SUT Journal of Mathematics
Vol. 35, No. 1 (1999), 81–112

# Smoothing effects for some derivative nonlinear Schrödinger equations without smallness condition

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(Received March 1, 1999)

**Abstract.** In this paper we study a smoothing property of solutions to the Cauchy problem for the nonlinear Schrödinger equations of type :

$$\text{(A)} \qquad \begin{array}{ll} \imath u_t + u_{xx} = \mathcal{N}(u, \overline{u}, u_x, \overline{u_x}), & t \in \mathbb{R}, x \in \mathbb{R}; \\ u(x,0) = u_0(x), \qquad x \in \mathbb{R}, & \end{array}$$

where

$$\begin{aligned} \mathcal{N}(u, \overline{u}, u_x, \overline{u_x}) = & K_1 |u|^2 u + K_2 |u|^2 u_x + K_3 u^2 \overline{u_x} + K_4 |u_x|^2 u + K_5 \overline{u} u_x^2 \\ & + K_6 |u_x|^2 u_x, \end{aligned}$$

the functions $K_j = K_j(|u|^2)$ satisfy $K_j \in \mathcal{C}^\infty([0, +\infty); \mathbb{C})$.
This equation has been derived from physics. For example if the nonlinear term is $\mathcal{N}(u) = \frac{\overline{u} u_x^2}{1+|u|^2}$ then equation (A) appears in the classical pseudospin magnet model [18]. The aim of this paper is to study the case : the nonlinearity depends on $u_x$ and $\overline{u_x}$, and satisfies the so called Gauge condition : $\mathcal{N}(e^{\imath\theta} u) = e^{\imath\theta} \mathcal{N}(u)$. We prove that if the initial data $u_0 \in \mathcal{H}^{3,l}$ for any $l \in \mathbb{N}$, then there exists a positive time $T > 0$ and a unique solution $u \in \mathcal{C}^\infty([-T, T] \backslash \{0\}; \mathcal{C}^\infty(\mathbb{R}))$ of the Cauchy problem (A). The result in this paper improves the previous one in [11] because we do not assume any size restriction on the data. Here $\mathcal{H}^{m,s} = \left\{ \varphi \in \mathcal{L}^2(\mathbb{R}); \|\varphi\|_{m,s} < +\infty \right\}$ and $\|\varphi\|_{m,s} = \|(1+x^2)^{s/2}(1-\partial_x^2)^{m/2}\varphi\|_{\mathcal{L}^2(\mathbb{R})}$, $\mathcal{H}^{m,\infty} = \cap_{s \geq 1} \mathcal{H}^{m,s}$.

*AMS* 1991 *Mathematics Subject Classification.* Primary 35Q55.

*Key words and phrases.* Schrödinger equation, smoothing effects.

## §1. Introduction

In this paper we study a smoothing property of solutions to the Cauchy problem for the derivative nonlinear Schrödinger equation of the following form

$$
\begin{cases} \imath u_t + u_{xx} = \mathcal{N}(u, \overline{u}, u_x, \overline{u_x}), & x \in \mathbb{R}, \quad t \in \mathbb{R}, \\ u(x,0) = u_0(x) , & x \in \mathbb{R}, \end{cases} \tag{1.1}
$$

where the nonlinearity is

$$
\mathcal{N}(u, \overline{u}, u_x, \overline{u_x}) = K_1|u|^2 u + K_2|u|^2 u_x + K_3 u^2 \overline{u_x} + K_4 |u_x|^2 u + K_5 \overline{u} u_x^2 + K_6 |u_x|^2 u_x.
$$

The functions $K_j = K_j(|u|^2)$ are such that $K_j(z) \in \mathcal{C}^{l+3}([0,+\infty);\mathbb{C})$. If $K_5(z) = \frac{1}{1+z}$ and $K_1 = K_2 = K_3 = K_4 = K_6 = 0$ then equation (1.1) appears in the classical pseudospin magnet model [18]. To state our main result we introduce some function spaces. The Lebesgue space is $\mathcal{L}^p(\mathbb{R}) = \{\varphi \in \mathcal{S}'(\mathbb{R}) : \|\varphi\|_p < \infty\}$, where $\|\varphi\|_p = \left(\int |\varphi(x)|^p dx\right)^{1/p}$ if $1 \le p < \infty$ and $\|\varphi\|_\infty = \text{ess.sup}\{|\varphi(x)|; x \in \mathbb{R}\}$ if $p = \infty$. For simplicity we let $\|\varphi\| = \|\varphi\|_2$. Weighted Sobolev space is $\mathcal{H}_p^{m,s} = \{\varphi \in \mathcal{S}'(\mathbb{R}) : \|\varphi\|_{m,s,p} = \left\|\left(1+x^2\right)^{s/2}\left(1-\partial_x^2\right)^{m/2}\varphi\right\|_p < \infty\}$, $m, s \in \mathbb{R}$, $1 \le p \le \infty$. For simplicity we write $\mathcal{H}^{m,s} = \mathcal{H}_2^{m,s}$ and $\|\phi\|_{m,s} = \|\phi\|_{m,s,2}$. We denote also $\mathcal{H}^{m,\infty} = \cap_{s=1}^{\infty}\mathcal{H}^{m,s}$. We let $\mathcal{C}(\mathcal{I};\mathcal{B})$ be the space of continuous functions from a time interval $\mathcal{I}$ to a Banach space $\mathcal{B}$.
Our main results of this paper are the followings.

**Theorem 1.1.** *If the initial data $u_0$ are such that $u_0 \in \mathcal{H}^{3,l}$ with $l \in \mathbb{N}$. Then for some time $T > 0$ there exists a unique solution*

$$
u \in \mathcal{C}\left([-T,T];\mathcal{H}^{2,0}\right) \cap \mathcal{L}^\infty\left(-T,T;\mathcal{H}^{3,0}\right) \cap \mathcal{C}\left([-T,T]\backslash\{0\};\mathcal{C}^{l+2}(\mathbb{R})\right)
$$

*of the Cauchy problem (1.1) such that*

$$
\sup_{t\in[-T,T]} |t|^k \left\|\left(1+x^2\right)^{-k/2}\partial_x^k u(t)\right\|_{2,0} < \infty \quad \text{for} \quad 0 \le k \le l.
$$

By virtue of Theorem 1.1, equation (1.1) and the Sobolev's embedding inequality (see [7]) we get

**Theorem 1.2.** *If the initial data satisfy $u_0 \in \mathcal{H}^{3,\infty}$. Then for some time $T > 0$ there exists a unique solution $u \in \mathcal{C}^\infty\left([-T,T]\backslash\{0\};\mathcal{C}^\infty(\mathbb{R})\right)$ of the Cauchy problem (1.1).*

Remark: Our method can be applied to the following problem

$$\text{(C)} \quad \begin{cases} \imath\partial_t u + \Delta u = \mathcal{N}(u, \nabla u, \bar{u}, \overline{\nabla u}) & (x,t) \in \mathbb{R}^n \times \mathbb{R} \\ u(x,0) = u_0(x) & x \in \mathbb{R}^n \end{cases}$$

where $\Delta = \sum_{i=1}^n \partial_{x_i}^2$, $\nabla u = (\partial_{x_1} u, \ldots, \partial_{x_n} u)$ and $\mathcal{N}$ satisfies the gauge condition and $\mathcal{N}(X) = o(|X|^2)$ around 0. More precisely, for $u_0 \in \mathcal{H}^{[\frac{n}{2}]+3,l}$ with $l \in \mathbb{N}$, there exists a unique solution $u$ of (C) satisfying,

$$\sup_{t\in[-T,T]} |t|^{|\alpha|} \left\| \left(1+|x|^2\right)^{-|\alpha|/2} \partial_x^\alpha u(t) \right\|_{[\frac{n}{2}]+2,0} < +\infty, .$$

for any multi-index $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq l$ where $|\alpha| = \alpha_1 + \ldots + \alpha_n$.
Same results were obtained by N. Hayashi, P.I. Naumkin and the author [11] for a nonlinearity which doesn't depend on $\overline{u_x}$. In case of nonlinearities depending on $\overline{u_x}$, they had to assume that the initial data $u_0$ is sufficiently small to obtain Theorem 1.1. In our case we do not consider any smallness condition on the data to get these results. After this work was essentially completed, we were informed that H. Chihara [3] studied the nonlinear Schrödinger equations of the derivative type

$$\text{(B)} \quad \begin{cases} \partial_t u - \imath\Delta u = f(u, \nabla u, \bar{u}, \overline{\nabla u}) & (x,t) \in \mathbb{R}^n \times \mathbb{R} \\ u(x,0) = u_0(x) & x \in \mathbb{R}^n \end{cases}$$

and he obtained similar results to ours for the cubic nonlinearities satisfying the condition for $w = e^{\imath\theta}$, $\theta \in \mathbb{R}$

$$f(wu, w\nabla u, w\bar{u}, w\nabla\bar{u}) = wf(u, \nabla u, \bar{u}, \nabla\bar{u}).$$

More precisely he showed that for $u_0 \in \mathcal{H}^{m,l}$ where $m$ is in an integer such that $m \geq [\frac{n}{2}] + 4$, then there exists a unique solution $u$ to the Cauchy problem (B) satisfying the growth : for any multi-index $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq l$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$,

$$(1+|x|^2)^{-\frac{|\alpha|}{2}} u \in \mathcal{C}([-T,T]\backslash\{0\}; \mathcal{H}^{l,0}).$$

His method depends on a theory of pseudo-differential operator and so our method in this paper is completely different from his. In our case, we consider the initial data $u_0 \in \mathcal{H}^{3,l}$ which is a lower order Sobolev space. Previously smoothing effects of solutions to the nonlinear Schrödinger equation (A) with $K_2 = \cdots = K_6 \equiv 0$ was studied in [10] and similar results to that of Theorem 1.1 were obtained by using the operator $\mathcal{J} = x + 2it\partial_x$, which commutes with the linear Schrödinger operator $\mathcal{L} = i\partial_t + \partial_x^2$. There are some results about the nonlinear Schrödinger equation of derivative type with the nonlinearity $\mathcal{N}(u,u_x) = i\left(|u|^2u\right)_x$

$$\begin{cases} iu_t + u_{xx} = i\left(|u|^2u\right)_x, & x \in \mathbb{R}, \quad t \in \mathbb{R}, \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.2}$$

By using some translation, the derivative nonlinear Schrödinger equation (1.2) can be translated to a system of nonlinear Schrödinger equations without derivatives of unknown functions (see, e.g., [8]). So in [12] the same results as in Theorem 1.1 were shown for the Cauchy problem (1.2).

The existence of solutions to the Cauchy problem (A) was established in [1] and [13] in the usual Sobolev space $\mathcal{H}^{3,0}$. Note that the transformation used in [1], [13], [20] is not sufficient to prove our results. In the present paper we apply a smoothing effect of the linear Schrödinger equation similar to the one used in [11]. But in [11], estimations on nonlinear terms including $\overline{u_x}$ lead to a smallness condition on the initial data in order to get Theorem 1.1. We explain this problem in a few words. A standard contraction mapping method was used in [11] to solve the Cauchy problem (1.1). In order to deal with the derivative terms inside the nonlinarities they applied a smoothing operator $\mathcal{S}$ (defined precisely below) and obtained the energy type inequality :

$$\begin{aligned}(1 - TC_1(\|u_0\|_{3,l}))\sum_{k=0}^{l}\sup_{t\in[0,T]}\left\|\mathcal{J}^k v_x(t)\right\|^2 \\ + (1 - C_2(\|u_0\|_{3,l})\delta e^{r(\frac{\|u_0\|_{3,l}}{\delta})})\sum_{k=0}^{l}\sup_{t\in[0,T]}\left\|w\mathcal{S}\sqrt{|\partial_x|}\mathcal{J}^k v_x(t)\right\|^2 \le C_3(\|u_0\|_{3,l}).\end{aligned}$$

where $\mathcal{J} = x + 2it\partial_x$, $\delta$ is an arbitrary parameter, $C_1(\|u_0\|_{3,l})$, $C_2(\|u_0\|_{3,l})$, $C_3(\|u_0\|_{3,l})$ are positive constants depending on $\|u_0\|_{3,l}$, $v = (1-\partial_x^2)u$ and $w$ is a positive function, $r$ is a continuous positive function such that the condition : there exists a $\delta > 0$ satisfying $(1 - C_2(\|u_0\|_{3,l})\delta e^{r(\frac{\|u_0\|_{3,l}}{\delta})}) \ge 0$, is fulfilled if and only if $\|u_0\|_{3,l}$ is small enough. Therefore the smallness condition on the initial data in [11] is needed. To overcome this problem, we remove nonlinear terms involving $\overline{u_x}$ by a diagonalization technique. Its main point is to conceal the whole bad terms by a linear tranformation. This method was used by H. Chihara [2] [3] and recently by N. Hayashi and E.I. Kaikina [9]. Smoothing properties of solutions to the linear Schrödinger equation were studied by many authors (see [4], [6], [19], [22]) and later the results in [6], [19], [22] were improved in [16]. We use in this paper some pseudo-differential operator of order 0. The history of such operators starts from Doi, who discovered in [6] the following operator $\exp\left(\int_{-\infty}^{x}\left(1+x'^2\right)^{-1}dx'\frac{D}{\langle D\rangle}\right)$, where $D = -i\partial_x$ and

