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CHANGES OF SIGN OF THE ERROR TERM IN THE PRIME NUMBER THEOREM[1]

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Dedicated to Professor Eduard Wirsing
on the occasion of his 75th birthday

Abstract: We assume the Riemann Hypothesis (RH). It is classical that there is an absolute constant $C > 1$ such that $\psi(x) - x$ changes sign in every interval $[x, Cx]$ for $x \geqslant 1$. We prove that $\psi(x) - x$ changes sign in $[x, 19x]$ for all $x \geqslant 1$, and also that for $x \geqslant x_0$, $\psi(x) - x$ changes sign in the interval $[x, Cx]$ where $C = 2.02$.
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1. Introduction

Ingham [1] showed (assuming RH) that there is a C such that $\pi(x) - \mathrm{li}(x)$ changes sign in every interval $[x, Cx]$ for which $x \geqslant 1$. This C is very large, possibly as large as 10^{1000} or even more. Ingham argued by elaborating on the method that Littlewood [4] devised to show that $\pi(x) - \mathrm{li}(x)$ has infinitely many sign changes. Let $\vartheta(x) = \sum_{p \leqslant x} \log p$. Ingham's method applies equally to sign changes of $\vartheta(x) - x$, but for $\psi(x) - x$ or $\Pi(x) - \mathrm{li}(x)$ the easier method of Littlewood [5] or Pólya [7] suffices. In this paper we consider $\psi(x) - x$ assuming the Riemann Hypothesis (RH).

Theorem 1. (Assume RH) *For every $x \geqslant 1$, the function $\psi(x) - x$ takes both positive and negative values in the interval $[x, 19\,x]$.*

The constant 19 is best possible, since $\psi(x) < x$ for $1 \leqslant x < 19$.

We turn now to the problem of finding C such that $\psi(x) - x$ changes sign in the interval $[x, Cx]$ for every $x \geqslant x_0$.

For $x > 1$, x not a prime power, we know that

$$\psi(x) = x - \sum_\rho \frac{x^\rho}{\rho} + \log 2\pi - \frac{1}{2}\log(1 - 1/x^2)\,. \tag{1}$$

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Here the sum over the zeros is uniformly convergent in closed subintervals of $(1,\infty)$ not containing a prime power. Put

$$f(y) \;=\; -\sum_\rho \frac{e^{i\gamma y}}{\rho}\,. \tag{2}$$

Thus under RH,

$$\psi\big(e^y\big) - e^y \;=\; f(y)e^{y/2} + O(1)\,,$$

and our object is to find c so that $f(y)$ changes sign in any interval of the form $[Y,\,Y+c\,]$. The most familiar argument in this direction involves putting

$$f_k(y) \;=\; -\sum_\rho \frac{e^{i\gamma y}}{\rho(i\gamma)^k}\,.$$

Thus $f_0(y) = f(y)$ and $f_k(y)' = f_{k-1}(y)$ for $k>0$. Let $0<\gamma_1 \leqslant \gamma_2 \leqslant \cdots$ denote the ordinates of the zeros of the zeta function in the upper half-plane, and put $y_r = \pi r/\gamma_1 + \phi$ where ϕ is chosen, $0 \leqslant \phi < 2\pi$, so that

$$e^{i\gamma_1\phi} \;=\; -\frac{\rho_1}{|\rho_1|}\,. \tag{3}$$

Thus the combined contribution of $\pm\gamma_1 = \pm 14.134725141734693790\ldots$ to $f(y)$ is largest when $y = y_{2r}$ for some r, and is smallest (i.e., most negative) when $y = y_{2r-1}$. If

$$\frac{1}{|\rho_1|\gamma_1^k} \;>\; \sum_{j=2}^{\infty} \frac{1}{|\rho_j|\gamma_j^k}\,, \tag{4}$$

