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THE ASYMPTOTIC BEHAVIOR OVER A SMALL PARAMETER OF A SERIES OF LARGE DEVIATION PROBABILITIES WEIGHTED WITH THE DIRICHLET DIVISORS FUNCTION

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Dedicated to Professor Eduard Wirsing
on the occasion of his 75th birthday

Abstract: We obtain the precise asymptotics of the series

$$\sum_{k=1}^{\infty} \frac{d_k}{k} \mathsf{P}\left(|S_k| \geqslant \varepsilon k\right)$$

as $\varepsilon \downarrow 0$ where S_k are partial sums of independent identically distributed random variables attracted to a stable law of index $\alpha > 1$.
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1. Introduction

Let $\{X_n, n \geqslant 1\}$ be independent identically distributed random variables and $\{S_n, n \geqslant 1\}$ be their partial sums. The backbone of the classical probability theory is the limit theorems for sums for various types of convergence. We deal with the so-called complete convergence in this paper. To be specific, the random variables S_n/n are said to converge to 0 completely if

$$\sum_{n=1}^{\infty} \mathsf{P}(|S_n/n| \geqslant \varepsilon) < \infty \qquad \text{for all } \varepsilon > 0. \tag{1.1}$$

By the Borel–Cantelli lemma, (1.1) implies the strong law of large numbers, that is the almost sure convergence of S_n/n to zero. The complete convergence is introduced by Hsu and Robbins [12] who proved, in particular, that (1.1) holds if

$$\mathsf{E}X_1 = 0, \qquad \mathsf{E}X_1^2 < \infty. \tag{1.2}$$

Erdös [5] was able to prove the converse, so that (1.1) and (1.2) are equivalent.

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Many results on series related to (1.1) have been obtained since then. For example, Spitzer [21] studied the series

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathsf{P}(|S_n/n| \geqslant \varepsilon) \tag{1.3}$$

and proved that it converges for all $\varepsilon > 0$ if and only if

$$\mathsf{E}X_1 = 0. \tag{1.4}$$

This condition is known to be equivalent to the strong law of large numbers, so that series (1.3) converges for all $\varepsilon > 0$ if and only if the strong law of large numbers holds.

Both series (1.1) and (1.3) depend on a parameter $\varepsilon > 0$. If the distribution of terms X_n is not degenerate, then the limit of both series as $\varepsilon \downarrow 0$ is infinite and the question arises on how to obtain their precise asymptotics.

Heyde [10] solved the problem for the Hsu–Robbins–Erdös series by proving that

$$\lim_{\varepsilon \downarrow 0} \varepsilon^2 \sum_{n=1}^{\infty} \mathsf{P}(|S_n/n| \geqslant \varepsilon) = \sigma^2$$

where $\sigma^2 = \mathsf{E}X_1^2$. Note that the condition for the asymptotics is the same as just for the convergence, namely the finiteness of the second moment.

The same problem for the Spitzer series have been studied in many papers (see, for example, [4], [20], [17], [2]). The problem is not yet solved in a final form, since sufficient conditions used to obtain the asymptotics of the Spitzer series are stronger than the assumption needed for just the convergence of it. One of the sufficient conditions for the asymptotics of the Spitzer series is

$$F \text{ belongs to the domain of attraction of an } \alpha\text{-stable law with } 1 < \alpha \leqslant 2 \tag{1.5}$$

which together with (1.4) yields

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\ln(1/\varepsilon)} \sum_{n=1}^{\infty} \frac{1}{n} \mathsf{P}(|S_n/n| \geqslant \varepsilon) = \frac{\alpha}{\alpha - 1}. \tag{1.6}$$

In this paper, we investigate a similar problem but for double sums. Namely let $\{X(m,n), m \geqslant 1, n \geqslant 1\}$ be independent identically distributed random variables and

$$S(m,n) = \sum_{i=1}^{m} \sum_{j=1}^{n} X(i,j).$$

Consider the series

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn} \mathsf{P}(|S(m,n)| \geqslant mn\varepsilon). \tag{1.7}$$

What is its asymptotics as $\varepsilon \downarrow 0$?

Double series (multiple series, in a more general setting) of probabilities related to the complete convergence have been already studied in the literature (see, for example [19], [9], [14]). The asymptotics of the series

$$R_2(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \mathsf{P}(S(m,n) < x) \tag{1.8}$$

is studied as $x \to \infty$ in [7]. By analogy with the one-dimensional case the latter series is called the renewal function (constructed from a random walk with two-dimensional time).

