SUT Journal of Mathematics
Vol. 35, No. 2 (1999), 247–262

EXACT ASYMPTOTIC BEHAVIOUR IN A RENEWAL THEOREM FOR CONVOLUTION EQUIVALENT DISTRIBUTIONS WITH EXPONENTIAL TAILS*

Mikhail S. Sgibnev

(Received April 18, 1999)

Abstract. We study the exact asymptotic behaviour of some special generalized renewal measures on the whole line $\mathbf{R}$ and, in particular, that of the ordinary renewal measure on $\mathbf{R}$. When the underlying distribution F is concentrated on $[0, \infty)$, these measures are closely related to the higher renewal moments $EN(t)^n$, where $N(t)$ is the number of renewals up to time t. The tail of F is assumed to possess the tail behaviour of a distribution from the class $\mathcal{S}(\gamma)$ for an arbitrary $\gamma > 0$.

AMS 1991 *Mathematics Subject Classification.* Primary 60K05; Secondary 60J15.

Key words and phrases. Renewal theorem, higher renewal moments, generalized renewal measures, tail behaviour, subexponential and related distributions, Banach algebras.

§1. Introduction

Let X_i, $i = 1, 2, \ldots$, be independent, identically distributed random variables with a common non-arithmetic distribution F and finite positive mean μ. Write $S_0 = 0$ and $S_k = \sum_{i=1}^{k} X_i$, $k = 1, 2, \ldots$.

In this paper we shall be concerned with the exact asymptotic behaviour of the generalized renewal measures of the following type:

$$(1.1) \qquad \Phi_n(A) = \sum_{k=0}^{\infty} \frac{n \cdot (n+k-1)!}{k!} F^{k*}(A), \qquad A \in \mathcal{B},$$

*This research was supported by Grant 96-01-01939 of the Russian Foundation for Fundamental Research.

where $n \geq 1$ is integer, F^{k*} is the k-fold convolution of F, $F^{1*} \stackrel{\text{def}}{=} F$, F^{0*} is the Dirac measure δ, i.e. the atomic measure of unit mass concentrated at the origin, and $\mathcal{B}$ is the σ-algebra of all Borel subsets of $\mathbf{R}$. The ordinary renewal measure $H = \sum_{k=0}^{\infty} F^{k*}$ is a particular case of (1.1): $H = \Phi_1$. The question arises whether the series in (1.1) will converge for, at least, bounded A, i.e. whether the measure Φ_n will be σ-finite. In this respect, the following assertion is true [22, Proposition]: the measure Φ_n is σ-finite if and only if $E(X_1^-)^n < \infty$; here $x^- = \max(0, -x)$. Moreover, the function $\Phi_n(t) \stackrel{\text{def}}{=} \Phi_n((-\infty, t\,])$ is finite if $E(X_1^-)^{n+1} < \infty$; see [27, Corollary 5.2].

In the class of all generalized renewal measures $\sum_{k=0}^{\infty} a_k F^{k*}$, the measures Φ_n are distinguished by their close relationship with the higher renewal moments $EN(t)^n$, $n \geq 1$, in the following sense. Suppose temporarily that the X_i are non-negative. Denote by $N(t) = \sup\{k \geq 1 : S_k \leq t\}$ the number of renewals up to and including time t. Obviously, $H(t) = EN(t) + 1$. Instead of $EN(t)^n$, it is convenient, from the technical point of view, to study the *renewal Φ-moments* defined by Smith [26] as

$$\Phi_n(t) = E\{[N(t)+1][N(t)+2]\cdots[N(t)+n]\}, \qquad n = 1, 2, \ldots\,,$$

since the Laplace-Stieltjes transform of $\Phi_n(t)$ is then given by the following simple expression:

$$\widehat{\Phi}_n(s) \stackrel{\text{def}}{=} \int_0^{\infty} \exp(sx)\, d\Phi_n(x) = n!\,[1 - E\exp(sX_1)]^{-n}, \quad \Re s < 0.$$

It is easy to verify that the measure Φ_n generated in the usual way by the non-decreasing function $\Phi_n(t)$ admits precisely the representation (1.1).

Suppose $E|X_1|^n < \infty$. Define coefficients $\gamma_k^{(n)}$, $k = 1, \ldots, n$, by the following asymptotic relation:

$$\frac{n!}{[1 - \widehat{F}(s)]^n} = \sum_{k=1}^{n} (-1)^k \frac{\gamma_k^{(n)}}{s^k} + o\left(\frac{1}{s}\right) \qquad \text{as } s \to 0. \tag{1.2}$$

If $E|X_1|^{n+1}$ is also finite, we slightly change this definition by replacing the right-hand side of (1.2) with $\sum_{k=0}^{n}(-1)^k \gamma_k^{(n)}/s^k + o(1)$ in order to define $\gamma_0^{(n)}$ as well. It is clear that the coefficient $\gamma_k^{(n)}$ depends on the first $n-k+1$ moments of F.

We shall be concerned with the exact asymptotic behaviour of the remainder term $\mathcal{R}_n$ in the representation

$$\Phi_n(A) = \sum_{k=1}^{n} \gamma_k^{(n)} \mathcal{L}^{k*}(A) + \mathcal{R}_n(A), \qquad A \in \mathcal{B}, \tag{1.3}$$

where the $\gamma_k^{(n)}$ are defined by (1.2) and $\mathcal{L}$ is the restriction of Lebesgue measure to the non-negative half-axis $[0,\infty)$. More precisely, we will study the exact tail behaviour of $\mathcal{R}_n$, i.e. the behaviour of $\mathcal{R}_n((t,\infty))$ as $t\to\infty$. This will allow us, by putting in (1.3) alternatively $A=(-\infty,t]$ or $A=[0,t]$, to obtain asymptotic expansions for the specific quantities of interest, i.e. $\Phi_n(t)$ or $\Phi_n([0,t])$. Setting $n=1$ will then yield a refinement of the following well-known renewal theorem on the whole line $\mathbf{R}$: $H(t)-t/\mu\to\mu_2/(2\mu^2)$ as $t\to\infty$, where $H(t)\overset{\text{def}}{=}\Phi_1((-\infty,t])$ is the renewal function and $\mu_2=EX_1^2$ (see Theorem 5 below). Finally, the choice $A=(t,t+h]$ for $h>0$ will lead us to an expansion for $\Phi_n((t,t+h])$ as $t\to\infty$, which in the case $n=1$ will turn out to be a refinement of Blackwell's renewal theorem.