then $(-1)^r f_k(y_r + \pi k/2) > 0$. Given $k+2$ consecutive such points, we have at least $k+1$ changes of sign of f_k. This gives at least k changes of sign of f_{k-1}, and so on, until we have at least one change of sign of $f_0 = f$. Since any interval of length $(k+2)\pi/\gamma_1$ contains at least $k+2$ of these points, it follows that we can take $c = (k+2)\pi/\gamma_1$. It is clear that (4) holds for all sufficiently large k. In §2 we show that it fails for $k=2$, but holds for $k=3$. Thus we can take $c = 5\pi/\gamma_1 = 1.111303\ldots$. It is also clear that there is an effectively computable constant $a>0$ such that

$$\max_{y\in I} f(y) \geqslant a, \qquad \min_{y\in I} f(y) \leqslant -a$$

provided that I is an interval of length at least c. Hence any $C > e^c = 3.0381149\ldots$ is admissible for $x \geqslant x_0$.

A somewhat more efficient variant of this approach involves setting

$$F_1(y) \;=\; \int_{y-\pi/(2\gamma_1)}^{y+\pi/(2\gamma_1)} f(u)\,du \;=\; -2\sum_\rho \frac{e^{i\gamma y}\sin\frac{\pi\gamma}{2\gamma_1}}{\rho\,\gamma}\,,$$

and in general

$$F_k(y) \;=\; \int_{y-\pi/(2\gamma_1)}^{y+\pi/(2\gamma_1)} F_{k-1}(u)\,du \;=\; -2^k \sum_\rho \frac{e^{i\gamma y}}{\rho}\Big(\frac{\sin\frac{\pi\gamma}{2\gamma_1}}{\gamma}\Big)^k .$$

If

$$\frac{1}{|\rho_1|\gamma_1^k} \;>\; \sum_{j=2}^{\infty} \frac{\big|\sin\frac{\pi\gamma_j}{2\gamma_1}\big|^k}{|\rho|\gamma_j^k}, \tag{5}$$

then $(-1)^r F_k(y_r) > 0$. Thus f takes positive values in any interval of the form

$$\big[y_{2r} - k\pi/(2\gamma_1),\, y_{2r} + k\pi/(2\gamma_1)\big]\,,$$

and negative values in any interval

$$\big[y_{2r-1} - k\pi/(2\gamma_1),\, y_{2r-1} + k\pi/(2\gamma_1)\big]\,.$$

If $c = (k+2)\pi/\gamma_1$, then any interval $[y,\, y+c\,]$ contains subintervals of both these sorts. In §2 we show that (5) fails for $k = 1$, but holds for $k = 2$. Thus we can take $c = 4\pi/\gamma_1 = 0.8890\ldots$, and $C > e^c = 2.43799\ldots$ for $x \geqslant x_0$.

With these classical arguments acknowledged, we propose a better method. Suppose that $K \in L^1(\mathbb{R})$ is a nonnegative function with support in $[-\alpha,\, \alpha\,]$. Then

$$\int_{-\alpha}^{\alpha} f(Y+y)K(y)\,dy \;=\; -\sum_\rho \frac{e^{i\gamma Y}}{\rho}\int_{-\alpha}^{\alpha} e^{i\gamma y}K(y)\,dy \;=\; -\sum_\rho \frac{e^{i\gamma Y}}{\rho}\,\widehat{K}\Big(\frac{-\gamma}{2\pi}\Big)$$

where $\widehat{K}(t)$ denotes the Fourier transform of K, $\widehat{K}(t) = \int_{-\alpha}^{\alpha} K(y)e(-ty)\,dy$. Here $e(\theta) = e^{2\pi i\theta}$ is the complex exponential with period 1. If K can be chosen so that

$$-\Re\Big(\frac{e^{i\gamma_1 Y}}{\rho_1}\widehat{K}\Big(\frac{-\gamma_1}{2\pi}\Big)\Big) \;-\; \sum_{j=2}^{\infty} \frac{\big|\widehat{K}\big(\frac{-\gamma_j}{2\pi}\big)\big|}{|\rho_j|} \;>\; 0\,, \tag{6}$$