Series (1.7) can easily be transformed into an usual series involving Dirichlet divisors function. Namely let d_k be the number of solutions (m,n) of the equation $mn = k$ (solutions (m,n) and (n,m) are considered to be different for $m \neq n$). Then series (1.7) is nothing else but

$$\sum_{k=1}^{\infty} \frac{d_k}{k} \mathsf{P}(|S_k| \geqslant k\varepsilon).$$

At the first glance, this series can be studied in a way similar to the one-dimensional case. This is not true, however, since the weights $w_k = d_k/k$ vary irregularly (the regular variation is one of the assumptions in the classical theory). Nevertheless we are able to obtain the result by using the fact that the partial sums of d_k vary regularly.

The paper is organized as follows. For the reader convenience, we collect in Section 2 the main notation used throughout the paper. Section 3 contains the main result and its proof. The idea of the proof is to show first that the behavior of the series in the general case is the same as in the case of stable terms and then to determine it for the stable case. We collect all auxiliary results needed for the proof in Section 4. We briefly discuss an unsolved problem in Section 5.

2. Main notation

Let d_k be the number of divisors of an integer number $k \geqslant 1$ and $D_k = \sum_{i=1}^k d_i$, $k \geqslant 1$. Note that d_k can equivalently be defined as the number of solutions (i,j) of the equation $ij = k$ (if solutions (i,j) and (j,i) are considered to be different for $i \neq j$).

Put $w_k = d_k/k$, $k \geqslant 1$; $W_0 = 0$ and $W_k = \sum_{i=1}^k w_i$ for $k \geqslant 1$. It is clear that

$$w_n = o(W_n), \qquad n \to \infty. \tag{2.1}$$

We extend the definition of W for real positive arguments:

$$W(x) = W_k + (x-k)(W_{k+1} - W_k), \qquad x \in [k, k+1), \quad k \geqslant 0. \tag{2.2}$$

We denote by $\{b_n, n \geqslant 1\}$ the normalizing sequence in the attraction of distributions of partial sums of independent identically distributed random variables to the limit stable law. These normalizations necessarily are of the form (4.9). Put

$$\psi(x) = \frac{x}{b(x)}. \tag{2.3}$$

If ψ is continuous and increasing, then its inverse ψ^{-1} exists. Let

$$U(t) = W\left(\psi^{-1}(x)\right) \tag{2.4}$$

in such a case.

3. Main result

Theorem 3.1. *Let X, $\{X_n, n \geqslant 1\}$ be independent identically distributed random variables with distribution function F. Let F be not concentrated at a single point. Assume that conditions* (1.4) *and* (1.5) *hold. Then*

$$\lim_{\varepsilon \downarrow 0} \frac{1}{(\ln(1/\varepsilon))^2} \sum_{k=1}^{\infty} \frac{d_k}{k} \mathsf{P}(|S_k| \geqslant \varepsilon k) = \left(\frac{\alpha}{\alpha - 1}\right)^2. \tag{3.1}$$

Proof. Let sequences $\{b_n, n \geqslant 1\}$ and $\{a_n, n \geqslant 1\}$ be such that the distributions of $S_n/b_n - a_n$ weakly converge to an α-stable limit law (such sequences exist in view of condition (1.5)). Then there is a stable law G_α such that the distributions of S_n/b_n weakly converge to G_α. Moreover if Z_α is a random variable with the distribution function G_α, then $\mathsf{E}Z_\alpha = 0$. This result can be found in several books (see translator's note on p. 175 in [8] or Theorem 3, §5, Chapter XVII in [6]). In any case, $b_n = n^{1/\alpha} h(n)$ and h is a slowly varying function (see (4.9)). Without loss of generality one can assume that h is continuous, since any slowly varying function has a continuous equivalent version (see Proposition at the end of Section 1.4 in [18]). The attraction to the limit laws holds for this version, too, in view of the Slutsky theorem (see, e.g., Theorem 1 in Chow and Teicher [3], p. 249).

Further, one can also assume that the normalizing sequence $\{b_n, n \geqslant 1\}$ is such that the function $x/b(x)$ is increasing. This follows from 4° of Section 1.4 in [18] where it is proved that any regularly varying function of a positive order has an increasing equivalent version.

Summarizing all we discussed above concerning the sequences $\{a_n, n \geqslant 1\}$ and $\{b_n, n \geqslant 1\}$, we assume without loss of generality that $b_n = n^{1/\alpha} h(n)$, h is a continuous slowly varying function, $x/b(x)$ is increasing, and S_n/b_n weakly converge to an α-stable random variable Z_α such that $\mathsf{E}Z_\alpha = 0$.

We split the further proof into five steps. Recall the definitions (2.2), (2.3), and (2.4) of the functions W, ψ, and U, respectively.