The tails of distributions from the classes $\mathcal{S}(\gamma)$, $\gamma\geq 0$, (see Definition 1 in Section 2) proved to be ideal comparison functions for determining exact asymptotic behaviour of various quantities of interest not only in renewal theory, but also in branching processes, random walks and infinite divisibility (more on that in Section 2). So our main assumption on F will be $F((t,\infty))\sim G((t,\infty))$ as $t\to\infty$ for some $G\in\mathcal{S}(\gamma)$ with $\gamma>0$ (the notation $a(t)\sim b(t)$ as $t\to\infty$ means that $\lim_{t\to\infty}a(t)/b(t)=1$).

The asymptotic behaviour of the generalized renewal functions $\Phi_n(t)$ was studied in detail by Smith [27] under moment assumptions on the underlying distribution F. In the one-sided case, i.e. when $F([0,\infty))=1$, the exact asymptotic behaviour of the renewal measure $H=\Phi_1$ was studied in detail in Frenk [14]. As far as the *subexponential* behaviour of Φ_n is concerned, i.e. when the "comparison" distribution $G\in\mathcal{S}(\gamma)$ with $\gamma=0$, the reader is referred to Sgibnev [22].

§2. Preliminaries

Definition 1 The distribution G of a non-negative random variable is said to *belong to the class* $\mathcal{S}(\gamma)$, $\gamma\geq 0$, if the following conditions are satisfied:

$$\lim_{x\to\infty}[1-G(x+y)]/[1-G(x)]=e^{-\gamma y}\qquad\forall y\in\mathbf{R};$$
$$\widehat{G}(\gamma)\overset{\text{def}}{=}\int_0^\infty e^{\gamma x}\,G(dx)<\infty;$$
$$\lim_{x\to\infty}[1-G^{2*}(x)]/[1-G(x)]=2\widehat{G}(\gamma).$$

The class $\mathcal{S}=\mathcal{S}(0)$ (later called the class of *subexponential* distributions) was introduced by Chistyakov [4], while the classes $\mathcal{S}(\gamma)$ for positive γ were first considered by Chover, Ney, and Wainger [5, 6]. The importance of such distributions has widely been illustrated by the fact that in many cases the exact asymptotic behaviour of probabilistic quantities of interest can be expressed in terms of the distributions of $\mathcal{S}(\gamma)$. There is a rather extensive literature concerning both the properties of $\mathcal{S}(\gamma)$-distributions themselves and their use in various areas of probability theory (branching processes, queueing theory, infinite divisibility, etc.); see, e.g. Athreya and Ney [1], Teugels [28], Veraverbeke [29], Embrechts, Goldie and Veraverbeke [11], Embrechts and Goldie [9, 10], Pitman [17], Embrechts and Veraverbeke [12], Cline [7, 8], Sgibnev [21, 22, 24], Klüppelberg [16], Bertoin and Doney [2].

We shall need some knowledge about Banach algebras of measures. A function $\varphi(x)$, $x \in \mathbf{R}$, is called *submultiplicative* if $\varphi(x)$ is finite, positive, Borel measurable and $\varphi(0) = 1$, $\varphi(x+y) \le \varphi(x)\,\varphi(y)$ $\forall x, y \in \mathbf{R}$. The limits $r_-(\varphi) \overset{\text{def}}{=} \lim_{x\to-\infty} \log\varphi(x)/x > -\infty$ and $r_+(\varphi) \overset{\text{def}}{=} \lim_{x\to\infty} \log\varphi(x)/x < \infty$ exist and $r_-(\varphi) \le r_+(\varphi)$ [15, Section 7.6].

The collection $S(\varphi)$ of all complex-valued measures κ such that

$$\|\kappa\|_\varphi \overset{\text{def}}{=} \int_{\mathbf{R}} \varphi(x)\,|\kappa|(dx) < \infty$$

(here $|\kappa|$ stands for the total variation of κ) is a Banach algebra with norm $\|\kappa\|_\varphi$ and the usual operations of addition and scalar multiplication of measures; the product of two elements $\nu, \kappa \in S(\varphi)$ is defined as their convolution $\nu * \kappa$. The unit element of $S(\varphi)$ is the atomic measure δ of unit mass at the origin [15, Section 4.16].

The following theorem of [19] describes the structure of homomorphisms of $S(\varphi)$ onto $\mathbf{C}$.

Theorem 1. *Let $m : S(\varphi) \to \mathbf{C}$ be an arbitrary homomorphism. Then the following representation holds*:

$$m(\nu) = \int \chi(x,\nu)\exp(\alpha x)\,\nu(dx), \qquad \nu \in S(\varphi),$$

where α is a real number such that $r_-(\varphi) \le \alpha \le r_+(\varphi)$ and the function $\chi(x,\nu)$ of the two variables $x \in \mathbf{R}$ and $\nu \in S(\varphi)$ is a generalized character.

Here we mention only one property of a generalized character which will be used later: $\nu - \operatorname{ess\,sup}_{x\in\mathbf{R}} |\chi(x,\nu)| \le 1$.

In the present paper we shall use the following system of submultiplicative functions: $\varphi_k(x) \overset{\text{def}}{=} (1+|x|)^k$, $x < 0$, and $\varphi_k(x) \overset{\text{def}}{=} \exp(\gamma x)$, $x \ge 0$, where $\gamma > 0$ and $k = 0, 1, \dots$. The Laplace transform of an element κ of $S(\varphi_0)$ is defined as follows: $\widehat{\kappa}(s) \overset{\text{def}}{=} \int_{\mathbf{R}} \exp(sx)\,\kappa(dx)$; this integral converges absolutely with respect to $|\kappa|$ for all s in the strip

$$\Pi(\gamma) \overset{\text{def}}{=} \{s \in \mathbf{C} : 0 \le \Re s \le \gamma\}.$$

Now choose an arbitrary distribution $G \in \mathcal{S}(\gamma)$. Put $\tau(x) = 1 - G(x)$. Define

$$Q(\nu) = \sup_{x \geq 0} |\nu|((x, \infty))/\tau(x), \qquad \nu \in S(\varphi_0).$$

Consider the collection $\mathfrak{S}l(\tau)$ of all measures $\nu \in S(\varphi_0)$ such that $Q(\nu) < \infty$ and there exists the limit

$$l(\nu) \stackrel{\text{def}}{=} \lim_{x \to \infty} \nu((x, \infty))/\tau(x) \in \mathbf{C}.$$

As shown in [21, Proposition 2], $\mathfrak{S}l(\tau)$ is a Banach algebra with respect to some norm $\|\nu\|'$ equivalent to the norm $\|\nu\|_{\varphi_0} + Q(\nu)$. Moreover, for any two elements ν, $\kappa \in \mathfrak{S}l(\tau)$, the following equality holds:

$$l(\nu * \kappa) = l(\nu)\widehat{\kappa}(\gamma) + l(\kappa)\widehat{\nu}(\gamma). \tag{2.1}$$

Let $\mathcal{A}$ be an arbitrary commutative complex Banach algebra with unit element e. The *spectrum* $\sigma(a)$ of an element $a \in \mathcal{A}$ is defined to be the set of all complex numbers λ such that the element $a - \lambda e$ does not have an inverse. If $f(z)$ is an analytic function in a domain containing the spectrum of an element $a \in \mathcal{A}$, then there exists an element $f(a) \in \mathcal{A}$ such that for each homomorphism $m : \mathcal{A} \to \mathbf{C}$ the following relation holds: $m(f(a)) = f(m(a))$ [30, Section 3]. The element $f(a) \in \mathcal{A}$ is called the *value of the analytic function $f(z)$ at the element $a \in \mathcal{A}$*.