$\langle D\rangle = \left(1-\partial_x^2\right)^{1/2}$ which is useful to gain a smoothing property of solutions of linear Schrödinger equations. Chihara [1] used the following modification of this pseudo-differential operator $\exp\left(\int_{-\infty}^{x} |u(t,x')|^2 dx' \frac{D}{\langle D\rangle}\right)$, to prove the local existence of solutions $u$ to the Cauchy problem for the nonlinear Schrödinger equations in higher order Sobolev space. He made use of some well known results concerning pseudo-differential operators, such as the $\mathcal{L}^2$ - boundedness theorem and the sharp Gårding inequality, and that is the reason why the higher order Sobolev space was needed. In this paper we apply a more simple operator $\mathcal{S}(\varphi) = \cosh(\varphi) + i\sinh(\varphi)\mathcal{H}$, where $\varphi$ is defined later as

$$\varphi(x,t) = \frac{1}{\delta}\sum_{k=0}^{l} \partial_x^{-1}\left(|\mathcal{J}^k\tilde{u}(x,t)|^2 + |\mathcal{J}^k\tilde{u}_x(x,t)|^2 + \sum_{2\le j\le 5} |\mathcal{J}^k K_j\tilde{u}|^2 + |\mathcal{J}^k K_6\tilde{u}_x|^2\right),$$

which enables us to avoid the use of the technique of the pseudo-differential operators and so by virtue of simple explicit computations we can treat the problem in the natural order Sobolev space $\mathcal{H}^{3,0}$. Note that by a different approach smoothing effects for the generalized KdV equation were studied in [5], [14].

The rest of the paper is organized as follows. First we give some notations. In Section 2 we describe a smoothing property of the linear Schrödinger equation. Then in Section 3 we prove in Lemma 3.1 the local existence of solutions to the Cauchy problem (A) in the function space $\{u \in \mathcal{C}([-T,T];\mathcal{L}^2(\mathbb{R}));\ \|\mathcal{U}(-t)u(t)\|_{3,l} < \infty\}$, where $\mathcal{U}(t)$ is the free Schrödinger evolution group. And as a simple consequence we obtain the result of Theorem 1.1.

*Notations.* We denote $\partial_x = \frac{\partial}{\partial x}$, $\partial_x^{-1} = \int_{-\infty}^{x} dx'$ and let $\mathcal{F}\phi$ or $\hat{\phi}$ be the Fourier transform of $\phi(x)$, namely $\hat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\xi x}\phi(x)dx$. We denote by $\mathcal{F}^{-1}\phi$ or $\check{\phi}$ the inverse Fourier transform of the function $\phi(\xi)$, indeed $\check{\phi}(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{i\xi x}\phi(\xi)d\xi$. In what follows we also use the following relation $|\partial_x| = \mathcal{F}^{-1}|\xi|\mathcal{F} = -\mathcal{H}\partial_x$. The Hilbert transformation $\mathcal{H}$ with respect to the variable $x$ is defined as follows

$$\mathcal{H}\phi(x) = \frac{1}{\pi}\mathrm{Pv}\int_{\mathbb{R}} \frac{\phi(z)}{x-z}dz = -i\mathcal{F}^{-1}\frac{\xi}{|\xi|}\mathcal{F}\phi,$$

where Pv means the principal value of the singular integral. We widely use the fact that the Hilbert transformation $\mathcal{H}$ is a bounded operator from $\mathcal{L}^2(\mathbb{R})$ to $\mathcal{L}^2(\mathbb{R})$. The fractional derivative $|\partial_x|^\alpha$, $\alpha \in (0,1)$ is equal to

$$|\partial_x|^\alpha\phi = \mathcal{F}^{-1}|\xi|^\alpha\mathcal{F}\phi = C\int_{\mathbb{R}} (\phi(x+z)-\phi(x))\frac{dz}{|z|^{1+\alpha}}$$

and similarly we have

$$|\partial_x|^\alpha \mathcal{H}\phi = -i\mathcal{F}^{-1}\mathrm{sign}\xi|\xi|^\alpha \mathcal{F}\phi = C\int_{\mathbb{R}} (\phi(x+z)-\phi(x))\,\frac{dz}{z|z|^\alpha},$$

with some constant $C$ (see [21] for the constants $C$, p. 160 and p. 161 n 6.15). Let $\mathcal{J} = \mathcal{J}(t) = x + 2it\partial_x = M(t)(2it\partial_x)M(-t)$, where $M = M(t) = \exp(ix^2/4t)$. We also freely use the following identities $[\mathcal{J},\partial_x] = -1$, $[\mathcal{L},\mathcal{J}] = 0$, where $\mathcal{L} = i\partial_t + \partial_x^2$ and $[A,B] = AB - BA$. To the operator $\mathcal{J}$, we associate the space $\mathfrak{Z}^k(\mathbb{R})$ for $k \in \mathbb{N}$ defined by

$$\mathfrak{Z}^k(\mathbb{R}) = \left\{\varphi \in \mathcal{S}'(\mathbb{R}) \mid \left\|(2\imath t\partial_x)^k M(-t)\varphi\right\| + \|\varphi\| < +\infty\right\}.$$

We also define for $s \in \mathbb{R}$

$$\mathcal{F}((1-\partial_x^2)^s\phi)(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}(1+y^2)^s\phi(y)e^{\imath x\cdot y}dy.$$

For $(X_1,\dots,X_p) \in \mathbb{C}^p$, $\mathbf{X}$ denotes $\mathbf{X} = (X_1,\dots,X_p)$. Then with this notation, for $\mathbf{h} = (h_1,\dots,h_p) \in \mathcal{L}(\mathbb{R})^p$ one has $\|\mathbf{h}\|^2 = \sum_{k=1}^p \|h_k\|^2$. Different positive constants might be denoted by the same letter $C$, when it does not cause any confusion.

## §2. Linear smoothing effect

The aim of this section is to obtain some smoothing effects for solutions to the Cauchy problem for the linear Schrödinger equation

$$\begin{cases} \imath u_t + u_{xx} = f(x,t), & x \in \mathbb{R}, \quad t \in \mathbb{R} \\ u(x,0) = u_0(x)\,, & x \in \mathbb{R}, \end{cases} \tag{2.1}$$

where the function $f(x,t)$ is a force. Below in Section 3, we will consider the nonlinearity introduced in the introduction. Lemma 2.3 and lemma 2.4 were proved by N. Hayashi and P.I. Naumkin and the author [11], they are reproduced in this part for the sake of completeness of this paper. Lemma 2.3 describe a simple and explicit modifications of the smoothing effects obtained by Doi [6].

In the next lemma, it is proved that the commutator $[\mathcal{J}^k,(1-\partial_x^2)^{-1}]$ is a bounded operator from $\mathfrak{Z}^k(\mathbb{R})$ to $\mathfrak{Z}^k(\mathbb{R})$ where $k \in \mathbb{N}$.

**Lemma 2.1.** *For any $k \in \mathbb{N}$ one has the commutator relations*

$$a)\qquad [\mathcal{J},(1-\partial_x^2)^{-k}] = -(k+1)\partial_x\,(1-\partial_x^2)^{-(k+1)},$$

$$b)\qquad [\mathcal{J}^k,(1-\partial_x^2)^{-1}] = \sum_{j=1}^{k}\frac{p_{j,k}(\partial_x)}{(1-\partial_x^2)^{j+1}}\mathcal{J}^{k-j},$$

*where $p_{j,k}(X)$ is a polynomial whose order is less than or equal to $j$. These commutators are continuous operators in $\mathfrak{Z}^k(\mathbb{R})$.*

*Proof of lemma 2.1.* We start by proving this relation

$$\mathcal{J}(1-\partial_x^2)^{-1} = (1-\partial_x^2)^{-1}\mathcal{J} - 2\partial_x(1-\partial_x^2)^{-2}. \tag{2.2}$$

By definition of $\mathcal{J}$ and by inverse Fourier Transform, for any $f \in \mathcal{S}(\mathbb{R})$ and $x \in \mathbb{R}$

$$\begin{aligned}\mathcal{J}(1-\partial_x^2)^{-1}f(x) =& \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \frac{xe^{\imath xy}}{(1+y^2)}\hat{f}(y)\,dy \quad + \frac{2\imath t}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{e^{\imath xy}}{(1+y^2)}(\imath y\hat{f}(y))dy\\ =& \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{\imath xy}\left(\frac{\imath\partial_y\hat{f}(y)}{(1+y^2)} - \frac{2y\imath}{(1+y^2)^2}\hat{f}(y)\right)dy\\ &+ \frac{2\imath t}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{e^{\imath xy}}{(1+y^2)}(\imath y\hat{f}(y))dy\\ =& \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{e^{\imath xy}}{(1+y^2)}\left(\imath\partial_y\hat{f}(y) + 2\imath t(\imath y)\hat{f}(y)\right)dy\\ &+ \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left(\frac{-2y\imath}{(1+y^2)^2}\right)\hat{f}(y)dy.\end{aligned}$$

Hence by inverse Fourier transform, (2.2) is infered. The first part of lemma 2.1 is proved straightly by induction and (2.2). We shall also prove the second equality of this lemma by induction. Let us assume that the result is true for $k \in \mathbb{N}$, then applying $\mathcal{J}$ to this relation and equation (2.2) imply

$$\begin{aligned}&\mathcal{J}^{k+1}(1-\partial_x^2)^{-1}g\\ &= \mathcal{J}(1-\partial_x^2)^{-1}\mathcal{J}^k g + \sum_{j=1}^{k}\mathcal{J}(1-\partial_x^2)^{-j-1}p_{j,k}(\partial_x)\mathcal{J}^{k-j}g\\ &= (1-\partial_x^2)^{-1}\mathcal{J}^{k+1}g - 2(1-\partial_x^2)^{-2}\partial_x\mathcal{J}^k g -\\ &\sum_{j=1}^{k}(j+2)(1-\partial_x^2)^{-j-2}\partial_x p_{j,k}(\partial_x)\mathcal{J}^{k-j}g + \sum_{j=1}^{k}(1-\partial_x^2)^{-j-1}\mathcal{J}p_{j,k}(\partial_x)\mathcal{J}^{k-j}g.\end{aligned}$$

But $[\mathcal{J},\partial_x] = -1$ implies $\mathcal{J}p_{j,k}(\partial_x) = p_{j,k}(\partial_x)\mathcal{J} - p'_{j,k}(\partial_x)$. Then

$$\begin{aligned}&\mathcal{J}^{k+1}(1-\partial_x^2)^{-1}g = (1-\partial_x^2)^{-1}\mathcal{J}^{k+1}g - 2(1-\partial_x^2)^{-2}\partial_x\mathcal{J}^k g\\ &+ \sum_{j=2}^{k+1} -(j+1)(1-\partial_x^2)^{-j-1}\partial_x p_{j-1,k}(\partial_x)\mathcal{J}^{k+1-j}g\\ &+ \sum_{j=1}^{k}(1-\partial_x^2)^{-j-1}p_{j,k}(\partial_x)\mathcal{J}^{k+1-j}g - \sum_{j=1}^{k}(1-\partial_x^2)^{-j-1}p'_{j,k}(\partial_x)\mathcal{J}^{k-j}g,\end{aligned}$$

so

$$\begin{aligned}\mathcal{J}^{k+1}(1-\partial_x^2)^{-1}g &= (1-\partial_x^2)^{-1}\mathcal{J}^{k+1}g + \left\{(1-\partial_x^2)^{-2}(-2\partial_x + p_{1,k}(\partial_x))\right\}\mathcal{J}^k g\\ &- (1-\partial_x^2)^{-k-2}\{-(k+2)\partial_x p_{k,k}(\partial_x) + (1-\partial_x^2)p'_{k,k}(\partial_x)\}g\\ &+ \sum_{j=2}^{k}(1-\partial_x^2)^{-j-1}\left(-(j+1)\partial_x p_{j-1,k}(\partial_x) + p_{j,k}(\partial_x) - (1-\partial_x^2)p'_{j-1,k}(\partial_x)\right)\\ &\quad \mathcal{J}^{k+1-j}g.\end{aligned}$$

Then the following polynomials are defined

- $p_{1,k+1}(X) = -2X + p_{1,k}(X)$,
- $p_{j,k+1}(X) = -(j+1)Xp_{j-1,k}(X) + p_{j,k}(X) - (1-X^2)p'_{j,k}(X)$ for $k \geq j \geq 2$,
- $p_{k+1,k+1}(X) = -(k+2)Xp_{k,k}(X) - (1-X^2)p'_{k,k}(X)$.