Step 1. Put

$$\Delta_n = \sup_x |\mathsf{P}(|S_n| \geqslant b_n x) - \mathsf{P}(|Z_\alpha| \geqslant x)|.$$

Then

$$\lim_{\varepsilon\downarrow 0} \frac{1}{U\,(1/\varepsilon)} \sum_{k\leqslant \psi^{-1}(1/\varepsilon)} w_k \Delta_k = 0. \tag{3.2}$$

Indeed, G_α is continuous (see [8], p. 183), and thus $\Delta_n \to 0$ as $n \to \infty$. Therefore

$$\lim_{m\to\infty} \frac{1}{W_m} \sum_{k=1}^{m} w_k \Delta_k = 0,$$

whence relation (3.2) follows, since both W and U are slowly varying functions (see Lemma 4.1).

Step 2. It holds

$$\lim_{\varepsilon\downarrow 0} \frac{1}{U\,(1/\varepsilon)} \sum_{k>\psi^{-1}(1/\varepsilon)} w_k \mathsf{P}\left(|Z_\alpha| \geqslant \varepsilon\psi_k\right) = 0. \tag{3.3}$$

Putting $t = \psi^{-1}(1/\varepsilon)$ we rewrite (3.3) in an equivalent form

$$\lim_{t\to\infty} \frac{1}{W(t)} \sum_{k>t} w_k \mathsf{P}\left(|Z_\alpha| \geqslant \frac{\psi_k}{\psi(t)}\right) = 0.$$

Using (4.6) we conclude that the latter relation follows from

$$\lim_{t\to\infty} \frac{\psi^\alpha(t)}{W(t)} \sum_{k>t} \frac{w_k}{\psi_k^\alpha} = 0.$$

Now we apply (4.3) and get

$$\sum_{k>t} \frac{w_k}{\psi_k^\alpha} \leqslant \mathrm{const} \frac{\ln(t)}{t^{\alpha-1}} h^\alpha(t),$$

whence the preceding relation follows (see Lemma (4.1)), thus relation (3.3), too.

Step 3. We show that

$$\lim_{\varepsilon\downarrow 0} \frac{1}{U\,(1/\varepsilon)} \sum_{k>\psi^{-1}(1/\varepsilon)} k w_k \mathsf{P}\left(|X| \geqslant k\varepsilon\right) = 0. \tag{3.4}$$

We put again $t = \psi^{-1}(1/\varepsilon)$ and rewrite (3.4) in an equivalent form

$$\lim_{t\to\infty} \frac{1}{W(t)} \sum_{k>t} k w_k \mathsf{P}\left(|X| \geqslant k/\psi(t)\right) = 0.$$

According to (4.10) this is equivalent to

$$\lim_{t\to\infty} \frac{\psi^\alpha(t)}{W(t)} \sum_{k>t} \frac{w_k}{k^{\alpha-1}} q(k/\psi(t)) = 0.$$

With $\theta = 1/\psi(t)$ we use (4.3) and get

$$\sum_{k>t} \frac{w_k}{k^{\alpha-1}} q(\theta k) \leqslant \mathrm{const} \frac{\ln(t)}{t^{\alpha-1}} q(\theta t),$$

whence

$$\frac{\psi^\alpha(t)}{W(t)} \sum_{k>t} \frac{w_k}{k^{\alpha-1}} q(k/\psi(t)) \leqslant \mathrm{const} \frac{\ln(t)}{W(t)} \cdot \frac{tq(b(t))}{b^\alpha(t)}.$$

The first factor is $o(1)$ as $t\to\infty$ (see Lemma 4.1), while the second one is bounded by (4.11). This proves (3.4).

Step 4. We will show that

$$\lim_{\varepsilon\downarrow 0} \frac{1}{U(1/\varepsilon)} \sum_{k>\psi^{-1}(1/\varepsilon)} w_k \mathsf{P}\left(|S_k| \geqslant \varepsilon k\right) = 0. \tag{3.5}$$

First we treat the case of $\alpha < 2$. It follows from (4.13) that $\mathsf{P}\left(|S_k| \geqslant \varepsilon k\right) \leqslant \mathrm{const} k \mathsf{P}(|X| \geqslant \varepsilon k)$ for all $\varepsilon > 0$ and $k \geqslant 1$. Thus (3.5) follows from (3.4).