We shall need the following result on the values of an analytic function at elements of $\mathfrak{S}l(\tau)$ [21, Theorem 3].

Theorem 2. *Let $f(z)$ be an analytic function in a domain containing the spectrum $\sigma(\nu)$ of an element $\nu \in S(\varphi_0)$, and let $f(\nu)$ be the value of $f(z)$ at $\nu \in S(\varphi_0)$. If $\nu \in \mathfrak{S}l(\tau)$, then $f(\nu) \in \mathfrak{S}l(\tau)$ and the following equality holds*:

$$l[f(\nu)] = f'[\widehat{\nu}(\gamma)] \cdot l(\nu).$$

Let ν be a finite measure. Define a σ-finite measure, say $T\nu$, by the formula

$$T\nu(A) = \int_A n(x)\, dx, \qquad A \in \mathcal{B},$$

where $n(x) = -\nu((-\infty, x])$ for $x < 0$ and $n(x) = \nu((x, \infty))$ for $x \geq 0$. Notice the following properties of the operator T. First, if $\int_{\mathbf{R}} |x|^k |\nu|(dx) < \infty$ for some positive integer k, then $\int_{\mathbf{R}} |x|^{k-1} |T\nu|(dx) < \infty$, so that the kth iteration $T^k\nu$ is a finite measure. Hence $T\nu \in S(\varphi_{k-1})$, provided $\nu \in S(\varphi_k)$ with $k > 0$. Second, the Laplace transform $(T\nu)\widehat{\ }(s)$ is given by $(T\nu)\widehat{\ }(s) = [\widehat{\nu}(s) - \widehat{\nu}(0)]/s$, $\Re s = 0$, provided $\int_{\mathbf{R}} |x||\nu|(dx) < \infty$. The value $(T\nu)\widehat{\ }(0)$ is defined by continuity as $\int_{\mathbf{R}} x\, \nu(dx)$.

The proof of the following lemma can be found in [24].

Lemma 1. *Let $G \in \mathcal{S}(\gamma)$ with $\gamma > 0$ and let $\tau(x) = 1 - G(x)$. If $\nu \in \mathfrak{S}l(\tau)$ and $\int_{-\infty}^{0} |x|\,|\nu|(dx) < \infty$, then $T\nu \in \mathfrak{S}l(\tau)$ and $l(T\nu) = l(\nu)/\gamma$.*

Theorem 3. *Let X_1, X_2, ... be independent, identically distributed random variables with a common non-arithmetic distribution F, $\mu = EX_1 > 0$ and $E|X_1|^n < \infty$. Then, for each $h > 0$, the following relations hold:*

$$\Phi_n((x, x+h]) \sim nhx^{n-1}/\mu^n \qquad \text{as } x \to \infty,$$

$$\Phi_n((x, x+h]) \to 0 \qquad \text{as } x \to -\infty.$$

The absolutely continuous part of an arbitrary distribution F will be denoted by F_c and its singular component, by F_s: $F_s = F - F_c$. In particular, $(F^{m*})\widehat{_s}(\gamma)$ will stand for the Laplace transform at point γ of the singular component $(F^{m*})_s$ of the m-fold convolution F^{m*}.

§3. Renewal theorem

In what follows we assume that in the Banach algebra $\mathfrak{S}l(\tau)$ the comparison function $\tau(x)$ is the tail of a distribution $G \in \mathcal{S}(\gamma)$ with $\gamma > 0$.

Theorem 4. *Let X_1, X_2, ... be independent, identically distributed random variables with a common distribution $F \in \mathfrak{S}l(\tau)$, $\mu = EX_1 > 0$ and $E|X_1|^n < \infty$. Suppose that $(F^{m*})\widehat{_s}(\gamma) < 1$ for some integer $m \geq 1$, and that in the strip $\Pi(\gamma)$ there are no roots of the equation $1 - \widehat{F}(s) = 0$, distinct from zero. Then the representation (1.3) holds, where the restriction of $\mathcal{R}_n$ to $[0, \infty)$ belongs to $\mathfrak{S}l(\tau)$ and*

$$\lim_{t\to\infty} \mathcal{R}_n((t, \infty))/\tau(t) = n \cdot n!\, l(F)/[1 - \widehat{F}(\gamma)]^{n+1}. \tag{3.1}$$

Proof. By the hypotheses of the theorem, $F \in S(\varphi_n)$. Choose $\varepsilon > \gamma$. Consider the function

$$v(s) \stackrel{\text{def}}{=} (s - \varepsilon)[1 - \widehat{F}(s)]/s, \qquad s \in \Pi(\gamma) \setminus \{0\}.$$

Define $v(0) = \varepsilon\mu$. By the properties of the operator T, we have $TF \in S(\varphi_{n-1})$, and hence $v(s)$ is the Laplace transform of the measure $V \stackrel{\text{def}}{=} \varepsilon TF + \delta - F \in S(\varphi_{n-1})$.

Lemma 2. *Let $F \in S(\varphi_{k+1})$, $k \geq 0$. Suppose that $(F^{m*})\widehat{_s}(\gamma) < 1$ for some integer $m \geq 1$. Then $V \in S(\varphi_k)$ and there exists $V^{-1} \in S(\varphi_k)$. Moreover, if $F \in \mathfrak{S}l(\tau)$, then $V^{-1} \in \mathfrak{S}l(\tau)$ and*

$$l(V^{-1}) = \frac{\gamma l(F)}{(\gamma - \varepsilon)[1 - \widehat{F}(\gamma)]^2}. \tag{3.2}$$

Proof of Lemma 2. As shown above, $V \in S(\varphi_k)$. Let $\mathcal{M}$ be the space of maximal ideals of the Banach algebra $S(\varphi_k)$. The following facts are well known from the theory of Banach algebras. Each maximal ideal $M \in \mathcal{M}$ induces a homomorphism of the Banach algebra $S(\varphi_k)$ onto the field of complex numbers $\mathbf{C}$; moreover, M is the kernel of this homomorphism. Denote by $\nu(M)$ the value of this homomorphism at $\nu \in S(\varphi_k)$. An element $\nu \in S(\varphi_k)$ has an inverse if and only if ν does not belong to any maximal ideal $M \in \mathcal{M}$. In other words, ν is invertible if and only if $\nu(M) \neq 0$ for all $M \in \mathcal{M}$.