One notices that the order of the $p_{j,k+1}(X)$ is less than or equal to $j$, then $\left\|(1-\partial_x^2)^{-j-1}p_{j,k+1}(\partial_x)g\right\| \leq C\,\|g\|$. So lemma 2.1 is proved. □

In the next lemma we prove that if $\phi$ is a sufficiently smooth function, then the commutators $[|\partial_x|^\alpha, \phi]$, and $[|\partial_x|^\alpha\mathcal{H}, \phi]$, are continuous operators from $\mathcal{L}^2(\mathbb{R})$ to $\mathcal{L}^2(\mathbb{R})$.

**Lemma 2.2.** *The following inequalities*

$$\|[|\partial_x|^\alpha, \phi]\,\psi\| \leq C\,\|\phi\|_{1,0,\infty}\,\|\psi\| \quad \textit{and} \quad \|[|\partial_x|^\alpha\mathcal{H}, \phi]\,\psi\| \leq C\,\|\phi\|_{1,0,\infty}\,\|\psi\|$$

*are valid, provided that the right hand sides are bounded.*

*Proof.* We have

$$\begin{aligned}\|[|\partial_x|^\alpha, \phi]\,\psi\| &= \left\|C\int_{\mathbb{R}}\psi(x+z)(\phi(x+z)-\phi(x))\frac{dz}{|z|^{1+\alpha}}\right\|\\ &\leq C\left\|\int_{|z|\leq 1}|\psi(x+z)|\int_0^z|\phi_x(x+\xi)|d\xi\frac{dz}{|z|^{1+\alpha}}\right\|\\ &\quad + C\|\phi\|_\infty\left\|\int_{|z|>1}|\psi(x+z)|\frac{dz}{|z|^{1+\alpha}}\right\|\\ &\leq C\,\|\phi\|_{1,0,\infty}\left\|\int_{\mathbb{R}}|\psi(x+z)|\frac{dz}{|z|^\alpha(1+|z|)}\right\| \leq C\,\|\phi\|_{1,0,\infty}\,\|\psi\|.\end{aligned}$$

The commutators $[|\partial_x|^\alpha\mathcal{H}, \phi]$ are estimated in the same way. Lemma 2.2 is proved. □

We define the smoothing operator used in [11] $\mathcal{S}(\varphi) = \cosh(\varphi) + i\sinh(\varphi)\mathcal{H}$, where the real-valued function $\varphi(x,t) \in \mathcal{L}^\infty\left([0,T];\mathcal{H}^{2,0}_\infty\right) \cap \mathcal{C}^1\left([0,T];\mathcal{L}^\infty(\mathbb{R})\right)$ and is positive. From its definition we easily see that the operator $\mathcal{S}$ acts continuously from $\mathcal{L}^2(\mathbb{R})$ to$\mathcal{L}^2(\mathbb{R})$ with the following estimate $\|\mathcal{S}(\varphi)\psi\| \le 2\exp\left(\|\varphi\|_\infty\right)\|\psi\|$. The inverse operator $\mathcal{S}^{-1}(\varphi) = \frac{1}{\cosh(\varphi)}\left(1 + i\tanh(\varphi)\mathcal{H}\right)^{-1}$ also exists and is continuous

$$\|\mathcal{S}^{-1}(\varphi)\psi\| \le \left(1 - \tanh(\|\varphi\|_\infty)\right)^{-1}\|\psi\| \le \exp(\|\varphi\|_\infty)\|\psi\|. \tag{2.3}$$

The operator $\mathcal{S}$ helps us to obtain a smoothing property of the Schrödinger-type equation (2.1) by virtue of the usual energy estimates. In the next lemma we prepare an energy estimate, involving the operator $\mathcal{S}$, in which we have an additional positive term giving us the norm of the half derivative of the unknown function $u$. We also assume that $\varphi(x)$ is written as $\varphi(x) = \partial_x^{-1}(\omega^2)$, so that $\omega(x) = \sqrt{(\partial_x\varphi)}$.

**Lemma 2.3.** *The following inequality*

$$\begin{aligned}\frac{d}{dt}\|\mathcal{S}u\|^2 + \left\|\omega\mathcal{S}\sqrt{|\partial_x|}u\right\|^2 &\le 2\left|Im(\mathcal{S}u,\mathcal{S}f)\right| \\ &+ C\|u\|^2 e^{2\|\varphi\|_\infty}\left(\|\omega\|_\infty^4 + \|\omega\|_\infty^6 + \|\omega\|_{1,0,\infty}\|\omega\|_\infty + \|\varphi_t\|_\infty\right)\end{aligned}$$

*is valid for the solution $u$ of the Cauchy problem (2.1).*

*Proof.* Multiplying equation (2.1) by the operator $\mathcal{S}(\varphi)$ we get

$$(i\partial_t + \partial_x^2)\mathcal{S}(\varphi)u - [\partial_x^2, \mathcal{S}(\varphi)]u - i[\partial_t, \mathcal{S}(\varphi)]u = \mathcal{S}(\varphi)f. \tag{2.4}$$

Via the property $(i\mathcal{H})^2 = 1$ we have $[\partial_x, \mathcal{S}(\varphi)] = i(\partial_x\varphi)\mathcal{S}(\varphi)\mathcal{H}$. Hence the Leibnitz rule yields

$$[\partial_x^2, \mathcal{S}(\varphi)] = -2i(\partial_x\varphi)\mathcal{S}(\varphi)|\partial_x| + (\partial_x\varphi)^2\mathcal{S}(\varphi) + i(\partial_x^2\varphi)\mathcal{S}(\varphi)\mathcal{H}.$$

Similarly we have $[\partial_t, \mathcal{S}(\varphi)] = i(\partial_t\varphi)\mathcal{S}(\varphi)\mathcal{H}$. Therefore (2.4) yields

$$(i\partial_t + \partial_x^2)\mathcal{S}u + \mathcal{M}u = \mathcal{R}u + \mathcal{S}f, \tag{2.5}$$

where $\mathcal{M} = -2i(\partial_x\varphi)\mathcal{S}(\varphi)|\partial_x| = -2i\omega^2\mathcal{S}(\varphi)|\partial_x|$ and

$$\mathcal{R} = \left(\omega^4\mathcal{S}(\varphi) + 2i\omega(\partial_x\omega)\mathcal{S}(\varphi)\mathcal{H} - (\partial_t\varphi)\mathcal{S}(\varphi)\mathcal{H}\right)$$

Since $|\partial_x| = -\partial_x\mathcal{H}$ the remainder term $\mathcal{R}$ is a bounded operator. Indeed we have

$$\|\mathcal{R}u\| \le 4\|u\|\exp\left(\|\varphi\|_\infty\right)\left(\|\omega\|_\infty^4 + \|\omega\|_{1,0,\infty}\|\omega\|_\infty + \|\varphi_t\|_\infty\right). \tag{2.6}$$

Now we apply the usual energy method to (2.5) (i.e. we multiply (2.5) by $\overline{\mathcal{S}(\varphi)u}$ integrate over $\mathbb{R}$ and take the imaginary part of the result) to get

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{S}u\|^2 + \operatorname{Im}(\mathcal{S}u, \mathcal{M}u) \le |(\mathcal{S}u, u)| + |\operatorname{Im}(\mathcal{S}u, \mathcal{S}f)| \,. \tag{2.7}$$

Then using the estimates of Lemma 2.3 we obtain

$$\begin{aligned}
&\operatorname{Im}(\mathcal{S}u, \mathcal{M}u)\\
&= 2\left(\mathcal{S}u, \omega^2\mathcal{S}|\partial_x|u\right) = 2\left(\omega\mathcal{S}u, \partial_x\omega\mathcal{S}\mathcal{H}u - [\partial_x, \omega\mathcal{S}]\,\mathcal{H}u\right)\\
&= -2\left(\omega\mathcal{S}\sqrt{|\partial_x|}u + \left[\sqrt{|\partial_x|}, \omega\mathcal{S}\right]u, -\omega\mathcal{S}\sqrt{|\partial_x|}u + \left[\sqrt{|\partial_x|}\mathcal{H}, \omega\mathcal{S}\right]\mathcal{H}u\right)\\
&\quad - 2\left(\omega\mathcal{S}u, [\partial_x, \omega\mathcal{S}]\,\mathcal{H}u\right) \ge 2\Bigg(\left\|\omega\mathcal{S}\sqrt{|\partial_x|}u\right\|^2\\
&\qquad - \left\|\omega\mathcal{S}\sqrt{|\partial_x|}u\right\|\left(\left\|\left[\sqrt{|\partial_x|}, \omega\mathcal{S}\right]u\right\| + \left\|\left[\sqrt{|\partial_x|}\mathcal{H}, \omega\mathcal{S}\right]\mathcal{H}u\right\|\right)\\
&\qquad - \left\|\left[\sqrt{|\partial_x|}, \omega\mathcal{S}\right]u\right\|\left\|\left[\sqrt{|\partial_x|}\mathcal{H}, \omega\mathcal{S}\right]\mathcal{H}u\right\| - \left|\left(\omega\mathcal{S}u, [\partial_x, \omega\mathcal{S}]\,\mathcal{H}u\right)\right|\Bigg)\\
&\ge \left\|\omega\mathcal{S}\sqrt{|\partial_x|}u\right\|^2 - C\|u\|^2 e^{2\|\varphi\|_\infty}\left(\|\omega\|_\infty^4 + \|\omega\|_\infty^6 + \|\omega\|_\infty\|\omega\|_{1,0,\infty}\right).
\end{aligned}\tag{2.8}$$

We have the lemma from (2.6) - (2.8). ∎

In the next lemma we prepare the estimate for the nonlinearity.

**Lemma 2.4.** *We have the following estimates*

$$\begin{aligned}
&|(\mathcal{S}u, \mathcal{S}\phi\psi\partial_x v)| \le \left\||\phi|\mathcal{S}\sqrt{|\partial_x|}u\right\|^2 + \left\||\psi|\mathcal{S}\sqrt{|\partial_x|}v\right\|^2\\
&+ C(\|u\|^2 + \|v\|^2)e^{6\|\varphi\|_\infty}\left(\|\phi\|_{1,0,\infty}^2 + \|\psi\|_{1,0,\infty}^2\right)\left(1 + \|\varphi\|_{1,0,\infty}^2\right),
\end{aligned}\tag{2.9}$$

*provided that the right hand sides are bounded.*

*Proof.* Since $\psi\partial_x = \partial_x\psi - \psi_x$, we get the estimate

$$\|\mathcal{S}\left(\phi\psi\partial_x v\right) - \mathcal{S}\left(\phi\partial_x\psi v\right)\| \le Ce^{2\|\varphi\|_\infty}\|\phi\|_\infty\|\psi_x\|_\infty\|v\|.$$

Using the identity

$$\mathcal{H}\left(\phi\sqrt{|\partial_x|}\right) - \sqrt{|\partial_x|}\left(\phi\mathcal{H}\right) = -\left[\sqrt{|\partial_x|}, \phi\right]\mathcal{H} + \left[\sqrt{|\partial_x|}\mathcal{H}, \phi\right] - \mathcal{H}\left[\sqrt{|\partial_x|}, \phi\right],$$

we write the representation

$$\begin{aligned}
&\mathcal{S}\phi\sqrt{|\partial_x|} = \sqrt{|\partial_x|}\phi\mathcal{S} - \left[\sqrt{|\partial_x|}, \cosh(\varphi)\phi\right] - i\left[\sqrt{|\partial_x|}, \sinh(\varphi)\right]\phi\mathcal{H}\\
&+ i\sinh(\varphi)\left(\mathcal{H}\phi\sqrt{|\partial_x|} - \sqrt{|\partial_x|}\phi\mathcal{H}\right) = \sqrt{|\partial_x|}\phi\mathcal{S} - i\sinh(\varphi)\mathcal{H}\left[\sqrt{|\partial_x|}, \phi\right] + \mathcal{R},
\end{aligned}$$

where the remainder operator

$$\begin{aligned}\mathcal{R} =& i\sinh(\varphi)\left[\sqrt{|\partial_x|}\mathcal{H},\phi\right] - i\sinh(\varphi)\left[\sqrt{|\partial_x|},\phi\right]\mathcal{H}\\ &-\left[\sqrt{|\partial_x|},\cosh(\varphi)\phi\right] - i\left[\sqrt{|\partial_x|},\sinh(\varphi)\right]\phi\mathcal{H}.\end{aligned}$$