Now let $\alpha = 2$. Fix $1 < \eta < 2$. Relation (3.5) is equivalent to

$$\lim_{t\to\infty} \frac{1}{W(t)} \sum_{k>t} w_k \mathsf{P}\left(|S_k| \geqslant k/\psi(t)\right) = 0.$$

We use the Markov inequality and (4.12):

$$\mathsf{P}\left(|S_k| \geqslant k/\psi(t)\right) \leqslant \psi^\eta(t) \frac{b_k^\eta}{k^\eta} \sup_{i\geqslant 1} \left[\frac{\mathsf{E}|S_i|^\eta}{b_i^\eta}\right] \leqslant \mathrm{const}\psi^\eta(t) \frac{b_k^\eta}{k^\eta}.$$

Now

$$\sum_{k>t} w_k \frac{b_k^\eta}{k^\eta} \leqslant \mathrm{const} \frac{\ln(t) h^\eta(t)}{t^{\eta(\alpha-1)/\alpha}}$$

by (4.3), whence

$$\frac{1}{W(t)} \sum_{k>t} w_k \mathsf{P}\left(|S_k| \geqslant k/\psi(t)\right) \leqslant \mathrm{const} \frac{\ln(t)}{W(t)} \cdot \frac{\psi^\eta(t) h^\eta(t)}{t^{\eta(\alpha-1)/\alpha}} \to 0 \qquad \text{as } t\to\infty.$$

By Lemma 4.1 this proves (3.5) for $\alpha = 2$.

Step 5. Using (3.2), (3.3), and (3.5) we prove that

$$\lim_{\varepsilon\downarrow 0} \frac{1}{U(1/\varepsilon)} \left| \sum_{k=1}^{\infty} w_k \mathsf{P}\left(|S_k| \geqslant \varepsilon k\right) - \sum_{k=1}^{\infty} w_k \mathsf{P}\left(|Z_\alpha| \geqslant \varepsilon \psi_k\right) \right| = 0.$$

Lemma 3.1 completes the proof of the theorem by Lemma 4.1. ■

Lemma 3.1. *The series*

$$\mathcal{Q}(\varepsilon) = \sum_{k=1}^{\infty} w_k \mathsf{P}(|Z_\alpha| \geqslant \varepsilon \psi_k) \tag{3.6}$$

converges for all $\varepsilon > 0$. *Moreover*

$$\lim_{\varepsilon\downarrow 0} \frac{\mathcal{Q}(\varepsilon)}{U(1/\varepsilon)} = 1. \tag{3.7}$$

Proof. To prove the convergence of series (3.6) let $1 < \eta < \alpha$. Then

$$\mathcal{Q}(\varepsilon) \leqslant \frac{\mathsf{E}|Z_\alpha|^\eta}{\varepsilon^\eta} \sum_{k=1}^{\infty} \frac{w_k}{\psi_k^\eta}$$

by (4.7) and the Chebyshev inequality, and thus $\mathcal{Q}(\varepsilon) < \infty$ in view of (4.1).

Now we pass to the proof of (3.7):

$$\sum_{k=1}^{\infty} w_k \mathsf{P}(|Z_\alpha| \geqslant \varepsilon \psi_k) = \sum_{k=1}^{\infty} w_k \sum_{j=k}^{\infty} \int_{I_j(\varepsilon)} f(t)\, dt,$$

where f is the density of $|Z_\alpha|$ and $I_j(\varepsilon) = [\varepsilon\psi_j, \varepsilon\psi_{j+1})$. Changing the order of summation we get

$$\begin{aligned}
\sum_{k=1}^{\infty} w_k \mathsf{P}(|Z_\alpha| \geqslant \varepsilon \psi_k) &= \sum_{j=1}^{\infty} W(j) \int_{I_j(\varepsilon)} f(t)\, dt \\
&= \sum_{j=1}^{\infty} \int_{I_j(\varepsilon)} W\left(\psi^{-1}(t/\varepsilon)\right) f(t)\, dt \\
&\quad + \sum_{j=1}^{\infty} \int_{I_j(\varepsilon)} \left[W(j) - W\left(\psi^{-1}(t/\varepsilon)\right)\right] f(t)\, dt \\
&\equiv \mathcal{Q}_1(\varepsilon) + \mathcal{Q}_2(\varepsilon).
\end{aligned}$$

We will prove that

$$\mathcal{Q}_1(\varepsilon) \sim U(1/\varepsilon) \qquad \text{as } \varepsilon \downarrow 0, \tag{3.8}$$

$$Q_2(\varepsilon) = o\left(U(1/\varepsilon)\right) \qquad \text{as } \varepsilon \downarrow 0. \tag{3.9}$$

First we prove (3.8). Without loss of generality let $h_1 = 1$. Then

$$\mathcal{Q}_1(\varepsilon) = \int_\varepsilon^\infty U\left(t/\varepsilon\right) f(t)\,dt.$$