The space $\mathcal{M}$ is split into two sets: $\mathcal{M}_1$ is the set of those maximal ideals which do not contain the collection $L(\varphi_k)$ of all absolutely continuous measures from $S(\varphi_k)$, and $\mathcal{M}_2 = \mathcal{M} \setminus \mathcal{M}_1$. If $M \in \mathcal{M}_1$, then the homomorphism $S(\varphi_k) \to \mathbf{C}$ induced by M is of the form $\nu \to \widehat{\nu}(s_0)$, $\nu \in S(\varphi_k)$, where s_0 is some complex number such that $0 \leq \Re s_0 \leq \gamma$. In this case, $M = \{\mu \in S(\varphi_k) : \widehat{\mu}(s_0) = 0\}$ [15, Chapter IV, Section 4]. If $M \in \mathcal{M}_2$, then $\nu(M) = 0$ for each absolutely continuous measure $\nu \in S(\varphi_k)$.

We now show that $V(M) \neq 0$ for each $M \in \mathcal{M}$; hence we will establish the existence of an inverse element $V^{-1} \in S(\varphi_k)$. Actually, if $M \in \mathcal{M}_1$, then, for some $s_0 \in \{0 \leq \Re s \leq \gamma\}$, we have $V(M) = \widehat{V}(s_0) \neq 0$. Let $M \in \mathcal{M}_2$. First of all, we note that the condition $(F^{m*})\widehat{_s}(\gamma) < 1$ implies the inequality $(F^{m*})\widehat{_s}(\alpha) < 1$ for all $\alpha \in [0,\gamma]$. In fact, the function $(F^{m*})\widehat{_s}(\alpha)$, $\alpha \in [0,\gamma]$, is convex and $(F^{m*})\widehat{_s}(0) < 1$ since the inequalities $(F^{m*})\widehat{_s}(\gamma) < 1$ and $(F^{m*})\widehat{\ }(\gamma) > 1$ imply that $(F^{m*})_c(\mathbf{R}) > 0$. Applying Theorem 1, we have that, for some $\alpha \in [0,\gamma]$,

$$\begin{aligned} |F(M)|^m &= |F^{m*}(M)| = |(F^{m*})_s(M)| \\ &= \left| \int \chi(x, (F^{m*})_s) \exp(\alpha x)\, (F^{m*})_s(dx) \right| \\ &\leq \int \exp(\alpha x)\, (F^{m*})_s(dx) < 1. \end{aligned}$$

Since $TF \in L(\varphi_k)$, we obtain that $|V(M)| = |1 - F(M)| > 0$. This means that there exists an inverse element $V^{-1} \in S(\varphi_k)$ and that the function $1/v(s)$, $s \in \Pi(\gamma)$, is the Laplace transform of V^{-1}.

Applying Lemma 1, we see that $TF \in \mathfrak{S}l(\tau)$ with $l(TF) = l(F)/\gamma$, and hence $V \in \mathfrak{S}l(\tau)$ with $l(V) = l(F)(\varepsilon - \gamma)/\gamma$. Applying Theorem 2 with $f(z) = 1/z$, we obtain $l(V^{-1}) = -l(V)/[\widehat{V}(\gamma)]^2$, whence (3.2) follows. This completes the proof of Lemma 2.

We return to the proof of Theorem 4. Denote $W = (V^{-1})^{n*}$. We have

$$\frac{n!}{[1 - \widehat{F}(s)]^n} = n! \left(\frac{s-\varepsilon}{s}\right)^n \widehat{W}(s) = n! \sum_{k=0}^{n} (-\varepsilon)^k \binom{n}{k} \frac{\widehat{W}(s)}{s^k}. \tag{3.3}$$

Further,

$$\frac{\widehat{W}(s)}{s^k} = \frac{w_1(s)}{s^{k-1}} + \frac{w_0(0)}{s^k} = \sum_{j=1}^{k} \frac{w_{k-l}(0)}{s^l} + w_k(s), \tag{3.4}$$

where $w_j(s) \stackrel{\text{def}}{=} [w_{j-1}(s) - w_{j-1}(0)]/s$, $j = 1, 2, \dots, k$, and $w_0(s) \stackrel{\text{def}}{=} \widehat{W}(s)$. Substituting (3.4) into (3.3) and collecting similar terms, we obtain by the uniqueness of the expansion (1.2) that

$$\frac{n!}{[1-\widehat{F}(s)]^n} = \sum_{k=1}^{n} (-1)^k \frac{\gamma_k^{(n)}}{s^k} + n! \sum_{k=0}^{n} (-\varepsilon)^k \binom{n}{k} w_k(s). \tag{3.5}$$

Denote

$$\mathcal{R}_n = n! \sum_{k=0}^{n} (-\varepsilon)^k \binom{n}{k} T^k W.$$

By Theorem 1 with $f(z) = z^n$, we have $W \in S(\varphi_{n-1})$ and $W \in \mathfrak{S}l(\tau)$ with

$$l(W) = \frac{n\, l(V^{-1})}{[\widehat{V}(\gamma)]^{n-1}} = \frac{n\gamma^n\, l(F)}{(\gamma-\varepsilon)^n [1-\widehat{F}(\gamma)]^{n+1}}.$$

Note that $w_k(s)$ is the Laplace transform of the measure $T^k W$. By the properties of the operator T, $T^k W \in S(\varphi_{n-k-1})$ and, by Lemma 1, $T^k W \in \mathfrak{S}l(\tau)$ with $l(T^k W) = l(W)/\gamma^k$, $k = 1, \dots, n-1$. We cannot, however, assert that $T^n W \in \mathfrak{S}l(\tau)$ since $|T^n W|((-\infty, 0))$ may be infinite. Nevertheless, by Lemma 1, the restriction $T^n W|_{[0,\infty)}$ of $T^n W$ to $[0, \infty)$ is an element of $\mathfrak{S}l(\tau)$ and $l(T^n W|_{[0,\infty)}) = l(W)/\gamma^n$. Therefore, the restriction $\mathcal{R}_n|_{[0,\infty)}$ belongs to $\mathfrak{S}l(\tau)$ and

$$l(\mathcal{R}_n|_{[0,\infty)}) = n! \sum_{k=0}^{n} (-\varepsilon)^k \binom{n}{k} l(T^k W) = \frac{n \cdot n!\, l(F)}{[1-\widehat{F}(\gamma)]^{n+1}}.$$

This proves the theorem in the case $F([0,\infty)) = 1$ since relation (3.5) for $\Re s < 0$ is precisely the Laplace-transform version of (1.3).