then $f(y) > 0$ for some $y \in [Y-\alpha,\, Y+\alpha\,]$. Similarly, if

$$-\Re\Big(\frac{e^{i\gamma_1 Y}}{\rho_1}\widehat{K}\Big(\frac{-\gamma_1}{2\pi}\Big)\Big) \;+\; \sum_{j=2}^{\infty} \frac{\big|\widehat{K}\big(\frac{-\gamma_j}{2\pi}\big)\big|}{|\rho_j|} \;<\; 0\,, \tag{7}$$

then $f(y) < 0$ for some $y \in [Y-\alpha,\, Y+\alpha\,]$. By simple choices of K of the form

$$K(y) = \max(0, 1-|y|/\alpha)\big(1 \pm \cos(\gamma_1 y + \theta)\big)\,, \tag{8}$$

we obtain

Theorem 2. (Assume RH) *Let $f(y)$ be defined as in* (2)*. If $Y \geqslant 0$, then the function $f(y)$ takes values of both signs in the interval $[Y, Y+c]$ where $c = 0.7$, and for $x \geqslant x_0$, $\psi(x) - x$ changes sign in the interval $[x, Cx]$ where $C = 2.02$.*

We obtain the above by arguing rather crudely. We claim that by taking more care in verifying (6) and (7), we could reduce 0.7 to 0.62.

To optimize our approach we would need kernels $K(y)$ that depend on Y modulo $2\pi/\gamma_1$, which is to say a continuum of kernels. In the case that Y is of the form $Y = y_r$, we define carefully chosen kernels that seem to be close to optimal, and thus obtain the following special results.

Theorem 3. (Assume RH) *Let $\delta_+ = 0.1375$, and $\delta_- = 0.28495$. There is a $y \in [y_{2r} - \delta_+, y_{2r} + \delta_+]$, such that $f(y) > 0$. There is a $y \in [y_{2r} - \delta_-, y_{2r} + \delta_-]$ such that $f(y) < 0$. Similarly, there is a $y \in [y_{2r-1} - \delta_+, y_{2r-1} + \delta_+]$ such that $f(y) < 0$, and a $y \in [y_{2r-1} - \delta_-, y_{2r-1} + \delta_-]$ such that $f(y) > 0$.*

If $c = 2\pi/\gamma_1 + 2\delta_+$, then any interval $[y, y+c]$ contains subintervals of the form $[y_{2r} - \delta_+, y_{2r} + \delta_+]$ and of the form $[y_{2r-1} - \delta_+, y_{2r-1} + \delta_+]$, and hence $f(y)$ takes values of both signs in such an interval. However, $2\pi/\gamma_1 + 2\delta_+ = 0.7195$, which is larger than the constant we obtained already in Theorem 2.

Let J be a positive integer. Our method applies to any sum of the form

$$F(y) = \sum_{j=1}^{J} \frac{\cos(\gamma_j y + \phi_j)}{|\rho_j|},$$

for arbitrary real ϕ_j. Thus the following result provides lower bounds for the constants that can be obtained by our method.

Theorem 4. *Let $\tau_+ = 0.0953$ and $\tau_- = 0.2431$. There exist functions $F_\pm$ of the form*

$$F_\pm(y) = \sum_{j=1}^{J} \varepsilon_\pm(j) \frac{\cos \gamma_j y}{|\rho_j|}$$

with $\varepsilon_\pm(j) = \pm 1$ for all j, $\varepsilon_+(1) = 1$, and $\varepsilon_-(1) = -1$ such that $F_+(y) < 0$ for $-\tau_+ \leqslant y \leqslant \tau_+$ and $F_-(y) < 0$ for $-\tau_- \leqslant y \leqslant \tau_-$.

It seems plausible that with enough work one could prove Theorem 2 with c replaced by $2\delta_-$. Thus it seems likely that the optimal constant c in Theorem 2 lies between $2\tau_- = 0.4862$ and $2\delta_- = 0.5699$.