Fix $A > 0$. For $0 < \varepsilon < A$, we have

$$\int_A^\infty U\left(t/\varepsilon\right) f(t)\,dt \leqslant \int_\varepsilon^\infty U\left(t/\varepsilon\right) f(t)\,dt$$

and thus

$$\liminf_{\varepsilon\downarrow 0} \frac{1}{U\left(1/\varepsilon\right)} \int_\varepsilon^\infty U\left(t/\varepsilon\right) f(t)\,dt \geqslant \lim_{\varepsilon\downarrow 0} \frac{1}{U\left(1/\varepsilon\right)} \int_A^\infty U\left(t/\varepsilon\right) f(t)\,dt = \int_A^\infty f(t)\,dt$$

in view of Theorem 4.1 (i) (with $x = 1/\varepsilon$) and condition 4.7. Since $A > 0$ is arbitrary,

$$\liminf_{\varepsilon\downarrow 0} \frac{1}{U\left(1/\varepsilon\right)} \int_\varepsilon^\infty U\left(t/\varepsilon\right) f(t)\,dt \geqslant \int_0^\infty f(t)\,dt = 1. \tag{3.10}$$

Next, fix $B > 0$ and write

$$\int_\varepsilon^\infty U\left(t/\varepsilon\right) f(t)\,dt \leqslant \int_0^B U\left(t/\varepsilon\right) f(t)\,dt + \int_B^\infty U\left(t/\varepsilon\right) f(t)\,dt,$$

whence

$$\begin{aligned}
&\limsup_{\varepsilon\downarrow 0} \frac{1}{U\left(1/\varepsilon\right)} \int_\varepsilon^\infty U\left(t/\varepsilon\right) f(t)\,dt \\
&\qquad\leqslant \lim_{\varepsilon\downarrow 0} \frac{1}{U\left(1/\varepsilon\right)} \int_0^B U\left(t/\varepsilon\right) f(t)\,dt + \lim_{\varepsilon\downarrow 0} \frac{1}{U\left(1/\varepsilon\right)} \int_B^\infty U\left(t/\varepsilon\right) f(t)\,dt \qquad (3.11)\\
&\qquad= \int_0^B f(t)\,dt + \int_B^\infty f(t)\,dt = 1,
\end{aligned}$$

according to Theorem 4.2 (i) together with condition 4.7 and Theorem 4.1 (i) together with condition (4.8). Therefore, (3.10) and (3.11) imply (3.8).

To prove (3.9), we estimate

$$|\mathcal{Q}_2(\varepsilon)| \leqslant \sum_{j=1}^\infty \left(W(j+1) - W(j)\right) \int_{I_j(\varepsilon)} f(t)\,dt = \sum_{j=1}^\infty w(j+1) \int_{I_j(\varepsilon)} f(t)\,dt,$$

since the function $W \circ \psi^{-1}$ is nondecreasing. Fix $\delta > 0$ and choose j_0 such that $w_{j+1} \leqslant \delta W_j$ for all $j \geqslant j_0$ (see (2.1)). Then

$$|\mathcal{Q}_2(\varepsilon)| \leqslant \sum_{j<j_0} w(j+1) \int_{I_j(\varepsilon)} f(t)\,dt + \delta \sum_{j\geqslant j_0} W_j \int_{I_j(\varepsilon)} f(t)\,dt. \tag{3.12}$$

For a fixed $j < j_0$ and $0 < \eta < 1$

$$\limsup_{\varepsilon\downarrow 0} \frac{1}{U(1/\varepsilon)} \int_{I_j(\varepsilon)} f(t)\,dt \leqslant \limsup_{\varepsilon\downarrow 0} \frac{\varepsilon^\eta \psi_{j+1}^\eta}{U(1/\varepsilon)} \int_{I_j(\varepsilon)} t^{-\eta} f(t)\,dt = 0 \tag{3.13}$$

in view of (4.8), Lemma 4.1, and

$$\int_{I_j(\varepsilon)} t^{-\eta} f(t)\,dt \leqslant \mathsf{E}|Z_\alpha|^{-\eta}.$$

Further

$$\sum_{j\geqslant j_0} W_j \int_{I_j(\varepsilon)} f(t)\,dt \leqslant \sum_{j\geqslant 1} \int_{I_j(\varepsilon)} U(t/\varepsilon) f(t)\,dt = \mathcal{Q}_1(\varepsilon).$$

Now we use (3.12), (3.13), and (3.8) and complete the proof of (3.9), since δ is arbitrary. ■

4. Auxiliary results

Dirichlet divisors function. Below are two properties of the Dirichlet divisors functions d and D:

$$d_k = o(k^\nu) \text{ for any } \nu > 0,\ k \to \infty, \tag{4.1}$$

$$D_k = k\ln(k) + (2\gamma - 1)k + O(k), \qquad k \to \infty, \tag{4.2}$$

where $\gamma = 0.577\ldots$ is an Euler constant. Properties (4.1) and (4.2) can be found in any textbook on number theory (see, for example, Theorem 7.3 in [16]).