In the general case, i.e. when $F([0,\infty)) < 1$, an additional argument is needed to justify the transition from (3.5) to (1.3). This argument is based on the theory of tempered distributions [20, Chapter 7]. Denote by $\mathcal{S}_1$ the space of rapidly decreasing functions defined on $\mathbf{R}$, and by $\mathcal{S}_1'$ its dual space. The elements of $\mathcal{S}_1'$ are called *tempered distributions*. All measures in (1.3) may be regarded as tempered distributions since they are slowly increasing measures [20, Section 7.12]. Recall that a σ-finite measure ν is *slowly increasing* if, for some $k \geq 0$, $\int_{\mathbf{R}} (1+x^2)^{-k} |\nu|(dx) < \infty$ [20, Section 7.12]. For example, the formula $\Phi_n(\psi) \stackrel{\text{def}}{=} \int_{\mathbf{R}} \psi(x)\, \Phi_n(dx)$, $\psi \in \mathcal{S}_1$, defines an element of $\mathcal{S}_1'$, which will again be denoted by Φ_n. The Fourier transform $\mathcal{F}(u) \in \mathcal{S}_1'$ of an element $u \in \mathcal{S}_1'$ is defined as follows:

$$\mathcal{F}(u)(\psi) \stackrel{\text{def}}{=} u(\mathcal{F}(\psi)), \qquad \psi \in \mathcal{S}_1,$$

where

$$\mathcal{F}(\psi)(t) \stackrel{\text{def}}{=} (2\pi)^{-1/2} \int_{\mathbf{R}} \psi(x) \exp(-itx)\, dx, \qquad t \in \mathbf{R},$$

is the Fourier transform of the function $\psi(x)$. If ν is a finite measure, then $\mathcal{F}(\nu)$ may be identified with the function $(2\pi)^{-1/2} \widehat{\nu}(-ix)$, $x \in \mathbf{R}$, i.e. $\mathcal{F}(u)(\nu) = (2\pi)^{-1/2} \int_{\mathbf{R}} \widehat{\nu}(-ix)\psi(x)\, dx$, $\psi \in \mathcal{S}_1$.

Let a σ-finite measure ν be an element of $\mathcal{S}_1'$. For $a > 0$, put $\nu_a(A) \stackrel{\text{def}}{=} \nu(A - a)$, $A \in \mathcal{B}$, where $A - a \stackrel{\text{def}}{=} \{x \in \mathbf{R} : x + a \in A\}$. We set $\Delta_a \nu \stackrel{\text{def}}{=} \nu - \nu_a \in \mathcal{S}_1'$. Then $\mathcal{F}(\Delta_a \nu) = (1 - e^{-iax})\mathcal{F}(\nu)$ in the sense that

$$\begin{aligned} \mathcal{F}(\Delta_a \nu)(\psi) &= \mathcal{F}(\nu)[(1 - e^{-iax})\psi(x)] \\ &= (2\pi)^{-1/2} \int_{\mathbf{R}} \int_{\mathbf{R}} (1 - e^{-iax})\psi(x) e^{-itx}\, dx\, \nu(dt). \end{aligned}$$

In what follows, the notation Δ_a^k will mean that the operator Δ_a is applied k times. If ν and κ are two measures for which the convolution $\nu * \kappa$ makes sense, then clearly $\Delta_a(\nu * \kappa) = (\Delta_a \nu) * \kappa = \nu * (\Delta_a \kappa)$. The measure $\Delta_a L$ is nothing else but the restriction of Lebesgue measure to the interval $[0, a]$. Therefore, the tempered distribution $\mathcal{F}(\Delta_a^k L^{k*})$ may be identified with the function $(2\pi)^{-1/2}(1 - e^{-iax})^k/(ix)^k$, $x \in \mathbf{R}$, and hence $\mathcal{F}(\Delta_a^n L^{k*})$ is defined by the function $(2\pi)^{-1/2}(1 - e^{-iax})^n/(ix)^k$, $x \in \mathbf{R}$.

Lemma 3. *Let ν be a finite measure. Then the tempered distribution $\mathcal{F}(\Delta_a T\nu)$ may be identified with the function*

$$(2\pi)^{-1/2}(1 - e^{-iax})[\widehat{\nu}(-ix) - \widehat{\nu}(0)]/(-ix), \qquad x \in \mathbf{R}.$$

Proof of Lemma 3. The measure $\Delta_a T\nu$ has density

$$\begin{aligned} f(t) &\stackrel{\text{def}}{=} \nu(\mathbf{R})\delta((-\infty, t\,]) - \nu((-\infty, t\,]) \\ &\qquad - [\nu(\mathbf{R})\delta((-\infty, t - a]) - \nu((-\infty, t - a])] \\ &= [\nu(\mathbf{R})\delta - \nu]((t - a, t\,]). \end{aligned}$$

We have

$$\mathcal{F}(\Delta_a T\nu)(\psi) = (2\pi)^{-1/2} \int_{\mathbf{R}} \int_{\mathbf{R}} \psi(x) e^{-itx}\, dx\, f(t)\, dt.$$

Since ν is a finite measure, it is easily seen that the function $\nu((t-a])$ is integrable and so is the function $f(t)$. By Fubini's theorem, we may interchange the order of integration to obtain

$$\begin{aligned}\mathcal{F}(\Delta_a T\nu)(\psi) &= (2\pi)^{-1/2}\int_{\mathbf{R}}\psi(x)\int_{\mathbf{R}} e^{-itx}\int_{t-a}^{t}[\nu(\mathbf{R})\delta-\nu](du)\,dt\,dx \\ &= (2\pi)^{-1/2}\int_{\mathbf{R}}\psi(x)\int_{\mathbf{R}}\int_{u}^{u+a} e^{-itx}\,dt\,[\nu(\mathbf{R})\delta-\nu](du)\,dx \\ &= (2\pi)^{-1/2}\int_{\mathbf{R}}\psi(x)\frac{e^{-iax}-1}{-ix}[\nu(\mathbf{R})-\widehat{\nu}(-ix)]\,dx.\end{aligned}$$

This completes the proof of Lemma 3 since obviously $\nu(\mathbf{R})=\widehat{\nu}(0)$.