Let $V(x)$ denote the number of sign changes of $\psi(u) - u$ for $1 \leqslant u \leqslant x$. Assuming RH, our results imply that $V(x) \gg \log x$. Indeed, Kaczorowski [2] has shown unconditionally that

$$\liminf_{x\to\infty} \frac{V(x)}{\log x} \geqslant \frac{\gamma_1}{\pi},$$

and later Kaczorowski [3] showed that the constant can be improved slightly. On the other hand, we expect that $V(x)$ is closer to the order of $\sqrt{x}$; possibly even

$$0 < \liminf_{x\to\infty} \frac{V(x)}{\sqrt{x}} < \limsup_{x\to\infty} \frac{V(x)}{\sqrt{x}} < \infty .$$

From a calculation of sign changes out to 10^8 we extract the following values.

Table 1: Values of $V(x)$.

x	$V(x)$	$V(x)/\sqrt{x}$
10^2	24	2.4
10^3	162	5.12
10^4	701	7.01
10^5	2351	7.43
10^6	7314	7.31
10^7	20,432	6.46
10^8	64,694	6.47

2. Numerical scrutiny of classical arguments

The ordinates of the first 100,000 zeros have been computed to within 10^{-10} by Odlyzko [6]. We set $J = 32{,}767$, and use the computed values of the first J ordinates to derive the first three columns of the following table.

Table 2: Test of relation (4) for $k = 1, 2, 3$.

k	$\frac{1}{\lvert\rho_1\rvert\gamma_1^k}$	$\sum_{j=2}^{J} \frac{1}{\lvert\rho_j\rvert\,\gamma_j^k}$	$\sum_{j=J+1}^{\infty} \frac{1}{\lvert\rho_j\rvert\,\gamma_j^k}$
1	0.0050021155	0.0180445096	< 0.0000537248
2	0.0003538884	0.0003753803	< 0.0000000020
3	0.0000250368	0.0000121178	$< 10^{-10}$

The fourth column of the above table is computed by means of the following reasoning. The zeta function and its derivatives are easily calculated, when $|s|$ is not too large, by means of the Euler–Maclaurin summation formula. The gamma function and its derivatives are likewise easily computed. Hence we can quickly compute

$$\xi(s) = \frac{1}{2}s(s-1)\zeta(s)\Gamma(s/2)\pi^{-s/2}$$

and its derivatives. But

$$\frac{\xi'}{\xi}(s) = \sum_\rho \frac{1}{s-\rho}$$

where the sum is conditionally convergent. On differentiating $k-1$ times, we deduce that

$$\Big(\frac{\xi'}{\xi}\Big)^{(k-1)}(s) = (-1)^{k-1}(k-1)!\sum_\rho \frac{1}{(s-\rho)^k}\,.$$

Here the sums are absolutely convergent for $k \geqslant 2$. We take $s = 1/2$ in the above. For k odd the sum vanishes, due to the symmetry of the zeros. However, for even k this gives the second column of the following table.

Table 3: Sums over zeros.

k	$\sum_\gamma \frac{1}{\gamma^k}$	$\sum_{j=1}^{J} \frac{1}{\gamma_j^k}$	$\sum_{j=J+1}^{\infty} \frac{1}{\gamma_j^k}$
2	0.04620998623084	0.0230512683	0.0000537248
4	0.00007434519893	0.0000371726	$< 10^{-10}$
6	0.00000028834786	0.0000001442	$< 10^{-10}$

After computing the third column above, we can difference to obtain the fourth column. For odd k we employ the crude inequality

$$\sum_{j=J+1}^{\infty} \frac{1}{\gamma_j^k} < \frac{1}{\gamma_J}\sum_{j=J+1}^{\infty} \frac{1}{\gamma_j^{k-1}}\,.$$

Thus we obtain the entries in the last column of Table 2. In a similar manner we construct the following table, from which we see that (5) fails for $k = 1$, but holds for $k = 2$.

