1'}^{N_1'+1}\int_{N_2}^{N_2+1}\cdots\int_{N_d}^{N_d+1} F(dN_1',N_2,\dots,N_d)\,\mathrm{d}t_1'\,\mathrm{d}t_2\cdots\mathrm{d}t_d.$$

We apply one more times Corollary 9.1:

$$\begin{aligned}
T_1 &\\
&= (1+o(1)) \sum_{\substack{A\leqslant dN_1', N_2,\dots,N_d<B \\ dN_1'\geqslant N_b+\Delta \\ b=2,\dots,d}} \int_{N_1'}^{N_1'+1}\int_{N_2}^{N_2+1}\cdots\int_{N_d}^{N_d+1} F(dt_1',t_2,\dots,t_d)\,\mathrm{d}t_1'\,\mathrm{d}t_2\cdots\mathrm{d}t_d. \\
&= (1+o(1)) \sum_{\frac{\Delta+A}{d}\leqslant N_1'<\frac{B}{d}} \int_{N_1'}^{N_1'+1}\int_{A}^{dN_1'-\Delta+1}\cdots\int_{A}^{dN_1'-\Delta+1} F(dt_1',t_2,\dots,t_d)\,\mathrm{d}t_1'\,\mathrm{d}t_2\cdots\mathrm{d}t_d.
\end{aligned}$$

Here the sum is

$$\leqslant \int_{\frac{A}{d}}^{\frac{B}{d}} \Big(\int_{A}^{dt_1'}\cdots\int_{A}^{dt_1'} F(dt_1',t_2,\cdots,t_d)\,\mathrm{d}t_2\cdots\,\mathrm{d}t_d\Big)\,\mathrm{d}t_1'$$

if $\Delta=1$ resp.

$$\begin{aligned}
&\geqslant \int_{\frac{A+d}{d}}^{\frac{B}{d}} \Big(\int_{A}^{dt_1'-d}\cdots\int_{A}^{dt_1'-d} F(dt_1',t_2,\cdots,t_d)\,\mathrm{d}t_2\cdots\,\mathrm{d}t_d\Big)\,\mathrm{d}t_1' \\
&= \int_{\frac{A}{d}}^{\frac{B-d}{d}} \Big(\int_{A}^{dt_1'}\cdots\int_{A}^{dt_1'} F(dt_1'+d,t_2,\cdots,t_d)\,\mathrm{d}t_2\cdots\,\mathrm{d}t_d\Big)\,\mathrm{d}t_1'
\end{aligned}$$

if $\Delta=0$. Taking into account Corollary 1.3, apart from $o(dp(n))$ partitions of n we can compute both cases together for fixed d as

$$\begin{aligned}
T_1 &= o(p(n)) + (1+o(1))p(n)d^{\frac{2+d}{2}}\Big(\frac{1}{2\sqrt{6n}}\Big)^{\frac{d-1}{2}} \\
&\quad\times \frac{1}{d}\int_A^B \Big(\int_A^{t_1}\cdots\int_A^{t_1} F(t_1,\dots,t_d)\,\mathrm{d}t_2\cdots,\,\mathrm{d}t_d\Big)\,\mathrm{d}t_1 \\
&= o(p(n)) + (1+o(1))p(n)d^{\frac{2+d}{2}}\Big(\frac{1}{2\sqrt{6n}}\Big)^{\frac{d-1}{2}} \\
&\quad\times \frac{1}{d}\int_A^B \exp\Big(-\frac{\pi}{\sqrt{6n}}t_1 - \frac{\sqrt{6n}}{\pi d}\exp\Big(-\frac{dt_1\pi}{\sqrt{6n}}\Big)\Big) \\
&\quad\times \Big\{\prod_{d=2}^{d}\int_A^{t_1}\exp\Big(-\frac{\pi}{\sqrt{6n}}rt - \frac{\sqrt{6n}}{\pi d}\exp\Big(-\frac{dt\pi}{\sqrt{6n}}\Big)\,\mathrm{d}t\Big\}\,\mathrm{d}t_1 \\
&= o(p(n)) + \frac{(1+o(1))p(n)}{\Gamma\Big(\frac{1}{d}\Big)\cdots\Gamma\Big(\frac{d}{d}\Big)}\int_0^\infty x^{\frac{1}{d}-1}\mathrm{e}^{-x}\Big(\prod_{r=2}^{d}\int_x^\infty y^{\frac{r}{d}-1}\mathrm{e}^{-y}\,\mathrm{d}y\Big)\,\mathrm{d}x
\end{aligned}$$

for fixed d. This ends the proof of Theorem 1.5 (i) and (ii) in the case $a=1$.