Via the estimates of Lemma 2.3 we have for $\mathcal{R}$

$$\|\mathcal{R}w\| \le Ce^{\|\varphi\|_\infty}\|\phi\|_{1,0,\infty}\left(1+\|\varphi\|_{1,0,\infty}\right)\big(\|w\|+\|\mathcal{H}w\|\big). \tag{2.10}$$

We note that

$$\mathcal{S}\sqrt{|\partial_x|}\mathcal{H}\psi = \psi\mathcal{S}\sqrt{|\partial_x|}\mathcal{H} + \cosh(\varphi)\left[\sqrt{|\partial_x|}\mathcal{H},\psi\right] + i\sinh(\varphi)\left[\sqrt{|\partial_x|},\psi\right].$$

Therefore by Lemma 2.3 we find

$$\left\|\mathcal{S}\sqrt{|\partial_x|}\mathcal{H}\psi w\right\| \le \left\|\psi\mathcal{S}\sqrt{|\partial_x|}\mathcal{H}w\right\| + Ce^{2\|\varphi\|_\infty}\|\psi\|_{1,0,\infty}\|w\|. \tag{2.11}$$

By virtue of Lemma 2.3 we have the estimate

(2.12)
$$\begin{aligned}\left\|g\mathcal{S}\sqrt{|\partial_x|}\mathcal{H}u\right\| &\le \left\|\sqrt{|\partial_x|}\mathcal{H}g\mathcal{S}u\right\| + \left\|\left[\sqrt{|\partial_x|}\mathcal{H},g\mathcal{S}\right]u\right\|\\ &\le \left\|g\mathcal{S}\sqrt{|\partial_x|}u\right\| + \left\|\left[\sqrt{|\partial_x|},g\mathcal{S}\right]u\right\| + \left\|\left[\sqrt{|\partial_x|}\mathcal{H},g\mathcal{S}\right]u\right\|\\ &\le \left\|g\mathcal{S}\sqrt{|\partial_x|}u\right\| + C\|u\|\exp(\|\varphi\|_\infty)\|g\|_{1,0,\infty}\left(1+\|\varphi\|_{1,0,\infty}\right).\end{aligned}$$

We have

$$\begin{aligned}|(\mathcal{S}u,\mathcal{S}\phi\psi\partial_x v)| &\le |(\mathcal{S}u,\mathcal{S}\phi\partial_x\psi v)| + \|\mathcal{S}u\|\|\mathcal{S}\phi\psi_x v\|\\ &\le |(\mathcal{S}u,\mathcal{S}\phi\partial_x\psi v)| + Ce^{2\|\varphi\|_\infty}\|\phi\|_\infty\|\psi\|_{1,0,\infty}\|u\|\|v\|.\end{aligned}$$

Then via the estimate (2.10) and (2.11) with $w=\sqrt{|\partial_x|}\mathcal{H}\psi v$ we get

(2.13)
$$\begin{aligned}&|(\mathcal{S}u,\mathcal{S}\phi\partial_x\psi v)| = \left|\left(\mathcal{S}u,\mathcal{S}\phi\sqrt{|\partial_x|}\sqrt{|\partial_x|}\mathcal{H}\psi v\right)\right|\\ &\le \left|\left(\phi\sqrt{|\partial_x|}\mathcal{S}u,\mathcal{S}\sqrt{|\partial_x|}\mathcal{H}\psi v\right)\right|\\ &\quad + \left|\left(\mathcal{H}\sinh(\varphi)\mathcal{S}u,\left[\sqrt{|\partial_x|},\phi\right]\sqrt{|\partial_x|}\mathcal{H}\psi v\right)\right| + \|\mathcal{S}u\|\left\|\sqrt{|\partial_x|}\mathcal{H}\psi v\right\|\\ &\le \left(\left\|\phi\sqrt{|\partial_x|}\mathcal{S}u\right\| + C\|u\|e^{3\|\varphi\|_\infty}\|\phi\|_{1,0,\infty}\left(1+\|\varphi\|_{1,0,\infty}\right)\right)\left\|\mathcal{S}\sqrt{|\partial_x|}\mathcal{H}\psi v\right\|,\end{aligned}$$

whence using (2.11) with $w=v$, and (2.12) with $g=\psi$, we find

$$\begin{aligned}|(\mathcal{S}u,\mathcal{S}\phi\partial_x\psi v)| &\le \left(\left\|\bar{\phi}\mathcal{S}\sqrt{|\partial_x|}u\right\| + C\|u\|e^{3\|\varphi\|_\infty}\|\phi\|_{1,0,\infty}\left(1+\|\varphi\|_{1,0,\infty}\right)\right)\\ &\quad\times\left(\left\|\psi\mathcal{S}\sqrt{|\partial_x|}v\right\| + C\|v\|e^{3\|\varphi\|_\infty}\|\psi\|_{1,0,\infty}\left(1+\|\varphi\|_{1,0,\infty}\right)\right).\end{aligned}$$

Then

$$
\begin{aligned}
|(\mathcal{S}u, \mathcal{S}\phi\partial_x\psi v)| \le & \left\|\bar{\phi}\mathcal{S}\sqrt{|\partial_x|}u\right\|^2 + \left\|\psi\mathcal{S}\sqrt{|\partial_x|}v\right\|^2 \\
& + C(\|u\|^2 + \|v\|^2)e^{6\|\varphi\|_\infty}\left(\|\phi\|_{1,0,\infty}^2 + \|\psi\|_{1,0,\infty}^2\right)\left(1 + \|\varphi\|_{1,0,\infty}^2\right).
\end{aligned}
$$

Thus the estimate of lemma 2.4 is proved.

## §3. Proof of the main theorem

Only the case $t > 0$ is considered since the case $t < 0$ can be treated similarly. Local existence of solutions is first achieved by a contraction mapping method.

**Lemma 3.1.** *We assume that the initial data satisfy* $u_0 \in \mathcal{H}^{3,l}$. *Then for some time* $T > 0$ *there exists a unique solution* $u$ *of the Cauchy problem (1.1) such that*

$$
\sup_{t\in[0,T]} \sum_{k=0}^{l} \|\mathcal{J}^k u(t)\|_{3,0} < +\infty.
$$

*Proof.* Applying the operator $(1 - \partial_x^2)$ to the equation (1.1), we get for the function $v = (1 - \partial_x^2)u$

$$
\begin{cases} Lv = \mathcal{N}_1(u, u_x)v_x + \mathcal{N}_2(u, u_x)\overline{v_x} + R_0(v) \\ v(x, 0) = (1 - \partial_x^2)u_0, \end{cases} \tag{3.1}
$$

where $\mathcal{N}_1(u, u_x) = \partial_{u_x}\mathcal{N}(u, u_x)$ and $\mathcal{N}_2(u, u_x) = \partial_{\overline{u_x}}\mathcal{N}(u, u_x)$.
The gist of this part is the combinaison of the smoothing operator defined in the previous section, its aim is to get rid of a half derivative in $\mathcal{N}_1(u, u_x)v_x$. But [11] showed that it is impossible to use it on $\mathcal{N}_2(u, u_x)\overline{v_x}$ without a smallness condition on the initial data. It is the reason why we use the diagonalization technique which allows us to conceal this kind of bad terms by a linear transform. The remainder $R_0(v)$ does not cause any problem that is to say there exists a constant $C > 0$ such that

$$
\sum_{k=0}^{l} \left\|\mathcal{J}^k R_0(v)\right\|_{1,0} \le C \sum_{k=0}^{l} \left\|\mathcal{J}^k v\right\|_{1,0}.
$$

From the definition of $\mathcal{N}$ we see that

$$
\mathcal{N}_1(u, u_x, \bar{u}, \overline{u_x}) = (K_2 u)\bar{u} + (K_4 u)\overline{u_x} + 2(K_5\bar{u})u_x + 2(K_6 u_x)\overline{u_x} \tag{3.2}
$$

$$
\mathcal{N}_2(u, u_x, \bar{u}, \overline{u_x}) = K_3 u^2 + (K_4 u)u_x + (K_6 u_x)u_x. \tag{3.3}
$$

Lemma 3.1 is achieved by a standard contraction mapping method. To aim this goal, let us consider the linearized version of equation (3.1) :

$$\begin{cases} Lv = \mathcal{N}_1(u^\dagger, u_x^\dagger)v_x + \mathcal{N}_2(u^\dagger, u_x^\dagger)\overline{v_x} + R_0(v) \\ v(x,0) = (1-\partial_x^2)u_0, \end{cases} \tag{3.4}$$

where the function $u^\dagger = (1-\partial_x^2)v^\dagger$ is defined by the known function $v^\dagger$ from the ball $\mathfrak{B}$ defined by

$$\mathfrak{B} = \Big\{ v^\dagger \in \mathcal{C}^1([0,T], \mathcal{L}^2(\mathbb{R}));\ \sup_{t\in[0,T]} \sum_{k=0}^{l} \left\| \mathcal{J}^k v^\dagger \right\| \le 2\rho,$$
$$\sup_{t\in[0,T]} \sum_{k=0}^{l} \left\| \mathcal{J}^k v^\dagger \right\|_{1,0} \le \mu;$$
$$\sum_{k=0}^{l} \sup_{t\in[0,T]} \left\| \partial_t \mathcal{J}^k u \right\| + \sup_{t\in[0,T]} \left\| \partial_t \partial_x (\mathcal{J}^k u) \right\| \le \kappa,$$
$$\sum_{k=0}^{l} \sup_{t\in[0,T]} (\|\partial_t \partial_x^{-1} |\mathcal{J}^k u^\dagger|^2\|_\infty + \|\partial_t \partial_x^{-1} |\mathcal{J}^k u_x^\dagger|^2\|_\infty) \le \nu \Big\},$$

with $\rho = \|u_0\|_{3,l}$, $\mu$ is a positive constant depending on $\rho$; $\nu$ and $\kappa$ are also positive constants but depending on both $\rho$ and $\mu$. $\mu$ and $\rho$ will be set later. The Cauchy problem (3.4) defines a mapping $\mathfrak{A} : \mathfrak{A}v = v^\dagger$. $\mathcal{J}$ can be considered as behaving like the derivative operator $\partial_x$ since we have the relation for $k \in \{1, \ldots, l\}$:

$$\mathcal{J}^k(\phi\psi\overline{w_x}) = \sum_{l_1=0}^{k} \left( \sum_{l_2=0}^{k-l_1} (-1)^{l_1} \binom{k}{l_1} \binom{k-l_1}{l_2} \mathcal{J}^{k-(l_1+l_2)}\phi \mathcal{J}^{l_2}\psi \right) \overline{\mathcal{J}^{l_1} w_x}, \tag{3.5}$$

where $\psi, \phi, w_x$ are functions such that $\mathcal{J}^{l_1}\phi$, $\mathcal{J}^{l_1}\psi$, $\mathcal{J}^{l_1}w_x$ have meaning for $0 \le l_1 \le l$. Hence by a classical energy method one has from the first line of equation (3.4)

$$\frac{d}{dt} \left\| \mathcal{J}^k v(t) \right\| \le C + C \left\| \mathcal{J}^k v_x(t) \right\|.$$

It implies

$$\sup_{t\in[0,T]} \sum_{k=0}^{l} \left\| \mathcal{J}^k v(t) \right\| \le \rho + \sqrt{T} C \sup_{t\in[0,T]} \sum_{k=0}^{l} \left\| \mathcal{J}^k v_x(t) \right\|, \tag{3.6}$$

where time $T > 0$ is choosen to be small enough. In order to obtain estimates of the norm $\sum_{k=0}^{l} \left\| \mathcal{J}^k v_x(t) \right\|$ we apply the so called "diagonalisation technique"

to get a system of equations such that the smoothing operator defined in [11] can be applied succesfully.
For $k \in \{0, \dots, l\}$ we apply $\partial_x \mathcal{J}^k$ to the first line of equation (3.4) and define $h_k = \partial_x \mathcal{J}^k v$ and $v_k = \mathcal{J}^k v$ thus one has

$$\begin{aligned} \imath\partial_t h_k + \partial_x^2 h_k =& \mathcal{N}_1(u^\dagger, u_x^\dagger)\partial_x h_k + (-1)^k \mathcal{N}_2(u^\dagger, u_x^\dagger)\partial_x \overline{h_k} \\ & + \sum_{j=0}^{k-1} \mathcal{C}_{k,j}(u^\dagger, u_x^\dagger)\partial_x(\overline{h_j}) + \mathcal{J}^k R_0(v^\dagger) + R_k(v^\dagger, v), \end{aligned} \tag{3.7}$$

where the coefficients $\mathcal{C}_{k,j}(u^\dagger, u_x^\dagger)$ are computed by equation (3.5) and $R_k(v^\dagger, v)$ is bounded in the way

$$\left\| R_k(v^\dagger, v) \right\| \leq C \left( \sum_{j=0}^{k} \left\| \mathcal{J}^j v^\dagger \right\|_{1,0} \right) \sum_{j=0}^{k} \left\| \mathcal{J}^j v \right\|_{1,0}.$$