Applying Lemma 3 with $\nu=T^{n-1}W$, we see that $\mathcal{F}(\Delta_a R_n)$ is identifiable with the function

$$(2\pi)^{-1/2}(1-e^{-iax})\,n!\sum_{k=0}^{n}(-\varepsilon)^k\binom{n}{k}w_k(-ix), \qquad x\in\mathbf{R},$$

and hence $\mathcal{F}(\Delta_a^n\mathcal{R}_n)$ is given by the function

$$(2\pi)^{-1/2}(1-e^{-iax})^n n!\sum_{k=0}^{n}(-\varepsilon)^k\binom{n}{k}w_k(-ix), \qquad x\in\mathbf{R}.$$

We now turn to (3.5). Put $s=-ix$, $x\in\mathbf{R}$, multiply both sides by $(2\pi)^{-1/2}(1-e^{-iax})^n\psi(x)$, $\psi\in\mathcal{S}_1$, and then integrate the equality thus obtained over the whole line $\mathbf{R}$. This yields

(3.6)
$$(2\pi)^{-1/2}\int_{\mathbf{R}}\frac{n!(1-e^{-iax})^n}{[1-\widehat{F}(-ix)]^n}\psi(x)\,dx=\sum_{k=1}^{n}\gamma_k^{(n)}\mathcal{F}(\Delta_a^n L^{k*})(\psi)+\mathcal{F}(\Delta_a^n\mathcal{R}_n)(\psi),$$

Lemma 4. *The left-hand side of* (3.6) *is equal to* $\mathcal{F}(\Delta_a^n\Phi_n)(\psi)$.

Proof of Lemma 4. Define a measure $\Phi_{n,z}$ for $z\in(0,1)$ by

$$\Phi_{n,z}(A)=\sum_{k=0}^{\infty}\frac{n\cdot(n+k-1)!}{k!}\,z^kF^{k*}(A), \qquad A\in\mathcal{B}.$$

This measure is finite and $\widehat{\Phi}_{n,z}(s)=n!/[1-z\widehat{F}(s)]^n$, $\Re s=0$. Hence $\mathcal{F}(\Phi_{n,z})\in\mathcal{S}_1'$ may be identified with the function $(2\pi)^{-1/2}\,n!/[1-z\widehat{F}(-ix)]^n$, $x\in\mathbf{R}$. Clearly $\Phi_{n,z}\to\Phi_n$ in the topology of the space $\mathcal{S}_1'$ as $z\to 1-$, and hence $\Delta_a^n\Phi_{n,z}\to\Delta_a^n\Phi_n$ as $z\to 1-$, whence it follows that $\mathcal{F}(\Delta_a^n\Phi_{n,z})\to\mathcal{F}(\Delta_a^n\Phi_n)$ in $\mathcal{S}_1'$ as $z\to 1-$. In order to complete the proof of the lemma, it remains to notice that we may take the limit $z \to 1-$ through the integral on the right-hand side of the equality

$$\mathcal{F}(\Delta_a^n \Phi_{n,z})(\psi) = (2\pi)^{-1/2}\, n! \int_{\mathbf{R}} \frac{(1-e^{-iax})^n \psi(x)}{[1 - z\widehat{F}(-ix)]^n}\, dx$$

since the integrand is bounded from above by the integrable function $C|\psi(t)|$, uniformly in $z \in (1/2, 1)$, where C is a positive constant (see the proof of Lemma 1 in [18]). In view of Lemma 4, we may replace the left-hand side of (3.6) by $\mathcal{F}(\Delta_a^n \Phi_n)(\psi)$ and then go over from the Fourier transforms to their inverse images. We obtain

$$\Delta_a^n \Phi_n = \sum_{k=1}^{n} \gamma_k^{(n)} \Delta_a^n L^{k*} + \Delta_a^n \mathcal{R}_n. \tag{3.7}$$

Let $\mathcal{D}(\mathbf{R})$ be the space of all infinitely differentiable functions with compact supports. Any tempered distribution is completely determined by its values at functions $\psi \in \mathcal{D}(\mathbf{R})$ since $\mathcal{D}(\mathbf{R})$ is dense in $\mathcal{S}_1$ [20, Theorem 7.10]. Let ψ be an arbitrary element of $\mathcal{D}(\mathbf{R})$ whose support is contained in a finite interval $[c, d\,]$. Then the left-hand side of (3.7) will be equal to $\int_{\mathbf{R}} \psi(x)\Phi_n(dx)$ plus a finite number of integrals of the form $\pm \int_{\mathbf{R}} \psi(x + ja)\Phi_n(dx)$, where $j > 0$ are integers. The latter integrals are estimated by the quantities $\Phi_n([c - ja, d - ja\,]) \max\{|\psi(x)| : x \in [c, d\,]\}$, which tend to zero as $a \to \infty$ (see Theorem 3). Similar reasoning is also valid for the summands in the right-hand side of (3.7). Letting $a \to \infty$, we obtain the representation (1.3). This completes the proof of Theorem (4).

We now present some results concerning the asymptotic behaviour of $\Phi_n(A)$ for various choices of the set A. We begin with the generalized renewal function $\Phi_n(x)$.

Theorem 5. *In addition to the hypotheses of Theorem* 4, *suppose that* $E|X_1|^{n+1} < \infty$. *Then*

$$\Phi_n(x) \stackrel{\text{def}}{=} \Phi_n((-\infty, x\,]) = \sum_{k=0}^{n} \gamma_k^{(n)} x^k/k! - \mathcal{R}_n((x, \infty)),$$

where $\mathcal{R}_n((x,\infty))$ *satisfies* (3.1).