Table 4: Test of relation (5) for $k = 1, 2$.

k	$\frac{1}{\lvert\rho_1\rvert\gamma_1^k}$	$\sum_{j=2}^{J} \frac{\lvert\sin\frac{\pi\gamma_j}{2\gamma_1}\rvert^k}{\lvert\rho_j\rvert\gamma_j^k}$	$\sum_{j=J+1}^{\infty} \frac{1}{\lvert\rho_j\rvert\gamma_j^k}$
1	0.0050021155	0.0110557294	< 0.0000537248
2	0.0003538884	0.0001566074	< 0.0000000020

3. Proof of Theorem 1

Put $g(y) = \big(\log 2\pi - \frac{1}{2}\log(1-e^{-2y})\big)\, e^{-y/2}$. Thus by (1) and (2) we see that

$$\frac{\psi(e^y) - e^y}{e^{y/2}} \;=\; f(y) + g(y)\,.$$

Put

$$G_1(y) \;=\; \int_{y-\pi/(2\gamma_1)}^{y+\pi/(2\gamma_1)} g(u)\,du,$$

$$G_2(y) \;=\; \int_{y-\pi/(2\gamma_1)}^{y+\pi/(2\gamma_1)} G_1(u)\,du.$$

From Table 2 we deduce that when $k = 2$, the left hand side of (5) minus the right hand side is > 0.000195. Hence $(-1)^r F_2(y_r) > 0.00156$. On the other hand, if $y \geqslant 10$, then $|g(y)| < 0.012385$. Hence if $y > 10 + \pi/\gamma_1$, then $|G_2(y)| < 0.012385 \cdot (\pi/\gamma_1)^2 < 0.0007$. Thus $\psi(x) - x$ takes both positive and negative values in every interval $[x, 2.44\,x]$, for all $x \geqslant 28{,}283$. To complete the proof it suffices to note that $\psi(x) - x$ changes sign at 19, at 359, at 6803, and at 128,981, not to mention many other sign changes in between.

4. Proof of Theorem 2

By an easy calculation we find that if $K(y)$ is defined as in (8), then

$$\widehat{K}(t) \;=\; \alpha\Big(\frac{\sin\pi\alpha t}{\pi\alpha t}\Big)^2 \pm \frac{\alpha}{2}\, e^{i\theta}\Big(\frac{\sin\pi\alpha(t-\frac{\gamma_1}{2\pi})}{\pi\alpha(t-\frac{\gamma_1}{2\pi})}\Big)^2 \pm \frac{\alpha}{2}\, e^{-i\theta}\Big(\frac{\sin\pi\alpha(t+\frac{\gamma_1}{2\pi})}{\pi\alpha(t+\frac{\gamma_1}{2\pi})}\Big)^2.$$

Hence

$$\begin{aligned}\widehat{K}\big(-\frac{\gamma}{2\pi}\big) \;=\; &\alpha\Big(\frac{\sin\alpha\gamma/2}{\alpha\gamma/2}\Big)^2 \pm \frac{\alpha}{2}e^{i\theta}\Big(\frac{\sin\alpha(\gamma+\gamma_1)/2}{\alpha(\gamma+\gamma_1)/2}\Big)^2 \\ &\pm \frac{\alpha}{2}e^{-i\theta}\Big(\frac{\sin\alpha(\gamma-\gamma_1)/2}{\alpha(\gamma-\gamma_1)/2}\Big)^2.\end{aligned} \tag{9}$$

In particular, it follows that

$$-\frac{e^{i\gamma_1 Y}}{\rho_1}\widehat{K}\big(-\frac{\gamma_1}{2\pi}\big) \;=\; -\alpha\frac{e^{i\gamma_1 Y}}{\rho_1}\Big(\frac{\sin\alpha\gamma_1/2}{\alpha\gamma_1/2}\Big)^2 \mp \alpha\frac{e^{i(\gamma_1 Y+\theta)}}{2\rho_1}\Big(\frac{\sin\alpha\gamma_1}{\alpha\gamma_1}\Big)^2 \mp \alpha\frac{e^{i(\gamma_1 Y-\theta)}}{2\rho_1}\,.$$

Let ϕ be defined as in (3), and set $\theta = \gamma_1(Y-\phi)$. Then the last term above is $\pm\alpha/(2|\rho_1|)$. Put

$$\Delta \;=\; 2\Big(\frac{\sin\alpha\gamma_1/2}{\alpha\gamma_1/2}\Big)^2 + \Big(\frac{\sin\alpha\gamma_1}{\alpha\gamma_1}\Big)^2 .$$