We define now the derivative operators :

$$a_k = \mathcal{N}_1(u^\dagger, u_x^\dagger)\partial_x \;, \qquad b_k = \mathcal{N}_2(u^\dagger, u_x^\dagger)\partial_x \;, \qquad c_{k,j} = \mathcal{C}_{k,j}(u^\dagger, u_x^\dagger)\partial_x.$$

So equation (3.7) can be rewritten in this way for $0 \leq k \leq l$

$$\begin{cases} \imath\partial_t h_k + (\partial_x^2 - a_k)h_k - b_k\overline{h_k} - \sum_{j=0}^{k-1} c_{k,j}\overline{h_j} & = r_k \\ \imath\partial_t\overline{h_k} + (-\partial_x^2 + \overline{a_k})\overline{h_k} + \overline{b_k}h_k + \sum_{j=0}^{k-1} \overline{c_{k,j}}h_j & = -\overline{r_k}, \end{cases}$$

where $r_k = \mathcal{J}^k R_0(v^\dagger) + R_k(v^\dagger, v)$. If we define

$$\mathbf{h} = \begin{pmatrix} h_l \\ \overline{h_l} \\ \vdots \\ h_0 \\ \overline{h_0} \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} v_l \\ \overline{v_l} \\ \vdots \\ v_0 \\ \overline{v_0} \end{pmatrix} \text{ and } \mathbf{R} = \begin{pmatrix} r_l \\ \overline{r_l} \\ \vdots \\ r_0 \\ \overline{r_0} \end{pmatrix}.$$

The previous set of $(2l+2)$ equations gives the existence of a $(2l+2)$ matrix operator $\mathcal{G}$ such that the equation :

$$\imath\partial_t \mathbf{h} + \mathcal{G}\mathbf{h} \;=\; \mathbf{R}. \tag{3.8}$$

Precisely $\mathcal{G}$ is considered as a $(l+1)$ matrix operator whose components are $2 \times 2$ block matrix :
For two integers $0 \leq i, j \leq l$

- if $j < i$ then $\mathcal{G}_{i,j} = 0$,
- if $j = i$ then $\mathcal{G}_{i,j} = \begin{pmatrix} +\partial_x^2 - a_{l-i} & -b_{l-i} \\ \overline{b_{l-i}} & -\partial_x^2 + \overline{a_{l-i}} \end{pmatrix}$,
- if $j > i$ then $\mathcal{G}_{i,j} = \begin{pmatrix} 0 & -c_{l-i,l-j} \\ \overline{c_{l-i,l-j}} & 0 \end{pmatrix}$.

It means $\mathcal{G}$ is

(3.9)

$$\begin{pmatrix} \begin{pmatrix} \partial_x^2 - a_l & -b_l \\ \overline{b_l} & -\partial_x^2 + \overline{a_l} \end{pmatrix} & \begin{pmatrix} 0 & -c_{l,l-1} \\ \overline{c_{l,l-1}} & 0 \end{pmatrix} & \cdots & \begin{pmatrix} 0 & -c_{l,0} \\ \overline{c_{l,0}} & 0 \end{pmatrix} \\ & \ddots & \ddots & \vdots \\ 0 & & \ddots & \begin{pmatrix} 0 & -c_{1,0} \\ \overline{c_{1,0}} & 0 \end{pmatrix} \\ & & & \begin{pmatrix} \partial_x^2 - a_0 & -b_0 \\ \overline{b_0} & -\partial_x^2 + \overline{a_0} \end{pmatrix} \end{pmatrix}$$

We define now our transformation operators : for $k \in \{0, \dots, l\}$ and $k < j \leq l$

$$\begin{aligned} \Lambda_k &= \begin{pmatrix} 1 & -\frac{1}{2} b_k \partial_x^{-2} \\ -\frac{1}{2}\overline{b_k}\partial_x^{-2} & 1 \end{pmatrix}, \qquad \Lambda'_k = \begin{pmatrix} 1 & \frac{1}{2} b_k \partial_x^{-2} \\ \frac{1}{2}\overline{b_k}\partial_x^{-2} & 1 \end{pmatrix} \\ \Omega_{k,j} &= \begin{pmatrix} 0 & c_{k,j}\partial_x^{-2} \\ \overline{c_{k,j}}\partial_x^{-2} & 0 \end{pmatrix}, \qquad \Omega'_{k,j} = -\Omega_{k,j}. \end{aligned} \tag{3.10}$$

Then $\Lambda$ and $\Lambda'$ are the transformation operators (to be considered as a $(l+1)$ matrix whose elements are $2 \times 2$ matrix) defined by : for $0 \leq j, k \leq l$

- if $k = j$ then $\Lambda_{k,j} = \Lambda_{l-k}$ and $\Lambda'_{k,j} = \Lambda'_{l-k}$,
- if $j < k$ then $\Lambda_{k,j} = 0$ and $\Lambda'_{k,j} = 0$,
- if $k < j$ then $\Lambda_{k,j} = \Omega_{l-k,l-j}$ and $\Lambda'_{k,j} = \Omega'_{l-k,l-j}$,

$$\Lambda = \begin{pmatrix} \Lambda_l & \Omega_{l,l-1} & \cdots & \Omega_{l,0} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \Omega_{1,0} \\ 0 & & & \Lambda_0 \end{pmatrix} \quad \text{and } \Lambda' = \begin{pmatrix} \Lambda'_l & \Omega'_{l,l-1} & \cdots & \Omega'_{l,0} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \Omega'_{1,0} \\ 0 & & & \Lambda'_0 \end{pmatrix}.$$

We apply the operator $\Lambda$ to equation (3.8)

$$\imath\partial_t \Lambda \mathbf{h} + \Lambda\,\mathcal{G}\mathbf{h} = \Lambda \mathbf{R} + [\imath\partial_t, \Lambda]\,\mathbf{h}. \tag{3.11}$$

The next step is not the explicit calculation of the commutator $[\Lambda, \mathcal{G}]$ as it could be expected because it leads to heavy calculations. The direct computation of the matrix product $\Lambda'\Lambda$ show that it can be expressed as $\Lambda'\Lambda = I - W$ where $I$ is the unit matrix and $W$ is a non threatening matrix operator (it means that $W$ satisfies $\|W\mathbf{h}\| \leq C(\|\mathbf{h}\| + \|\mathbf{v}\|)$). We show now this point. We write

$$\Lambda'\Lambda = \begin{pmatrix} \Lambda_l'\Lambda_l & K_{l,l-1} & \cdots & K_{l,0} \\ & \ddots & & \vdots \\ & & \ddots & K_{1,0} \\ 0 & & & \Lambda_0'\Lambda_0 \end{pmatrix}.$$

Given $k \in \{0, \dots, l\}$

$$\Lambda_k'\Lambda_k = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{4} b_k \partial_x^{-2} \overline{b_k} \partial_x^{-2} & 0 \\ 0 & \frac{1}{4} \overline{b_k} \partial_x^{-2} b_k \partial_x^{-2} \end{pmatrix}.$$

One has

$$b_k \partial_x^{-2} \overline{b_k} \partial_x^{-2} h_k \ = \ \mathcal{N}_2(u^\dagger, u_x^\dagger) \partial_x^{-1} \overline{\mathcal{N}_2(u^\dagger, u_x^\dagger)} v_k.$$

Then

$$\begin{aligned} \left\| b_k \partial_x^{-2} \overline{b_k} \partial_x^{-2} h_k \right\| &\leq \left\| \mathcal{N}_2(u^\dagger, u_x^\dagger) \partial_x^{-1} (\overline{\mathcal{N}_2(u^\dagger, u_x^\dagger)} v_k) \right\| \\ &\leq \|\partial_x^{-1} (\overline{\mathcal{N}_2(u^\dagger, u_x^\dagger)} v_k)\|_\infty \left\| \mathcal{N}_2(u^\dagger, u_x^\dagger) \right\| \\ &\leq \|v_k\| \left\| \mathcal{N}_2(u^\dagger, u_x^\dagger) \right\| \left\| \mathcal{N}_2(u^\dagger, u_x^\dagger) \right\| \\ &\leq C(\rho) \|v_k\|. \end{aligned} \tag{3.12}$$

Therefore the diagonal terms of $\Lambda'\Lambda$ are harmless. Let us pay attention to the other terms. Standard matrix product's rules show that for any $i \in \{0, \dots, l\}$ and for any $j \in \{0, \dots, l-1\}$ satisfying $j > i$

$$K_{i,j} = \Lambda_i' \Omega_{i,j} + \sum_{k=1}^{i-(j+1)} \Omega_{i,i-k}' \Omega_{i-k,j} + \Omega_{i,j}' \Lambda_j.$$

For any $k \in \{1, \dots, i-(j+1)\}$,

$$\begin{aligned} \Omega_{i,i-k}' \Omega_{i-k,j} &= \begin{pmatrix} 0 & -c_{i,i-k} \partial_x^{-2} \\ -\overline{c_{i,i-k}} \partial_x^{-2} & 0 \end{pmatrix} \begin{pmatrix} 0 & -c_{i,i-k} \partial_x^{-2} \\ -\overline{c_{i,i-k}} \partial_x^{-2} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -c_{i,i-k} \partial_x^{-2} \overline{c_{i,i-k}} \partial_x^{-2} \\ -\overline{c_{i,i-k}} \partial_x^{-2} c_{i,i-k} \partial_x^{-2} & 0 \end{pmatrix}. \end{aligned}$$

So the next inequality is proved in the same way as in (3.12)

$$\left\|c_{i,i-k}\partial_x^{-2}\left(\overline{c_{i,i-k}}\partial_x^{-2}h_i\right)\right\| \leq C(\rho)\left\|v_i\right\|.$$

Hence $\Omega'_{i,i-k}\Omega_{i-k,j}$ is harmless too. By definition

$$\Lambda_i = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}0 & -\frac{b_i\partial_x^{-2}}{2}\\ -\frac{\overline{b_i}\partial_x^{-2}}{2} & 0\end{pmatrix}.$$

So

$$\Lambda'_i\Omega_{i,j} + \Omega'_{i,j}\Lambda'_j =$$
$$(\Omega_{i,j} + \Omega'_{i,j}) + \begin{pmatrix}0 & \frac{b_i\partial_x^{-2}}{2}\\ \frac{\overline{b_i}\partial_x^{-2}}{2} & 0\end{pmatrix}\Omega_{i,j} + \Omega'_{i,j}\begin{pmatrix}0 & -\frac{b_i\partial_x^{-2}}{2}\\ -\frac{\overline{b_i}\partial_x^{-2}}{2} & 0\end{pmatrix}.$$

As $\Omega_{i,j} = -\Omega'_{i,j}$ and the estimations $\left\|b_i\partial_x^{-2}(\overline{c_{i,j}}\partial_x^{-2}h_i)\right\| \leq C(\rho)\left\|v_i\right\|$ and $\left\|c_{i,j}\partial_x^{-2}(b_i\partial_x^{-2}h_i)\right\| \leq C(\rho)\left\|v_i\right\|$ can be obtained as (3.12), $\Lambda'_i\Omega_{i,j} + \Omega'_{i,j}\Lambda'_j$ is harmless too. If one defines $W$ by

$$W = \begin{pmatrix}W_l & K_{l,l-1} & \cdots & K_{l,0}\\ & \ddots & \ddots & \vdots\\ & & & K_{1,0}\\ 0 & & & W_0\end{pmatrix}$$

where

$$W_j = \begin{pmatrix}-\frac{1}{4}b_j\partial_x^{-2}\overline{b_j}\partial_x^{-2} & 0\\ 0 & -\frac{1}{4}\overline{b_j}\partial_x^{-2}b_j\partial_x^{-2}\end{pmatrix},$$

then above estimations imply that $\Lambda'\Lambda = I + W$ where $W$ satisfy $\|W\mathbf{h}\| \leq C(\rho)\|\mathbf{v}\|$. This identity is substituted into (3.11),

$$\imath\partial_t\Lambda\mathbf{h} + \Lambda\mathcal{G}\Lambda'\,\Lambda\mathbf{h} = \Lambda\mathbf{R} + \Lambda\mathcal{G}W\mathbf{h} + \imath[\partial_t,\Lambda]\mathbf{h}. \tag{3.13}$$