Proof. The assertion of the theorem is an immediate consequence of Theorem 4. We only need to verify the equality $\gamma_0^{(n)} = \mathcal{R}_n(\mathbf{R})$. Tracing the proof of Theorem 4, we see that the additional hypothesis $E|X_1|^{n+1} < \infty$ implies that $T^n W$ is a finite measure and so is the remainder $\mathcal{R}_n$. Moreover, the Fourier transform $\mathcal{F}(\Delta_a^n \mathcal{R}_n)$ may be identified with the function

$$(2\pi)^{-1/2}(1 - e^{-iax})^n \left\{ n!/[1 - \widehat{F}(-ix)]^n - \sum_{k=1}^{n} \gamma_k^{(n)}/(ix)^k \right\}, \qquad x \in \mathbf{R}.$$

On the other hand, since $\mathcal{R}_n$ is a finite measure, $\mathcal{F}(\Delta_a^n \mathcal{R}_n)$ is identifiable with $(2\pi)^{-1/2}(1-e^{-iax})^n \widehat{\mathcal{R}}_n(-ix)$. But if any two locally integrable functions define one and the same tempered distribution, they must coincide almost everywhere. Hence

$$\widehat{\mathcal{R}}_n(-ix) = n!/[1-\widehat{F}(-ix)]^n - \sum_{k=1}^{n} \gamma_k^{(n)}/(ix)^k \tag{3.8}$$

almost everywhere and, since both sides of (3.8) are continuous functions, this equality holds for all $x \in \mathbf{R} \setminus \{0\}$. Letting $x \to 0$ in (3.8) and recalling the definition of $\gamma_0^{(n)}$, we obtain $\mathcal{R}_n(\mathbf{R}) = \widehat{\mathcal{R}}_n(0) = \gamma_0^{(n)}$. The proof of Theorem 5 is complete.

Further, we consider the asymptotic behaviour of $\Phi_n([0,x])$ as $x \to \infty$. We shall need some additional notation. Let $\eta = \min\{k \geq 1 : S_k > 0\}$ and let F_+ be the distribution of the first positive sum S_η. Put

$$\widetilde{\Phi}_n \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{n\cdot(n+k-1)!}{k!} F_+^{k*}.$$

Since $EX_1 > 0$, both distributions F and F_+ possess finite moments of the same order. This may be seen as follows. By Theorem 10 I of [3, Chapter 4], $F((x,\infty)) \leq F_\oplus((x,\infty)) \leq cF((x,\infty))$, where $F_\oplus$ is the distribution of the first *non-negative* sum and c is a positive constant. Hence F and $F_\oplus$ possess finite moments of the same order. Taking into account the connection $F_\oplus = a + bF_+$, a, $b > 0$, between F_+ and $F_\oplus$ [13, Section XVIII.6], we arrive at the desired conclusion.

Define coefficients $\widetilde{\gamma}_k^{(n)}$ in complete analogy with the $\gamma_k^{(n)}$, taking the distribution F_+ as a starting point instead of F. Let Q_n be the measure with Laplace transform

$$\widehat{Q}_n(s) \stackrel{\text{def}}{=} n!/[1-\widehat{F}_+(s)]^n - \sum_{k=1}^{n} (-1)^k \widetilde{\gamma}_k^{(n)}/s^k, \qquad \Re s < 0.$$

Denote by D_- the distribution of $\inf_{k\geq 0} S_k$. Put

$$\Gamma_0^{(n)} \stackrel{\text{def}}{=} a^n \int_{-\infty}^{0} Q_n([-x,\infty))\, D_-^{n*}(dx),$$

where $a = \exp\{\sum_{k=1}^{\infty} \mathbf{P}(S_k \leq 0)/k\}$.

Theorem 6. *Under the hypotheses of Theorem* 4,

$$\Phi_n([0,x]) = \Gamma_0^{(n)} + \sum_{k=1}^{n} \gamma_k^{(n)} x^k/k! - \mathcal{R}_n((x,\infty)),$$

where $\mathcal{R}_n((x,\infty))$ *satisfies* (3.1).

Proof. The assertion of the theorem is an immediate consequence of Theorem 4. We only need to verify the equality $\Gamma_0^{(n)} = \mathcal{R}_n([0,\infty))$. First, we observe that the condition $(F^{m*})\widehat{_s}(\gamma) < 1$ for some integer $m = m(F) \geq 1$ is equivalent to $(F_+^{m*})\widehat{_s}(r) < 1$, where $m = m(F_+) \geq 1$ [25]. In particular, both distributions F and F_+ have non-vanishing absolutely continuous components. Tracing the proof of Theorem 4, we conclude that

$$\widetilde{\Phi}_n(A) = \sum_{k=1}^{n} \widetilde{\gamma}_k^{(n)} L^{k*}(A) + Q_n(A), \qquad A \in \mathcal{B}. \tag{3.9}$$

Moreover, Q_n is a *finite* measure. Putting $A = [0,t]$ in the lemma of [23], we have

$$\Phi_n([0,t]) = a^n \int_{-\infty}^{0} \widetilde{\Phi}_n([-x, t-x])\, D_-^{n*}(dx). \tag{3.10}$$

Putting $A = [-x, t-x]$ in (3.9) and substituting it into (3.10), we obtain after simple calculations that

$$\Phi_n([0,t]) = \sum_{k=1}^{n} \gamma_k^{(n)} \frac{t^k}{k!} + a^n \int_{-\infty}^{0} Q_n([-x, t-x])\, D_-^{n*}(dx).$$

The fact that, as a results of these steps, we will obtain $\sum_{k=1}^n \gamma_k^{(n)} t^k/k!$ as the polynomial part of $\Phi_n([0,t])$ follows from the uniqueness of the expansion $\Phi_n([0,t]) = \sum_{k=1}^n a_k t^k + o(t)$ as $t \to \infty$. Therefore, $\mathcal{R}_n([0,t]) = a^n \int_{-\infty}^0 Q_n([-x,t-x])\, D_-^{n*}(dx)$. Letting $t \to \infty$, we obtain the desired equality $\Gamma_0^{(n)} = \mathcal{R}_n([0,\infty))$.

Finally, let us consider the behaviour of $\Phi_n((x, x+h])$, $h > 0$, as $x \to \infty$.

Theorem 7. *Under the hypotheses of Theorem* 4*,*

$$\lim_{x\to\infty} \frac{\Phi_n((x,x+h]) - \sum_{k=1}^n \gamma_k^{(n)}[(x+h)^k - x^k]/k!}{\tau(x)} = \frac{(1-e^{-\gamma h})\, n \cdot n!\, l(F)}{[1-\widehat{F}(\gamma)]^{n+1}}.$$

Proof. The ratio under the limit is equal to $\mathcal{R}_n((x,x+h])/\tau(x)$. We have

$$\begin{aligned} \frac{\mathcal{R}_n((x,x+h])}{\tau(x)} &= \frac{\mathcal{R}_n((x,\infty))}{\tau(x)} - \frac{\mathcal{R}_n((x+h,\infty))}{\tau(x+h)} \frac{\tau(x+h)}{\tau(x)} \\ &\to \frac{(1-e^{-\gamma h})\, n \cdot n!\, l(F)}{[1-\widehat{F}(\gamma)]^{n+1}} \qquad \text{as } x \to \infty. \end{aligned}$$

This proves the theorem.