Then we find that

$$\frac{(\pm 1-\Delta)\alpha}{2|\rho_1|} \;\leqslant\; -\Re\frac{e^{i\gamma_1 Y}}{\rho_1}\widehat{K}\big(-\frac{\gamma_1}{2\pi}\big) \;\leqslant\; \frac{(\pm 1+\Delta)\alpha}{2|\rho_1|}\,. \tag{10}$$

From (9) we see that

$$\sum_{j=2}^{\infty}\frac{\big|\widehat{K}\big(\frac{-\gamma_j}{2\pi}\big)\big|}{|\rho_j|} \;\leqslant\; \alpha\sum_{j=2}^{\infty}E_j \tag{11}$$

where

$$E_j \;=\; \frac{1}{|\rho_j|}\bigg(\Big(\frac{\sin\alpha\gamma_j/2}{\alpha\gamma_j/2}\Big)^2 + \frac{1}{2}\Big(\frac{\sin\alpha(\gamma_j+\gamma_1)/2}{\alpha(\gamma_j+\gamma_1)/2}\Big)^2 + \frac{1}{2}\Big(\frac{\sin\alpha(\gamma_j-\gamma_1)/2}{\alpha(\gamma_j-\gamma_1)/2}\Big)^2\bigg).$$

In particular $E_1 = \frac{1}{2|\rho_1|}(\Delta+1)$. From (10) and (11) we see that if

$$\sum_{j=1}^{\infty}E_j < \frac{1}{|\rho_1|}, \tag{12}$$

then (6) holds if we take the plus sign, and (7) holds if we take the minus sign. With $\alpha = 0.35$, we compute that $\sum_{j=1}^{J}E_j = 0.0703064889$. Also, since $E_j < 74/(|\rho_j|\gamma_j^2))$ for $j > J$, it follows from the second line of Table 2 that $\sum_{j>J}E_j < 0.00000015$. On the other hand, $1/|\rho_1| = 0.0707035277$, so (12) holds and the proof is complete.

In deriving (10) and (11) we used the triangle inequality, which is somewhat wasteful. If we kept strictly to (6) and (7), then a numerical check would have to be made for each Y modulo $2\pi/\gamma_1$. Such checks for closely spaced Y suggest that (6) and (7) hold when α is smaller, say close to 0.31. However, it seems that a rigorous argument along these lines would involve a lot of work.

5. Proof of Theorem 3

We employ (6) and (7) where $K(y)$ is of the form

$$K(y) = \sum_{m=-M}^{M} a_m \max(0, 1-|y-m\delta|/\delta) \tag{13}$$

where the a_m are nonnegative real numbers with $a_{-m} = a_m$. Thus $K(y)$ is an even nonnegative piecewise linear function with $K(m\delta) = a_m$. Moreover, the support of K is contained in $(-(M+1)\delta, (M+1)\delta)$. For such K we find that

$$\widehat{K}(t) = \Big(a_0 + 2\sum_{m=1}^{M} a_m \cos 2\pi m t\delta\Big) \frac{(\sin \pi t\delta)^2}{\pi^2 t^2 \delta}\,.$$

Let K_+ be of this type, with $a_0 = 1$, $M = 10$, and the a_m as in Table 5.

Table 5: Choice of a_m for $K_+(y)$.

m	a_m	m	a_m
1	0.94400	6	0.57043
2	0.90721	7	0.46654
3	0.84179	8	0.36138
4	0.76204	9	0.25600
5	0.67160	10	0.17640

We take $\delta = 0.0125$, with the result that K_+ has support in $(-\delta_+, \delta_+)$. We compute that

$$\sum_{j=2}^{J} \frac{|\widehat{K}_+(\frac{-\gamma_j}{2\pi})|}{|\rho_j|} = 0.0073809339\,.$$