One by one, it is aimed at proving that all the terms in this equation are harmless and we want to transform it in an equation where the Schrödinger operator is obvious and where bad terms like $\mathcal{J}^k h_x$ are explictly excluded. The next term to estimate is $[\partial_t,\Lambda]$. We start by estimating its diagonal terms. For $k \in \{0,\ldots,l\}$ one has

$$\partial_t(b_k)\partial_x^{-2}h_k = (-1)^k\partial_t(\mathcal{N}_2(u^\dagger,u_x^\dagger))\partial_x^{-1}h_k.$$

Then

$$\left\|\partial_t(b_k)\partial_x^{-2}h_k\right\| \leq \left\|\partial_t(\mathcal{N}_2(u^\dagger,u_x^\dagger))\right\|\cdot\left\|\partial_x^{-1}h_k\right\|_\infty.$$

By definition of $\mathcal{N}_2$

$$\left\|\partial_t(\mathcal{N}_2(u^\dagger, u_x^\dagger))\right\| \leq C(\rho)(\left\|\partial_t u^\dagger\right\| + \left\|\partial_t u_x^\dagger\right\|),$$

as $u$ is in the ball $\mathfrak{B}$, one gets

$$\left\|\partial_t(\mathcal{N}_2(u^\dagger, u_x^\dagger))\right\| \leq \kappa C(\rho).$$

Then

$$\left\|\partial_t(b_k)\partial_x^{-2} h_k\right\| \leq \kappa C(\rho)\left\|h\right\|.$$

Now we turn our attention to the remaining elements of $[\partial_t, \Lambda]$. Once more by definition of $\mathcal{C}_{i,j}(u^\dagger, u_x^\dagger)$, one has for $0 \leq j < k$

$$\begin{aligned}\left\|\partial_t(\mathcal{C}_{i,j}(u^\dagger, u_x^\dagger))\right\| &\leq C(\rho)(\sum_{s=0}^{k}\left\|\partial_t \mathcal{J}^s u^\dagger\right\| + \left\|\partial_t \mathcal{J}^s u_x^\dagger\right\|),\\ &\leq C(\rho)\kappa\end{aligned}$$

as $c_{k,j}\partial_x^{-2} h_j = \mathcal{C}_{i,j}(u^\dagger, u_x^\dagger)\partial_x^{-1} h_k$, as above one gets

$$\left\|\partial_t(c_{k,j})\partial_x^{-2} h_j\right\| \leq C(\rho)\kappa\left\|h_j\right\|.$$

These inequalities lead to $\|[\partial_t, \Lambda]\mathbf{h}\| \leq C(\rho)\kappa\|h\|$. The operators $\mathcal{B}, \tilde{\mathcal{B}}, \tilde{\Delta}, \tilde{\mathcal{A}}$ are introduced :

$$\mathcal{B} = 2I - 2\Lambda$$

and

$$\tilde{\Delta} = \begin{pmatrix} \begin{pmatrix} \partial_x^2 & 0 \\ 0 & -\partial_x^2 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} \partial_x^2 & 0 \\ 0 & -\partial_x^2 \end{pmatrix} \end{pmatrix},$$

and

$$\tilde{\mathcal{A}} = \begin{pmatrix} \begin{pmatrix} a_l & 0 \\ 0 & -\overline{a_l} \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} a_0 & 0 \\ 0 & -\overline{a_0} \end{pmatrix} \end{pmatrix},$$

last one is

$$\tilde{\mathcal{B}} = \begin{pmatrix} \begin{pmatrix} 0 & b_l \\ -\overline{b_l} & 0 \end{pmatrix} & \begin{pmatrix} 0 & c_{l,l-1} \\ -\overline{c_{l,l-1}} & 0 \end{pmatrix} & \cdots & \begin{pmatrix} 0 & c_{l,0} \\ -\overline{c_{l,0}} & 0 \end{pmatrix} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \begin{pmatrix} 0 & c_{1,0} \\ -\overline{c_{1,0}} & 0 \end{pmatrix} \\ 0 & & & \begin{pmatrix} 0 & b_0 \\ -\overline{b_0} & 0 \end{pmatrix} \end{pmatrix}$$

So $\mathcal{G}$ is decomposed in $\mathcal{G} = \tilde{\Delta} - \tilde{\mathcal{A}} - \tilde{\mathcal{B}}$. Then

$$\begin{aligned} \Lambda\mathcal{G}W &= (I - \frac{1}{2}\mathcal{B})(\tilde{\Delta} - \tilde{\mathcal{A}} - \tilde{\mathcal{B}})W \\ &= (I - \frac{1}{2}\mathcal{B})(\tilde{\Delta}W - \tilde{\mathcal{B}}W - \tilde{\mathcal{A}}W) \\ &= \left(\tilde{\Delta}W - \tilde{\mathcal{B}}W - \tilde{\mathcal{A}}W\right) - \frac{1}{2}\mathcal{B}\tilde{\Delta}W + \frac{1}{2}\mathcal{B}\tilde{\mathcal{B}}W + \frac{1}{2}\mathcal{B}\tilde{\mathcal{A}}W. \end{aligned}$$

In this equation, the most difficult term to estimate is $\tilde{\Delta}W$. We consider only its diagonal terms because estimating the other ones is done in the same way. One has for any $k \in \{0, \dots, l\}$

$$\begin{aligned} \partial_x^2 b_k \partial_x^{-2} \overline{b_k} \partial_x^{-2} h_k =& ((-1)^k \partial_x^2 \mathcal{N}_2(u^\dagger, u_x^\dagger))\partial_x^{-1} \overline{b_k} \partial_x^{-2} h_k + (-1)^k b_k \overline{b_k} \partial_x^{-2} h_k \\ &+ 2((-1)^k \partial_x \mathcal{N}_2(u^\dagger, u_x^\dagger))\overline{b_k} \partial_x^{-2} h_k. \end{aligned}$$

The most annoying term to estimate is

$$\begin{aligned} b_k \overline{b_k} \partial_x^{-2} h_k &= \mathcal{N}_2(u^\dagger, u_x^\dagger)\partial_x \left(\overline{\mathcal{N}_2(u^\dagger, u_x^\dagger)} \partial_x^{-1} h_k\right) \\ &= \mathcal{N}_2(u^\dagger, u_x^\dagger)(\partial_x \overline{\mathcal{N}_2(u^\dagger, u_x^\dagger)})v_k + \mathcal{N}_2(u^\dagger, u_x^\dagger)\overline{\mathcal{N}_2(u^\dagger, u_x^\dagger)} h_k. \end{aligned}$$

Thus

$$\begin{aligned} \left\| b_k \overline{b_k} \partial_x^{-2} h_k \right\| &\leq C(\mu) \|v_k\| + C(\rho)^2 \|h_k\| \\ &\leq (C(\mu) + C(\rho)^2) \|v_k\|. \end{aligned}$$

As

$$(\partial_x \mathcal{N}_2(u^\dagger, u_x^\dagger))\overline{b_k} \partial_x^{-2} h_k = (\partial_x \mathcal{N}_2(u^\dagger, u_x^\dagger))\overline{\mathcal{N}_2(u^\dagger, u_x^\dagger)} \partial_x^{-1} h_k,$$

one gets

$$\left\| (\partial_x \mathcal{N}_2(u^\dagger, u_x^\dagger))\overline{b_k} \partial_x^{-2} h_k \right\| \leq C(\rho) \|h_k\|.$$

Next term to estimate is $(\partial_x^2 \mathcal{N}_2(u^\dagger, u_x^\dagger))\partial_x^{-1} \overline{b_k} \partial_x^{-2} h_k$. The highest derivative term in $\partial_x^2 \mathcal{N}_2(u^\dagger, u_x^\dagger)$ is $u_{xxx}^\dagger$ and if $\partial_x^2(\mathcal{N}_2)$ can be considered as a polynomial with $u^\dagger_{xxx}$ as a variable, its degree is one. It means that $\partial_x^2(\mathcal{N}_2(u^\dagger,u^\dagger_x)) = u^\dagger_{xxx} f(u^\dagger,u^\dagger_x,u^\dagger_{xx})$ where $f$ is a continuous function. Hence by Cauchy-Schwarz and Sobolev's inequalities

$$\begin{aligned}\left\|(\partial_x^2\mathcal{N}_2(u^\dagger,u^\dagger_x))\partial_x^{-1}\overline{b_k}\partial_x^{-2}h_k\right\| &\le C(\mu)\left\|u^\dagger_{xxx}\right\|\cdot\left\|\partial_x^{-1}(\mathcal{N}_2(u^\dagger,u^\dagger_x)v_k)\right\|_\infty\\ &\le C(\mu)^2\left\|\mathcal{N}_2(u^\dagger,u^\dagger_x)\right\|\|v_k\|\\ &\le C(\mu)^2C(\rho)^2\|v_k\|.\end{aligned}$$

From these estimates one infers :

$$\left\|\partial_x^2 b_k\partial_x^{-2}\overline{b_k}\partial_x^{-2}h_k\right\| \le (C(\rho)+C(\mu))\|h_k\|.$$

It implies $\left\|\tilde\Delta W\mathbf{h}\right\| \le (C(\rho)+C(\mu))\|\mathbf{h}\|$. But the inequalities $\left\|\mathcal{B}\tilde\Delta W\mathbf{h}\right\| \le C(\mu)\|h\|$ and $\left\|\mathcal{B}\tilde{\mathcal{B}}W\mathbf{h}\right\| \le C(\mu)\|\mathbf{h}\|$ and $\left\|\mathcal{B}\tilde{\mathcal{A}}W\mathbf{h}\right\| \le C(\mu)\|\mathbf{h}\|$ are obtained similarly, it means $\|\Lambda\mathcal{G}W\mathbf{h}\| \le C(\mu)\|\mathbf{h}\|$. We focus our mind now on $\Lambda' G\Lambda$ in order to find a suitable decomposition of this matrix operator. By definition,

$$\begin{aligned}\Lambda\mathcal{G}\Lambda' &= (I-\frac{1}{2}\mathcal{B})(\tilde\Delta-\tilde{\mathcal{B}}-\tilde{\mathcal{A}})(I+\frac{1}{2}\mathcal{B})\\ &= (I-\frac{1}{2}\mathcal{B})(\tilde\Delta+\frac{1}{2}\tilde\Delta\mathcal{B}-\tilde{\mathcal{B}}-\frac{1}{2}\tilde{\mathcal{B}}\mathcal{B}-\tilde{\mathcal{A}}-\frac{1}{2}\tilde{\mathcal{A}}\mathcal{B})\\ &= (\tilde\Delta-\tilde{\mathcal{A}})+(\frac{1}{2}\tilde\Delta\mathcal{B}-\tilde{\mathcal{B}}-\frac{1}{2}\mathcal{B}\tilde\Delta)+\mathcal{R}_a.\end{aligned}\tag{3.14}$$

where $\mathcal{R}_a$ is a matrix operator such that $\|\mathcal{R}_a\mathbf{h}\| \le C(\mu)\|\mathbf{h}\|$. One can check that

$$\frac{1}{2}\tilde\Delta\mathcal{B} = \frac{1}{2}\tilde{\mathcal{B}}+\mathcal{R}_b \quad\text{and}\quad \mathcal{B}\tilde\Delta = -\tilde{\mathcal{B}},$$

where $\mathcal{R}_b$ satisfies $\|\mathcal{R}_b\mathbf{h}\| \le C(\mu)\|\mathbf{h}\|$ too. Hence (3.14) is simplified :

$$\Lambda\mathcal{G}\Lambda' = (\tilde\Delta-\tilde{\mathcal{A}})+\mathcal{R}_a+\mathcal{R}_b$$

So equation (3.13) can be written in the following way ($\mathcal{R}_c = \mathcal{R}_a+\mathcal{R}_b$):

$$\imath\partial_t\Lambda\mathbf{h}+(\tilde\Delta-\tilde{\mathcal{A}})\Lambda\mathbf{h} = \Lambda\mathbf{R}-\Lambda\mathcal{G}W\mathbf{h}+\mathcal{R}_c\Lambda\mathbf{h}+\imath[\partial_t,\Lambda]\mathbf{h}.\tag{3.15}$$