Related to the Dirichlet divisors function D are the functions W and U defined by (2.2) and (2.4), respectively. The function U depends on the normalizing function b involved in the attraction to the stable law via ψ defined by (2.3). The function b necessarily is such that (4.9) holds.

Lemma 4.1. *If ψ is continuous and increasing, then both W and U are slowly varying functions. Moreover*

$$\lim_{x\to\infty} \frac{W(x)}{\ln^2(x)} = 1,$$

$$\lim_{x\to\infty} \frac{U(x)}{\ln^2(x)} = \left(\frac{\alpha}{\alpha - 1}\right)^2.$$

Proof. The assumption that ψ is continuous and increasing guarantees that ψ^{-1} exists and thus U is well defined. The first relation follows from (4.2) and Abel's partial summation formula:

$$\sum_{k=m}^{n} \Delta[A_k] B_k = A_n B_n - A_{m-1} B_{m-1} + \sum_{k=m}^{n-1} A_k \nabla[B_k]$$

where $\{A_n, n \geqslant 1\}$ and $\{B_n, n \geqslant 1\}$ are arbitrary sequences and $\Delta[A_k] = A_k - A_{k-1}$ and $\nabla[B_k] = B_k - B_{k+1}$. The second relation follows from the first one and (2.4), since ψ^{-1} is regularly varying of index $\alpha/(\alpha-1)$. ■

In the proof of the main result we need upper bounds for the series

$$\sum_{k>t} \frac{d_k}{k^\alpha L(k)}$$

for $\alpha > 1$ and slowly varying functions L. The estimates are easily obtained via Abel's partial summation formula for $A_k = D_k$ and $B_k = 1/k^\alpha L(k)$. Since $A_k \nabla[B_k] \leqslant \mathrm{const} k \ln(k) \nabla[B_k]$ by (4.2),

$$\sum_{k=m}^{n-1} A_k \nabla[B_k] \leqslant \mathrm{const} \sum_{k=m}^{n-1} k \ln(k) \nabla[B_k].$$

Applying the Abel partial summation formula once more, now with $A_k = k\ln(k)$ and the same B_k, we get

$$\sum_{k=m}^{n} \Delta[A_k] B_k \leqslant A_n B_n + \mathrm{const} \left[\sum_{k=m}^{n} \Delta[k\ln(k)] B_k + (m-1)\ln(m-1) B_{m-1} \right].$$

Passing to the limit as $n \to \infty$ we obtain

$$\sum_{k=m}^{\infty} \frac{d_k}{k^\alpha L(k)} \leqslant \mathrm{const} \frac{\ln(m)}{m^{\alpha-1} L(m)}, \tag{4.3}$$

since $\Delta[k\ln(k)] = O(\ln(k))$ and $\ln(x)/L(x)$ is a slowly varying function. Note that we used property (4.14) for $\alpha > 1$ and the slowly varying function $M(x) = \ln(x)/L(x)$ at the very last stage of the proof of (4.3).

Stable random variables and their distribution functions. A random variable Z_α is called stable of index α, $0 < \alpha \leqslant 2$ if its characteristic function h_α is such that

$$h_\alpha(t) = \exp\left\{iat - c|t|^\alpha \left(1 - i\beta \mathrm{sign}(t) \omega_\alpha(t)\right)\right\}, \tag{4.4}$$

where a is a real number, c and β are real numbers such that $c \geqslant 0$, $|\beta| \leqslant 1$, and

$$\mathrm{sign}(t) = \begin{cases} -1, & t < 0, \\ 0, & t = 0, \\ 1, & t > 0, \end{cases} \qquad \omega_\alpha(t) = \begin{cases} \mathrm{tg} \frac{\pi\alpha}{2}, & \alpha \neq 1, \\ -\frac{2}{\pi} \log|t|, & \alpha = 1. \end{cases}$$

By G_α and h_α we denote the distribution function and characteristic function of Z_α, respectively.

It is well known that

$$G_\alpha \text{ possesses the continuous density } g_\alpha \tag{4.5}$$

(see [8], p. 183) and there is a finite constant $C_\alpha \geqslant 0$ ($C_\alpha > 0$ if $\alpha \neq 2$) such that

$$\lim_{x\to\infty} x^\alpha \mathsf{P}(|Z_\alpha| \geqslant x) = C_\alpha \tag{4.6}$$

(see [8], p. 182). This implies, in particular, that

$$\mathsf{E}|Z_\alpha|^\eta < \infty \tag{4.7}$$

for any $0 < \eta < \alpha$. In addition,

$$\mathsf{E}|Z_\alpha|^{-\eta} < \infty \qquad \text{for all } 0 \leqslant \eta < 1, \tag{4.8}$$

so that (4.7) holds for $-1 < \eta < \alpha$. We prove (4.8) below.