Also, we note that

$$(\sin \pi u)^2 \left|1 + 2\sum_{m=1}^{M} a_m \cos 2\pi m u\right| \,<\, 0.2$$

uniformly in u. Hence $|\widehat{K}_+(t)| \leqslant a/t^2$ with $a = 16/\pi^2$, and consequently $|\widehat{K}_+(t/(2\pi))| \leqslant 64/t^2$. Thus by the second line of Table 2 we conclude that

$$\sum_{j=J+1}^{\infty} \frac{|\widehat{K}_+(\frac{-\gamma_j}{2\pi})|}{|\rho_j|} \,<\, 0.0000002\,.$$

On the other hand,

$$\frac{\widehat{K}_+(\frac{-\gamma_1}{2\pi})}{|\rho_1|} \,=\, 0.0078655340\,,$$

so we have (6) when $Y = y_{2r}$, and (7) when $Y = y_{2r-1}$.

Similarly, let K_- be of the form (13), with $a_0 = 0$, $M = 40$, and the a_m as in Table 6.

Table 6: Choice of a_m for $K_-(y)$.

m	a_m	m	a_m	m	a_m	m	a_m
1	0.00000	11	0.02329	21	0.31653	31	0.38884
2	0.00000	12	0.04391	22	0.34228	32	0.36259
3	0.00000	13	0.06627	23	0.36906	33	0.33571
4	0.00000	14	0.09174	24	0.39403	34	0.30907
5	0.00000	15	0.12148	25	0.41247	35	0.27871
6	0.00000	16	0.15803	26	0.42705	36	0.23867
7	0.00000	17	0.18862	27	0.43328	37	0.20743
8	0.00000	18	0.22393	28	0.42587	38	0.16867
9	0.00000	19	0.25615	29	0.41624	39	0.12862
10	0.00361	20	0.28579	30	0.40593	40	0.09620

We take $\delta = 0.00695$, with the result that K_- has support in $(-\delta_-, \delta_-)$. We compute that

$$\sum_{j=2}^{J} \frac{|\widehat{K}_-(\frac{-\gamma_j}{2\pi})|}{|\rho_j|} = 0.0054491548 .$$

Also, we note that

$$2(\sin \pi u)^2 \Big| \sum_{m=1}^{M} a_m \cos 2\pi m u \Big| < 0.1$$

uniformly in u. Hence $|\widehat{K}_-(t/(2\pi))| \leqslant 60/t^2$. Thus by the second line of Table 2 we conclude that

$$\sum_{j=J+1}^{\infty} \frac{|\widehat{K}_-(\frac{-\gamma_j}{2\pi})|}{|\rho_j|} < 0.0000002 .$$

On the other hand,

$$\frac{\widehat{K}_-(\frac{-\gamma_1}{2\pi})}{|\rho_1|} = -0.0055318015 ,$$

so we have (7) when $Y = y_{2r}$, and (6) when $Y = y_{2r-1}$.

6. Proof of Theorem 4

It suffices to take $J = 36$ and $\varepsilon_\pm(j)$ as in Table 7.

Table 7: Choices of $\varepsilon_\pm(j)$.

j	$\varepsilon_+(j)$	$\widehat{K}_+\left(\frac{\gamma_j}{2\pi}\right)$	$\varepsilon_-(j)$	$\widehat{K}_-\left(\frac{\gamma_j}{2\pi}\right)$	j	$\varepsilon_+(j)$	$\widehat{K}_+\left(\frac{\gamma_j}{2\pi}\right)$	$\varepsilon_-(j)$	$\widehat{K}_-\left(\frac{\gamma_j}{2\pi}\right)$
1	1	0.11125	−1	−0.07824	19	−1	0.00095	−1	0.00192
2	−1	0.06692	1	−0.04652	20	−1	0.00024	1	0.00139
3	−1	0.04282	1	0.00000	21	−1	−0.00073	−1	−0.00001
4	−1	0.01663	−1	0.03139	22	1	−0.00187	−1	−0.00243
5	−1	0.00780	−1	0.02861	23	−1	−0.00219	−1	−0.00309
6	1	−0.00269	−1	0.00959	24	1	−0.00232	1	−0.00292
7	1	−0.00594	1	−0.00154	25	−1	−0.00223	1	−0.00234
8	1	−0.00648	1	−0.00443	26	1	−0.00155	1	−0.00001
9	1	−0.00453	1	−0.00036	27	1	−0.00093	−1	0.00118
10	−1	−0.00318	−1	0.00212	28	1	−0.00054	1	0.00159
11	1	−0.00046	−1	0.00457	29	1	0.00046	−1	0.00164
12	1	0.00225	−1	0.00311	30	1	0.00124	−1	0.00080
13	1	0.00392	−1	−0.00000	31	−1	0.00186	−1	−0.00031
14	−1	0.00449	1	−0.00149	32	1	0.00220	1	−0.00098
15	1	0.00497	1	−0.00304	33	1	0.00242	1	−0.00136
16	−1	0.00466	1	−0.00219	34	−1	0.00248	−1	−0.00097
17	−1	0.00388	−1	−0.00038	35	−1	0.00241	1	−0.00069
18	−1	0.00277	1	0.00135	36	−1	0.00208	1	0.00022