Case $a \geqq 2$**.** The term corresponding to T_1 is

$$T_a := \sum_{\substack{A \leqslant dN_1', \ldots, N_d < B \\ N_a \geqslant \Delta + dN_1' \\ N_a \geqslant \Delta + \max_{b \notin \{1,a\}} N_b}} F(dN_1', N_2, \ldots, N_d).$$

We use the integral representation and we apply Corollary 9.1:

$$T_a = (1 + o(1)) \\ \times \sum_{A+\Delta \leqslant N_a < B} \int_{N_a}^{N_a+1} \int_{A/d}^{\frac{N_a-\Delta}{d}+1} \Big[\int_A^{N_a-\Delta+1} \cdots \int_A^{N_a-\Delta+1} F(dt_1', \ldots, t_d) \prod_{j \neq 1,a} \mathrm{d}t_j \Big] \mathrm{d}t_1' \, \mathrm{d}t_a.$$

By Corollary 1.3 we see that we can handle the cases $\Delta = 0$ and 1 together and we do the same computations as in the case $a = 1$.

14. Some properties of truncated Gamma functions; end of the proof of Theorem 1.5

For $1 \leqslant a \leqslant d$, let us consider the integrals

$$J_a = \int_0^\infty x^{\frac{a}{d}-1} \mathrm{e}^{-x} \Big(\prod_{\substack{r=1 \\ r \neq a}}^{d} \int_x^\infty y^{\frac{r}{d}-1} \mathrm{e}^{-y} \, \mathrm{d}y \Big) \, \mathrm{d}x.$$

We have

$$\prod_{j=1}^{d} \Gamma\Big(\frac{j}{d}\Big) = \prod_{j=1}^{d} \Big(\int_0^\infty x_j^{\frac{j}{d}-1} \mathrm{e}^{-x_j} \, \mathrm{d}x_j \Big) = J_1 + J_2 + \cdots + J_d,$$

since

$$\{(x_1, \ldots, x_d) \in [0, \infty[^d\} = \cup_{a=1}^d \{(x_1, \ldots, x_d) \in [0, \infty[^d, x_a = \min_{1 \leqslant j \leqslant d} x_j\}.$$

For $1 < a \leqslant d$,

$$J_1 - J_a = \int_0^\infty \Big(\int_x^\infty \mathrm{e}^{-x-y} (xy)^{\frac{1}{d}-1} \big(y^{\frac{a-1}{d}} - x^{\frac{a-1}{d}} \big) \Big(\prod_{\substack{r=2 \\ r \neq a}}^{d} \int_x^\infty z^{\frac{r}{d}-1} \mathrm{e}^{-z} \, \mathrm{d}z \Big) \, \mathrm{d}y \Big) \, \mathrm{d}x > 0.$$

Therefore,

$$J_1 > \frac{1}{d} \Gamma\Big(\frac{1}{d}\Big) \Gamma\Big(\frac{2}{d}\Big) \cdots \Gamma\Big(\frac{d}{d}\Big)$$

and

$$o(p(n)) + (1+o(1))p(n)\frac{J_1}{\Gamma\left(\frac{1}{d}\right)\Gamma\left(\frac{2}{d}\right)\cdots\Gamma\left(\frac{d}{d}\right)} \sim p(n)\frac{J_1}{\Gamma\left(\frac{1}{d}\right)\Gamma\left(\frac{2}{d}\right)\cdots\Gamma\left(\frac{d}{d}\right)}$$