Proof of (4.8). Note that (4.8) is obvious for $c = 0$ in (4.4), so that we treat the case of $c > 0$. It follows from (4.4) that $|h_\alpha(t)| = \exp\{-c|t|^\alpha\}$ and h_α is absolute integrable on $\mathbb{R}$. Now, the function

$$h_\alpha(t) \int_{-1}^{1} |x|^{-\eta} e^{-itx}\, dx$$

is absolute integrable on $\mathbb{R}$, since $0 \leqslant \eta < 1$. According to (4.5) Z_α possesses the density and moreover

$$g_\alpha(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} h_\alpha(t)\, dt.$$

Thus

$$\begin{aligned} 2\pi \int_{-1}^{1} |x|^{-\eta} g_\alpha(x)\, dx &= \int_{-1}^{1} |x|^{-\eta} \int_{-\infty}^{\infty} e^{-itx} h_\alpha(t)\, dt\, dx \\ &= \int_{-\infty}^{\infty} \left[h_\alpha(t) \int_{-1}^{1} |x|^{-\eta} e^{-itx}\, dx \right] dt < \infty. \end{aligned}$$

This completes the proof of (4.8), since $\mathsf{E}|Z_\alpha|^{-\eta} = \int_{-\infty}^{\infty} |x|^{-\eta} g_\alpha(x)\, dx$ and

$$\int_{|x|>1} |x|^{-\eta} g_\alpha(x)\, dx \leqslant \int_{|x|>1} g_\alpha(x)\, dx \leqslant 1. \qquad \blacksquare$$

Domains of attraction. A distribution function F is said to belong to the domain of attraction of a distribution function G if there are sequences $\{b_n, n \geqslant 1\}$ and $\{a_n, n \geqslant 1\}$ of real numbers such that $b_n > 0$ and the distributions of $S_n/b_n - a_n$ weakly converge to G where $\{S_n, n \geqslant 1\}$ is the sequence of partial sums constructed from a sequence of independent identically distributed random variables with the distribution function F. It is well known that G necessarily is stable and there exists $0<\alpha\leqslant 2$ such that

$$b_n = n^{1/\alpha}h(n) \text{ where } h \text{ is a slowly varying function.} \tag{4.9}$$

The function G corresponds to the characteristic function given by (4.4) and the number α there is the same as α in (4.9). If a distribution function F is attracted to a stable law, then

$$\mathsf{P}(|X|\geqslant x)=\frac{q(x)}{x^\alpha} \qquad \text{with a slowly varying function } q \tag{4.10}$$

where X is a random variable distributed by the law F (Theorem 2.6.1 in [13]). The normalizing sequence $\{b_n, n\geqslant 1\}$ and the function $q(x)$ in (4.10) are related each to other as follows

$$\lim_{n\to\infty} n\mathsf{P}(|X|\geqslant b_n)=\lim_{n\to\infty}\frac{nq\left(b_n\right)}{b_n^\alpha}=P_\alpha \tag{4.11}$$

where $0\leqslant P_\alpha<\infty$.

One of the useful facts about S_k follows from Lemma 5.2.2 in [13], namely

$$\sup_{n\geqslant 1}\mathsf{E}\left[\frac{|S_n|}{b_n}\right]^\eta<\infty \tag{4.12}$$

if $0<\eta<\alpha$.

Finally, we mention another bound for large deviation probabilities:

$$\sup_{x\geqslant 0}\sup_{n\geqslant 1}\frac{\mathsf{P}(|S_n|\geqslant xb_n)}{n\mathsf{P}(|X|\geqslant xb_n)}<\infty. \tag{4.13}$$

This bound is proved in [2] where a result of [11] is generalized.

Slowly varying functions. A positive function ℓ defined for $x\geqslant 0$ is called slowly varying (at infinity) in the Karamata sense if

$$\lim_{t\to\infty}\frac{\ell(ct)}{\ell(t)}=1 \qquad \text{for all } c>0.$$

Any function $f(t)=x^\nu\ell(t)$ is called regularly varying of order ν. If ℓ is a slowly varying function and $\alpha>1$, then

$$\sum_{k=m}^\infty\frac{\ell(k)}{k^\alpha}\leqslant \text{const}\frac{\ell(m)}{m^{\alpha-1}}. \tag{4.14}$$

For the sake of completeness we prove this result, despite it is well known in the theory of regular variation. Fix $\nu>0$ such that $\alpha-\nu>1$. Then according to 4° of Section 1.4 in [18] introduce an increasing equivalent version of $x^\nu/\ell(x)$ (let it be $x^\nu/\ell_1(x)$). Thus

$$\sum_{k=m}^\infty\frac{\ell(k)}{k^\alpha}\leqslant\frac{\ell_1(m)}{m^\nu}\sum_{k=m}^\infty\frac{1}{k^{\alpha-\nu}}$$

and result follows, since $\ell_1\sim\ell$.