If the $\varepsilon_+(j)$ and $K_+(y)$ are both chosen optimally, then we would expect that $\varepsilon_+(j) = -\operatorname{sgn} \widehat{K}_+(\gamma_j/(2\pi))$ for all $j > 1$. Similarly, if the $\varepsilon_-(j)$ and $K_-(y)$ are chosen optimally, then we would expect that $\varepsilon_-(j) = -\operatorname{sgn} \widehat{K}_-(\gamma_j/(2\pi))$ for all $j > 1$. In Table 7 we see that these relations hold for most of the smaller j, which suggests that our choices are at least moderately close to optimal. It is to be expected that better choices of functions $K_\pm(y)$ can be found by using larger values of M, and that better $F_\pm(y)$ can be constructed by using more zeros.

Graphs of $F_\pm(y)$ can be found in Figures 1 and 2.

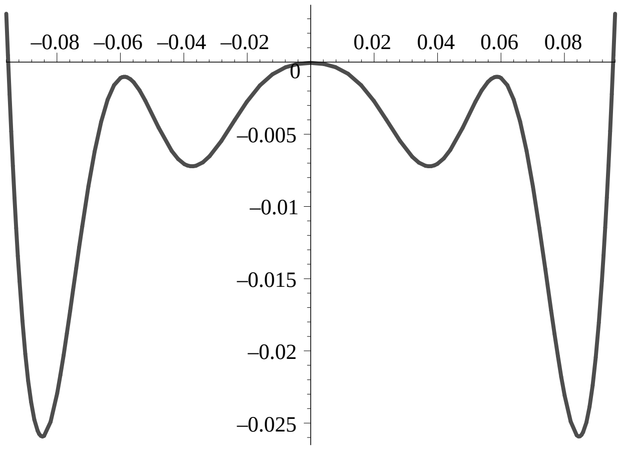

Figure 1. Graph of $F_+(y)$ for $-0.096 \leqslant y \leqslant 0.096$.

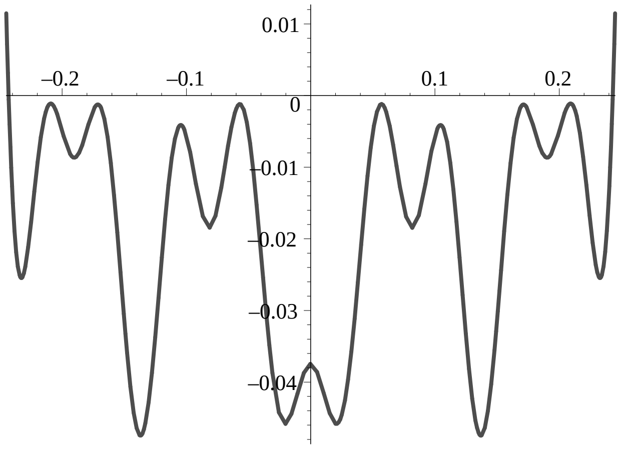

Figure 2. Graph of $F_-(y)$ for $-0.245 \leqslant y \leqslant 0.245$.

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