The diagonalization technique is efficient now : the operator $\tilde{\mathcal{B}}$ is not in (3.15) explicitly any more. It is important to get rid of it because this operator implies nonlinear terms including $\overline{\partial_x h_k}$. Classical energy method is not enough to get a satisfying estimation like $\|\Lambda\mathbf{h}\| \le C(\rho)+\sqrt{T}C(\mu)\|\mathbf{h}\|$, even though one applies the smoothing method described in [11], one has to assume a smallness condition on the initial data in order to make it work. Only the operator $\tilde{\mathcal{A}}$ is remaining in equation 3.15 and it does not include bothering terms like $\overline{\partial_x h_k}$. To deal with this equation and in order to obtain an inequality like the one described above, we now make use of the smoothing operator defined in [11]. The function $\phi$ is defined by

$$\phi(x,t) = \frac{1}{\delta}\sum_{k=0}^{l}\partial_x^{-1}\Big(|\mathcal{J}^k u^\dagger(x,t)|^2 + |\mathcal{J}^k u_x^\dagger(x,t)|^2 + \sum_{2\le j\le 5}|\mathcal{J}^k K_j u^\dagger(x,t)|^2 + |\mathcal{J}^k K_6 u_x^\dagger(x,t)|^2\Big).$$

$\phi$ is in the space $\mathcal{L}^\infty\left([0,T];\mathcal{C}^1(\mathbb{R})\right) \cap \mathcal{C}^1\left([0,T];\mathcal{L}^\infty(\mathbb{R})\right)$ and $\delta$ is a positive parameter set later.
$\omega$ is denoted by

$$\omega(x,t) = \frac{1}{\sqrt{\delta}}\Big(\sum_{k=0}^{l}\partial_x^{-1}[|\mathcal{J}^k u^\dagger(x,t)|^2 + |\mathcal{J}^k u_x^\dagger(x,t)|^2 + \sum_{2\le j\le 5}|\mathcal{J}^k K_j u^\dagger(x,t)|^2 + |\mathcal{J}^k K_6 u_x^\dagger(x,t)|^2]\Big)^{\frac{1}{2}}.$$

$\phi$ and $\omega$ satisfy $\|\phi\|_\infty \le \frac{C(\rho)}{\delta}$, $\|\phi_t\|_\infty \le \frac{C(\nu)}{\delta}$, $\|\omega\|_\infty \le \frac{C(\rho)}{\sqrt{\delta}}$ and $\|w\|_{1,0,\infty} \le \frac{C(\mu)}{\sqrt{\delta}}$.
The smoothing operator $S_0(\phi) = \cosh(\phi) + \imath\sinh(\phi)\mathcal{H}$ is introduced now, it satisfies all the proprieties described in section 2 (that is to say lemma 2.3 amd lemma 2.4) and similarly $S(\phi)$ is denoted by

$$S(\phi)\mathbf{g} = \begin{pmatrix} S_0(\phi)g_{2l+1} \\ \vdots \\ S_0(\phi)g_0 \end{pmatrix} \quad \text{where} \quad \mathbf{g} \in L^2(\mathbb{R})^{2l+2}.$$

$p$ is the projection operator defined by : for any $\mathbf{x} \in \mathbb{R}^{2l+2}$

$$p(\mathbf{x}) = \begin{pmatrix} x_{2l} \\ x_{2l-2} \\ \vdots \\ x_0 \end{pmatrix} \quad \text{so } p(\mathbf{x}) \in \mathbb{R}^{l+1}.$$

Thus equation (3.15) is rewritten in the following way :

$$\begin{cases} \imath\partial_t p(\mathbf{w}) + \partial_x^2 p(\mathbf{w}) = p(\Lambda\mathbf{R}) + p(\mathcal{R}_d\mathbf{h}) + p(\tilde{\mathcal{A}}\mathbf{w}) \\ \mathbf{w} = \Lambda\mathbf{h}, \end{cases} \tag{3.16}$$

where $\mathcal{R}_d\mathbf{h}$ is such that $\|\mathcal{R}_d\mathbf{h}\| \leq (C(\rho)\kappa + C(\mu))\,\|\mathbf{h}\|$. Therefore applying Lemma 2.3 to the first line of equation (3.16), this energy type inequality is obtained :

$$\begin{aligned} &\frac{d}{dt}\|S\,p(\mathbf{w})\|^2 + \left\|\omega S\sqrt{|\partial_x|}p(\mathbf{w})\right\|^2 \leq 2|\mathrm{Im}(Sp(\mathbf{w}), Sp(\Lambda\mathbf{R}))| \\ &\quad + 2|\mathrm{Im}(Sp(\mathbf{w}), Sp(\mathcal{R}_d\mathbf{h}))| + 2|\mathrm{Im}(Sp(\mathbf{w}), Sp(\tilde{\mathcal{A}}))| \\ &\quad + Ce^{\|\phi\|_\infty}\left(\|\omega\|_\infty^4 + \|\omega\|_\infty^6 + \|\omega\|_{1,0,\infty}\|w\|_\infty + \|\phi_t\|_\infty\right)\|p(\mathbf{w})\|^2. \end{aligned} \tag{3.17}$$

By Sobolev's inequality

$$\begin{aligned} |\mathrm{Im}(Sp(\mathbf{w}), Sp(\Lambda\mathbf{R}))| &\leq \|Sp(\mathbf{w})\|\,.\,\|Sp(\Lambda\mathbf{R})\| \\ &\leq e^{2\|\phi\|_\infty}.\,\|p(\mathbf{w})\|\,\|p(\Lambda\mathbf{R})\| \\ &\leq C(\mu)e^{2\|\phi\|_\infty}.\,\|p(\mathbf{w})\|\,.\,\|\mathbf{h}\| \\ &\leq C(\mu)C(\frac{\rho}{\delta})\,\|\mathbf{h}\|^2 + \frac{1}{2}\|p(\mathbf{w})\|^2. \end{aligned} \tag{3.18}$$

In the same way and by definition of $\mathcal{R}_d$

$$|\mathrm{Im}(Sp(\mathbf{w}), Sp(\mathcal{R}_d\mathbf{h}))| \leq C(\frac{\rho}{\delta})\kappa\,\|\mathbf{h}\|^2 + \frac{1}{2}\|p(\mathbf{w})\|^2, \tag{3.19}$$

and

$$\begin{aligned} &Ce^{\|\phi\|_\infty}\left(\|\omega\|_\infty^4 + \|\omega\|_\infty^6 + \|\omega\|_{1,0,\infty}\|w\|_\infty + \|\phi_t\|_\infty\right)\|p(\mathbf{w})\|^2 \\ &\qquad \leq C(\mu, \frac{1}{\delta})\,\|p(\mathbf{w})\|^2. \end{aligned} \tag{3.20}$$

Since

$$(Sp(\mathbf{w}), Sp(\tilde{\mathcal{A}})) = \sum_{k=0}^{l}(S_0\mathbf{w}_k, S_0\mathcal{N}_1(u^\dagger, u_x^\dagger)\partial_x\mathbf{w}_k),$$

this main term is estimated by lemma 2.4 (for complete details see [11]) :

$$\left|(Sp(\mathbf{w}), Sp(\tilde{\mathcal{A}}))\right| \leq 2\delta\sum_{k=0}^{l}\left\|S_0\sqrt{|\partial_x|}\mathbf{w}_k\right\|^2 + C(\mu)\,\|p(\mathbf{w})\|^2. \tag{3.21}$$

Substitution of (3.18), (3.19), (3.20) and (3.21) into (3.17) yields

$$\begin{aligned} &\frac{d}{dt}\|Sp(\mathbf{w})\|^2 + (1-2\delta)\left\|\omega S\sqrt{|\partial_x|}p(w)\right\|^2 \\ &\leq \left(C(\nu) + C(\rho) + C(\mu)C(\frac{1}{\delta})\right)\left(\|p(\mathbf{w})\|^2 + \|h\|^2\right). \end{aligned} \tag{3.22}$$

If $\delta$ is set to be $\delta = \frac{1}{2}$, then the energy inequality (3.22) becomes

$$\frac{d}{dt}\|Sp(\mathbf{w})\|^2 \le (C(\nu)+C(\mu)+C(\kappa))\left(\|p(\mathbf{w})\|^2+\|\mathbf{h}\|^2\right). \tag{3.23}$$

By definition $\mathbf{w} = \Lambda\mathbf{h}$ or $\mathbf{h} = \mathbf{w}+\frac{1}{2}\mathcal{B}\mathbf{h}$ or $\mathbf{h} = S^{-1}\left(S\mathbf{w}+\frac{1}{2}S\mathcal{B}\mathbf{h}\right)$. One notices that $\|\mathbf{h}\| = 2\|p(\mathbf{h})\|$. But by definition of $\Lambda$ one has $\|\mathbf{w}\| \le C(\rho)\|\mathbf{h}\|$. Then the previous inequality implies

$$\frac{d}{dt}\|Sp(\mathbf{w})\|^2 \le C(\rho,\mu,\nu,\kappa)(\|\mathbf{v}\|^2+\|\mathbf{h}\|^2),$$

hence by equation (3.6)

$$\frac{d}{dt}\|Sp(\mathbf{w})\|^2 \le C(\rho,\mu,\nu,\kappa)(\rho^2T+1)\|\mathbf{h}\|^2, \tag{3.24}$$

then by integrating (3.24) with respect to time $t$ ($\mathcal{S}$ is a continuous operator in $\mathcal{C}([0,T]);\mathcal{L}^2(\mathbb{R}))$ so that $\lim_{t\to 0}\mathcal{S}p(\mathbf{w})(t) = \mathcal{S}p(\mathbf{w})(0))$

$$\|Sp(\mathbf{w})(t)\|^2 \le \|Sp(\mathbf{w})(0)\|^2 + C(\rho,\mu,\nu,\kappa)(\rho^2T^2+T)\sup_{\tau\in[0,T]}\|\mathbf{h}(\tau)\|^2.$$

Or $\mathbf{h} = S^{-1}\left(S\mathbf{w}+\frac{1}{2}S\mathcal{B}\mathbf{h}\right)$, then

$$\begin{aligned}\sum_{k=0}^{l}\|h_k\|^2 &\le \left\|S^{-1}Sp(\mathbf{w})\right\|^2 + C\left\|S^{-1}Sp(\mathcal{B}\mathbf{h})\right\|^2 \\ &\le |||S^{-1}|||\left(\|Sp(\mathbf{w})\|^2 + C(\rho)e^{2\|\phi\|_\infty}\|\mathbf{v}\|\right),\end{aligned} \tag{3.25}$$

one obtains

$$\sum_{k=0}^{l}\|h_k\|^2 \le C_0(\rho)\left(\|Sp(\mathbf{w})\|^2 + C(\rho)\|\mathbf{v}\|^2\right).$$

But

$$\begin{aligned}\|\mathcal{S}p(\mathbf{w})(0)\| &\le e^{2\|\phi\|_\infty}\|\mathbf{w}(0)\|^2 \\ &\le C(\rho)\|\mathbf{w}(0)\| \\ &\le C(\rho)C(\rho)\|\mathbf{w}(0)\|.\end{aligned}$$

From the previous equation, (3.6), (3.24), (3.25), there exists $C_1(\rho)$ such that

$$\sup_{t\in[0,T]}\sum_{k=0}^{l}\|h_k(t)\|^2 \le C_0\left(C_1(\rho)+C(\rho,\mu,\nu,\kappa)(T^2+C(\rho)T)\sup_{t\in[0,T]}\|\mathbf{h}(t)\|^2\right).$$

If it is decided that $\mu^2 = 4C_0C_1$ and $T$ is such that $C(\rho,\mu,\nu,\kappa)(T^2+C(\rho)T) \leq \frac{1}{2}$, the desired estimations are obtained :

$$\sum_{k=0}^{l} \left\| J^k v(t) \right\|^2 \leq \rho^2 + C(\rho,\mu,\nu)T \text{ and } \quad \sum_{k=0}^{l} \left\| \partial_x(\mathcal{J}^k v(t)) \right\|^2 \leq \mu^2, \tag{3.26}$$

where $\lim_{T\to 0} C(\rho,\mu,\nu,\kappa,T) = 0$.
We want to show now that we can define $\nu$ such that

$$\sum_{k=0}^{l} \|\partial_t\partial_x^{-1}|\mathcal{J}^k u|^2\|_\infty + \|\partial_t\partial_x^{-1}|\mathcal{J}^k u_x|^2\|_\infty \leq \frac{1}{2}\nu + C(\rho,\mu,\nu,\kappa)T.$$

Now from equation (3.4) and lemma 2.1, u satisfies the equation

$$\begin{aligned} L\mathcal{J}^k u &= (1-\partial_x^2)^{-1}\mathcal{N}_1(u^\dagger,u_x^\dagger)(\mathcal{J}^k v)_x \\ &+ (1-\partial_x^2)^{-1}\mathcal{N}_2(u^\dagger,u_x^\dagger)\overline{(\mathcal{J}^k v)_x} + (1-\partial_x^2)^{-1}\mathcal{J}^k R(v^\dagger) \\ &+ \sum_{j=1}^{k}(1-\partial_x^2)^{-j-1}p_{j,k}(\partial_x)\left(\mathcal{N}_1(u^\dagger,u_x^\dagger)(\mathcal{J}^{k-j}v)_x + \mathcal{J}^{k-j}R(v^\dagger)\right) \\ &+ \sum_{j=1}^{k}(1-\partial_x^2)^{-j-1}p_{j,k}(\partial_x)\mathcal{N}_2(u^\dagger,u_x^\dagger)\overline{(\mathcal{J}^{k-j}v)_x}. \end{aligned} \tag{3.27}$$

Since $\partial_t\partial_x^{-1}|\mathcal{J}^k u|^2 = 2\Re(\partial_x^{-1}\overline{\mathcal{J}^k u}\cdot(\mathcal{J}u)_t)$, the following inequality is obtained by mutliplying the previous equation by $\overline{\mathcal{J}^k u}$ and taking its imaginary part and then integration $\int_{-\infty}^{x}\cdots dx'$.