for fixed $d \geqslant 2$. We can estimate

$$\frac{J_1}{\Gamma\left(\frac{1}{d}\right)\Gamma\left(\frac{2}{d}\right)\cdots\Gamma\left(\frac{d}{d}\right)} - \frac{1}{d} = \frac{\sum_{a=2}^{d}(J_1 - J_a)}{d\Gamma\left(\frac{1}{d}\right)\Gamma\left(\frac{2}{d}\right)\cdots\Gamma\left(\frac{d}{d}\right)}$$

from below in the following way. For any $\delta > 0$ and $2 \leqslant a \leqslant d$,

$$\begin{aligned}
&J_1 - J_a \\
&> \int_0^\infty \Big(\int_{x(1+\delta)}^\infty \mathrm{e}^{-x-y}(xy)^{\frac{1}{d}-1}\Big(y^{\frac{a-1}{d}} - \Big(\frac{y}{1+\delta}\Big)^{\frac{a-1}{d}}\Big) \\
&\quad \times \Big(\prod_{\substack{r=2\\ r\neq a}}^{d} \int_x^\infty z^{\frac{r}{d}-1}\mathrm{e}^{-z}\,\mathrm{d}z\Big)\,\mathrm{d}y\Big)\,\mathrm{d}x \\
&= \Big(1 - \frac{1}{(1+\delta)^{\frac{a-1}{d}}}\Big)\Big\{ J_1 \\
&\quad - \int_0^\infty x^{\frac{1}{d}-1}\mathrm{e}^{-x}\Big(\int_x^{x(1+\delta)} y^{\frac{a}{d}-1}\mathrm{e}^{-y}\,\mathrm{d}y\Big)\Big(\prod_{\substack{r=2\\ r\neq a}}^{d} \int_x^\infty z^{\frac{r}{d}-1}\mathrm{e}^{-z}\,\mathrm{d}z\Big)\,\mathrm{d}x\Big\} \\
&> \Big(1 - \frac{1}{(1+\delta)^{\frac{a-1}{d}}}\Big)\Big\{\frac{1}{d}\Gamma\Big(\frac{1}{d}\Big)\cdots\Gamma\Big(\frac{d}{d}\Big) - \delta\Gamma\Big(\frac{1+a}{d}\Big)2^{-\frac{1+a}{d}} \prod_{\substack{r=2\\ r\neq a}}^{d} \Gamma\Big(\frac{r}{d}\Big)\Big\},
\end{aligned}$$

$$\begin{aligned}
\frac{J_1 - J_a}{d\Gamma\left(\frac{1}{d}\right)\cdots\Gamma\left(\frac{d}{d}\right)} &> \Big(1 - \frac{1}{(1+\delta)^{\frac{a-1}{d}}}\Big)\Big\{\frac{1}{d^2} - \frac{\delta\Gamma\left(\frac{1+a}{d}\right)}{2^{\frac{1+a}{d}}d\Gamma\left(\frac{1}{d}\right)\Gamma\left(\frac{a}{d}\right)}\Big\} \\
&> \frac{\exp\left(\frac{a-1}{d}\log(1+\delta)\right) - 1}{(1+\delta)^{\frac{a-1}{d}}}\Big\{\frac{1}{d^2} - \frac{\delta}{d^2}\Big\} \\
&> \frac{a-1}{d^3}\,\frac{(1-\delta)\log(1+\delta)}{1+\delta}.
\end{aligned}$$

Choosing $\delta := 0.364$ we obtain that

$$\sum_{a=2}^{d} \frac{J_1 - J_a}{d\Gamma\left(\frac{1}{d}\right)\Gamma\left(\frac{2}{d}\right)\cdots\Gamma\left(\frac{d}{d}\right)} > \sum_{a=2}^{d}\frac{a-1}{7d^3} = \frac{1}{14}\Big(\frac{1}{d} - \frac{1}{d^2}\Big).$$

This ends the proof of Theorem 1.5.

Similar arguments yield estimates for the case $N_1 > N_2 > \ldots > N_d$, *i. e.*, for the number of "d-regular" partitions of n, and more generally to obtain estimates for Theorem 1.7.

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