Remark 1 In the renewal-measure case, i.e. when $n = 1$, the $\mathcal{S}(\gamma)$-behaviour of the remainder term was considered in [14] for distributions F concentrated on the positive half-axis. Theorem 7 generalizes Theorem 3.1.14 of [14] to arbitrary $n \geq 1$ and to the whole line $\mathbf{R}$.

Remark 2 The condition $\widehat{(F^{m*})_s}(\gamma) < 1$ for some integer $m = m(F) \geq 1$ is also necessary in order to ensure that $\int_0^\infty e^{\gamma x}|\mathcal{R}_n|(dx) < \infty$. Actually, suppose the contrary. First, we observe that, by Theorem 3, $\Phi_n((x, x+h]) \to 0$ as $x \to -\infty$. Hence $\int_{-\infty}^0 e^{\gamma x}\Phi_n(dx) = \int_{-\infty}^0 e^{\gamma x}|\mathcal{R}_n|(dx) < \infty$ since $\gamma > 0$. Then, choosing $A_m \in \mathcal{B}$ of Lebesgue measure zero such that $\int_{A_m} e^{\gamma x}(F^{m*})_s(dx) \geq 1$, we would obtain from (1.3) that, on one hand, $\Phi_n(\cup_{m=1}^\infty A_m) = \infty$ and, on the other hand, $\left|\int_{\cup_{m=1}^\infty A_m} e^{\gamma x}\mathcal{R}_n(dx)\right| \leq \int_{\mathbf{R}} e^{\gamma x}|\mathcal{R}_n|(dx) < \infty$. This contradiction proves the necessity of the indicated condition in Theorem 4.

Acknowledgement

I wish to express gratitude to the referee for a careful reading of the manuscript and helpful comments.

References

[1] Athreya, K.B., and Ney, P.E. (1972). *Branching Processes*. Springer-Verlag, Berlin, Heidelberg, New York.

[2] Bertoin, J. and Doney, R.A. (1996). Some asymptotic results for transient random walks. *J. Appl. Probab.* **28** 206–226.

[3] Borovkov, A.A. (1976). *Stochastic processes in queueing theory*. Springer, New York.

[4] Chistyakov, V.P. (1964). A theorem on sums of independent positive random variables and its applications to branching processes. *Theory Probab. Appl.* **9** 640–648.

[5] Chover, J., Ney, P. and Wainger, S. (1973). Degeneracy properties of subcritical branching processes. *Ann. Probab.* **1** 663–673.

[6] Chover, J., Ney, P. and Wainger, S. (1973). Functions of probability measures. *J. Analyse Math.* **26** 255–302.

[7] Cline, D.B.H. (1986). Convolution tails, product tails and domains of attraction. *Probab. Theory Related Fields* **72** 529–557.

[8] Cline, D.B.H. (1987). Convolutions of distributions with exponential and subexponential tails. *J. Austral. Math. Soc.* (Series A) **43** 347–365.

[9] Embrechts, P. and Goldie, C.M. (1980). On closure and factorization properties of subexponential and related distributions. *J. Austral. Math. Soc.* (Series A) **29** 243–256.

[10] Embrechts, P. and Goldie, C.M. (1982). On convolution tails. *Stochastic Process. Appl.* **13** 263–278.

[11] Embrechts, P., Goldie, C.M. and Veraverbeke, N. (1979). Subexponentiality and infinite divisibility. *Z. Wahrscheinlichkeitstheor. Verw. Geb.* **49** 335–347.

[12] Embrechts, P. and Veraverbeke, N. (1982). Estimates for the probability of ruin with special emphasis on the possibility of large claims. *Insurance: Mathematics and Economics* **1** 55–72.

[13] Feller, W. (1966). *An Introduction to Probability Theory and Its Applications II.* Wiley, New York.

[14] Frenk, J.B.G. (1987). *On Banach Algebras, Renewal Measures and Regenerative Processes.* CWI Tracts **38**, Centre for Mathematics and Computer Science, Amsterdam.

[15] Hille, E. and Phillips, R.S. (1957). *Functional Analysis and Semi-Groups.* AMS Colloquium Publications 31, Providence.

[16] Klüppelberg, C. (1989). Subexponential distributions and characterizations of related classes. *Probab. Theory Related Fields* **82** 259–269.

[17] Pitman, E.J.G. (1980). Subexponential distribution functions. *J. Austral. Math. Soc.* (Series A) **29** 337–347.

[18] Rogozin, B.A. (1976). Asymptotics of renewal functions, *Theory Probab. Appl.* **21** 669–686.

[19] Rogozin, B.A. and Sgibnev, M.S. (1980). Banach algebras of measures on the line. *Siberian Math. J.* **21** 265–273.

[20] Rudin, W. (1973). *Functional Analysis.* McGraw-Hill, New York.

[21] Sgibnev, M.S. (1988). Banach algebras of measures of class $\mathcal{S}(\gamma)$. *Siberian Math. J.* **29** 647–655.

[22] Sgibnev, M.S. (1991). Asymptotic behaviour of higher renewal moments. *Theory Probab. Appl.* **36** 508–518.

[23] Sgibnev, M.S. (1997). Asymptotics of generalized renewal functions in the presence of a finite variance. *Teor. Veroyatnost. i Primenen.* **42** 632–637 (in Russian).

[24] Sgibnev, M.S. (1998). On the asymptotic behaviour of the harmonic renewal measure. *J. Theoretic. Probab.* **11** 371–382.

[25] Sgibnev, M.S. (1998). Equivalence of two conditions on singular components. *Statist. Probab. Lett.* **40** 127–131.

[26] Smith W.L. (1959). On the cumulants of renewal processes, *Biometrika* **46** 1–29.

[27] Smith, W.L. (1967). A theorem on functions of characteristic functions and its application to some renewal theoretic random walk problems. In *Proc. Fifth Berkeley Symp. Math. Statist. Probab.* Vol. 2, Pt II, 265–309. University of California Press, Berkeley.

[28] Teugels, J.L. (1975). The class of subexponential distributions. *Ann. Probab.* **3** 1000–1011.

[29] Veraverbeke, N. (1977). Asymptotic behaviour of Wiener-Hopf factors of a random walk. *Stochastic Process. Appl.* **5** 27–37.

[30] Wermer, J. (1976). *Banach Algebras and Several Complex Variables*, Second Edition. Springer, New York.

Mikhail S. Sgibnev
Institute of Mathematics, Siberian Branch of the Russian Academy of Sciences,
Novosibirsk 90, 630090 Russia
E-mail: `sgibnev@math.nsc.ru`