Our proof of Theorem 3.1 makes use of the following two results due to Aljanĉić, Bojanić and Tomić [1] (also see [18]).

Theorem 4.1. *Assume that f is a real function and U is a slowly varying function. Let the Lebesgue integral*

$$\int_A^\infty t^\eta f(t)\,dt \tag{4.15}$$

be well-defined for some $\eta \geqslant 0$ and $A > 0$. Then,

$$\int_A^\infty U(xt)f(t)\,dt$$

is well-defined if either
(i) $\eta > 0$, *or*
(ii) $\eta = 0$ *and $U(t)$ is eventually non-increasing on $(0,\infty)$. In either case,*

$$\int_A^\infty U(xt)f(t)\,dt \sim U(x)\int_A^\infty f(t)\,dt \qquad \text{as } x \to \infty\,.$$

Theorem 4.2. *Assume that f is a real function and U is a slowly varying function. Let the Lebesgue integral*

$$\int_0^B t^\eta f(t)\,dt \tag{4.16}$$

be well-defined for some $\eta \leqslant 0$ and $B > 0$. Then,

$$\int_0^B U(xt)f(t)\,dt \sim U(x)\int_0^B f(t)\,dt \qquad \text{as } x \to \infty$$

if either
(i) $\eta < 0$, *or*
(ii) $\eta = 0$ *and $U(t)$ is eventually non-decreasing on $(0,\infty)$.*

5. Concluding remark

Let $R(x) = R_1(x)$ be the renewal function defined by

$$R_1(x) = \sum_{n=1}^\infty \mathsf{P}(S_n < x)$$

constructed from a random walk $\{S_n\}$ and $R(x) = R_2(x)$ be defined by (1.8) in the two-dimansional case. Let $V_1(x)$ be the precise asymptotics of $R_1(x)$ and let $V_2(x)$ be the precise asymptotics of $R_2(x)$ (that is, $R_1(x) \sim V_1(x)$ and $R_2(x) \sim V_2(x)$ as $x \to \infty$). The exact form of functions V_1 and V_2 is known but it does not matter for the below discussion. What is important is that $V_2(x) = V_1(x)\ln(x)$ (see [15]).

This result requires only the "rought" asymptotics of D like $D(x) \sim x\ln(x)$. However one can also prove that

$$R_2(x) - V_1(x)\mathcal{P}(\ln(x)) = o(x^r) \tag{5.1}$$

where $\mathcal{P}(x) = x + (2\gamma - 1)$ and $0 < r < 1$ is a number such that $D(x) = x\mathcal{P}(\ln(x)) + o(x^r)$ (see [15]). Since $V_1(x) = cx$ for some $c > 0$, relation (5.1) tells us somewhat more than just the asymptotics $R_2(x) \sim V_2(x)$. Of course, there is a price we pay for relation (5.1) (the price can be stated in terms of the existence of moments of random variables; again the form of the assumptions do not matter for this discussion). Relation (5.1) implies that the exact approximation in the renewal theorem for random walks with multidimensional time depends on the solution of the Dirichlet divisors problem which, in turn, depends on the solution of the Riemann conjecture on zeros of the ζ function.

Now we come back to the function studied in this paper. The proof of Theorem 3.1 also uses the fact that the Dirichlet divisors function $D(x)$ is equivalent to $x\ln(x)$ as $x \to \infty$ and does not use any further information on the remainder term in this approximation. Denoting by $U_1(\varepsilon)$ the normalizing function for the one-dimensional case, we see from (1.6) that $U_1(\varepsilon) = \ln(1/\varepsilon)$, while $U_2(\varepsilon)$ denoting the normlizing function for series (1.7) is $\ln^2(1/\varepsilon)$ (see Theorem 3.1), that is $U_2(\varepsilon) = U_1(\varepsilon)\ln(1/\varepsilon)$. Having in mind an analogy between the series

$$\sum_{n=1}^{\infty} \mathsf{P}(|S_n| \geqslant n\varepsilon)$$

and the renewal function (1.8) and the above discussion of relation (5.1), a natural question arises on whether or not an additional information (like expansion (4.2)) may help to sharpen the result of Theorem 3.1 in a direction like (5.1)?

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