(3.28)

$$\begin{aligned} &\|\partial_t\partial_x^{-1}|\mathcal{J}^k u|^2\|_\infty \leq 2\|\overline{\mathcal{J}^k u}(\mathcal{J}^k u)_x\|_\infty + \\ &2\left\|\mathcal{J}^k u\right\|\cdot\left\{\left\|(1-\partial_x^2)^{-1}\mathcal{N}_1(u^\dagger,u_x^\dagger)(\mathcal{J}^k v)_x\right\| + \left\|(1-\partial_x^2)^{-1}\mathcal{N}_2(u^\dagger,u_x^\dagger)\overline{(\mathcal{J}^k v)_x}\right\|\right\} \\ &+ 2\sum_{j=1}^{k}\left\|\mathcal{J}^j u\right\|\cdot\left\{\left\|(1-\partial_x^2)^{-j-1}p_{j,k}(\partial_x)\mathcal{N}_1(u^\dagger,u_x^\dagger)(\mathcal{J}^{k-j}v)_x\right\|\right. \\ &\left.+\left\|(1-\partial_x^2)^{-j-1}p_{j,k}(\partial_x)\mathcal{N}_2(u^\dagger,u_x^\dagger)\overline{(\mathcal{J}^{k-j}v)_x}\right\|\right\} \\ &+ 2\sum_{j=1}^{k}\left\|\mathcal{J}^j u\right\|\cdot\left\|(1-\partial_x^2)^{-j-1}p_{j,k}(\partial_x)\mathcal{J}^{k-j}R(v^\dagger)\right\| + \left\|(1-\partial_x^2)^{-1}\mathcal{J}^k R(v^\dagger)\right\|. \end{aligned}$$

Since $(1-\partial_x^2)^{-j-1}p_{j,k}(\partial_x)$ are bounded operators in $\mathcal{L}^2(\mathbb{R})$ and $\sum_{k=1}^{l}\left\|\mathcal{J}^j R(v^\dagger)\right\| \leq C(\rho)$ and $\left\|\mathcal{J}^j u\right\| \leq C(\rho)\left\|\mathcal{J}^j v\right\|$; there exists a positive constant $C_3(\rho)$ depending on $\rho$ such that

$$2\sum_{j=1}^{k}\left\|(1-\partial_x^2)^{-j-1}p_{j,k}(\partial_x)\mathcal{J}^{k-j}R(v^\dagger)\right\| + \left\|(1-\partial_x^2)^{-1}\mathcal{J}^k R(v^\dagger)\right\| \le C_3(\rho).$$

As

$$\mathcal{N}_1(u^\dagger,u_x^\dagger) = \partial_x\left(\mathcal{N}_1(u^\dagger,u_x^\dagger)\mathcal{J}^k v\right) - \partial_x(\mathcal{N}_1(u^\dagger,u_x^\dagger)),$$

and $\partial_x(\mathcal{N}_1(u^\dagger,u_x^\dagger))$ can be considered as a polynomial such that $\partial_{u_{xx}^\dagger}\partial_x(\mathcal{N}_1(u^\dagger,u_x^\dagger)) = 0$ and $\partial_{\overline{u_{xx}^\dagger}}\partial_x(\mathcal{N}_1(u^\dagger,u_x^\dagger)) = 0$, one has

$$\left\|(1-\partial_x^2)^{-1}\partial_x(\mathcal{N}_1(u^\dagger,u_x^\dagger))\mathcal{J}^k v\right\| \le \|u_{xx}^\dagger\|_\infty \left\|\mathcal{J}^k v\right\|$$

and

$$\left\|(1-\partial_x^2)^{-1}\partial_x\left((\mathcal{N}_1(u^\dagger,u_x^\dagger))\mathcal{J}^k v\right)\right\| \le c(\rho)\left\|\mathcal{J}^k v\right\|.$$

The other terms in the right handside of (3.28) can be estimated in the same way. We have also $\|u_{xx}^\dagger\|_\infty \le C(\mu)$. So that there exist two positive constants $C_{4,k}(\rho)$ and $C_{5,k}(\mu)$ depending respectively on $\rho$ and $\mu$ such that

$$\|\partial_t\partial_x|\mathcal{J}^k u|^2\|_\infty \le C_{4,k}(\rho) + C_{5,k}(\mu) + 2\left\|\mathcal{J}^k v\right\|^2,$$

then (3.26) and the previous inequality imply

$$\|\partial_t\partial_x^{-1}|\mathcal{J}^k u|^2\|_\infty \le C_{4,k}(\rho) + C_{5,k}(\mu) + 2\rho + C(\rho,\mu,\nu,T)$$

where

$$\lim_{T\to 0} C(\rho,\mu,\nu,T) = 0.$$

Since $(\mathcal{J}^k u)_x$ satisfies

$$\begin{aligned}
&L(\mathcal{J}^k u)_x = \partial_x(1-\partial_x^2)^{-1}\mathcal{N}_1(u^\dagger,u_x^\dagger)\overline{(\mathcal{J}^k v)_x}\\
&\quad + \partial_x(1-\partial_x^2)^{-1}\mathcal{N}_2(u^\dagger,u_x^\dagger)(\mathcal{J}^k v)_x + \partial_x(1-\partial_x^2)^{-1}\mathcal{J}^k R(v^\dagger)\\
&\quad + \sum_{j=1}^{k}\partial_x(1-\partial_x^2)^{-j-1}p_{j,k}(\partial_x)\{\mathcal{N}_1(u^\dagger,u_x^\dagger)(\mathcal{J}^{k-j}v)_x + \mathcal{J}^{k-j}R(v^\dagger)\}\\
&\quad + \sum_{j=1}^{k}\partial_x(1-\partial_x^2)^{-j-1}p_{j,k}(\partial_x)\mathcal{N}_2(u^\dagger,u_x^\dagger)\overline{(\mathcal{J}^{k-j}v)_x}.
\end{aligned}$$

As above one finds the inequality

$$\begin{aligned}
&\|\partial_t\partial_x^{-1}|(\mathcal{J}^k u)_x|^2\|_\infty \le 2\|(\mathcal{J}^k u)_x\|_\infty\|(\mathcal{J}^k u)_{xx}\|_\infty \\
&+ 2\left\|(\mathcal{J}^k u)_x\right\| \cdot \left\{\sum_{j=1,2}\left\|(1-\partial_x^2)^{-1}\mathcal{N}_j(u^\dagger, u_x^\dagger)(\mathcal{J}^k v)_x\right\|\right\} \\
&+ 2\sum_{j=1}^{k}\left\|(\mathcal{J}^k u)_x\right\| \cdot \left\{ \left\|(1-\partial_x^2)^{-j-1}p_{j,k}(\partial_x)\mathcal{N}_1(u^\dagger, u_x^\dagger)(\mathcal{J}^{k-j} v)_x\right\|\right. \\
&\quad \left. + \left\|(1-\partial_x^2)^{-j-1}p_{j,k}(\partial_x)\mathcal{N}_2(u^\dagger, u_x^\dagger)\overline{(\mathcal{J}^{k-j} v)_x}\right\| \right\} \\
&+ 2\sum_{j=1}^{k}\left\|(\mathcal{J}^k u)_x\right\| \cdot \left\|(1-\partial_x^2)^{-j-1}p_{j,k}(\partial_x)\mathcal{J}^{k-j}R(v^\dagger)\right\| \\
&+ \left\|(1-\partial_x^2)^{-1}\mathcal{J}^k R(v^\dagger)\right\|.
\end{aligned}$$

Thus there exist two positive constants $C_{6,k}(\rho)$ and $C_{7,k}(\mu)$ depending respectively on $\rho$ and $\mu$ such that

$$\|\partial_t\partial_x^{-1}|(\mathcal{J}^k u)_x|^2\|_\infty \le C_{6,k}(\rho) + C_{7,k}(\mu) + 2\left\|\mathcal{J}^k v\right\|^2 + \left\|(\mathcal{J}^k v)_x\right\|^2.$$

Therefore

$$\begin{aligned}
&\sum_{k=0}^{l}\left(\|\partial_t\partial_x^{-1}|\mathcal{J}^k u|^2\|_\infty + \|\partial_t\partial_x^{-1}|(\mathcal{J}^k u)_x|^2\|_\infty\right) \le \\
&\sum_{k=0}^{l}\left(C_{4,k}(\rho) + C_{5,k}(\mu) + C_{6,k}(\rho) + C_{7,k}(\mu)\right) + 2\rho + 2\mu + C(\rho,\mu,\nu,T).
\end{aligned}$$

Now $\nu$ is set : $\nu = 2\sum_{k=0}^{l}(C_{4,k}(\rho) + C_{5,k}(\mu) + C_{6,k}(\rho) + C_{7,k}(\mu)) + 4(\rho+\mu)$. In order to show that our Cauchy mapping $(A)$ is such that $\mathfrak{A}(\mathfrak{B}) \subset \mathfrak{B}$ the only remaining taks is to show that

$$\sum_{k=0}^{l}\left\|\partial_t\mathcal{J}^k u\right\| + \left\|\partial_t(\mathcal{J}^k u)_x\right\| \le \frac{1}{2}\kappa + C(\rho,\mu,\nu,\kappa)T,$$

for a suitable $\kappa$. Now for any $k \in \{0,\dots,l\}$, according to (3.27) and (3.28), $\mathcal{J}^k u$ satisfies the equation

$$\imath\partial_t\mathcal{J}^k u = -\partial_x^2\mathcal{J}^k u + (1-\partial_x^2)^{-1}P(v^\dagger, v),$$

where $P(v^\dagger, v)$ is such that

$$\left\| P(v^\dagger, v) \right\| \leq C(\mu) \sum_{j=0}^{k} \|h_j\| \,.$$

Then

$$\left\| \partial_t \mathcal{J}^k u \right\| \leq \|\mathbf{v}\| + C(\mu) \|\mathbf{h}\| \,.$$

From (3.26), one has

$$\left\| \partial_t \mathcal{J}^k u \right\| \leq \rho + \mu C(\rho) + C(\rho, \mu, \nu, \kappa).$$

Since $\partial_x(\mathcal{J}^k u)$ satisfies the equation

$$\imath \partial_t \partial_x (\mathcal{J}^k u) = -\partial_x^3 \mathcal{J}^k u + \partial_x (1 - \partial_x^2)^{-1} P(v^\dagger, v),$$

as above, from (3.26), one has

$$\left\| \partial_t \partial_x (\mathcal{J}^k u) \right\| \leq \|h\| + C(\rho) \|h\| \,.$$

Hence there exist two positive constants $C_{8,k}(\rho)$ and $C_{9,k}(\mu)$ such that

$$\left\| \partial_t \partial_x (\mathcal{J}^k u) \right\| + \left\| \partial_t \mathcal{J}^k u \right\| \leq C_{8,k}(\rho) + C_{9,k}(\mu) + C(\rho, \mu, \nu, \kappa, T).$$

If we set $\kappa = 2 \displaystyle\sum_{k=0}^{l} (C_{8,k}(\rho) + C_{9,k}(\mu))$. Then we have

$$\sum_{k=0}^{l} \left\| \partial_t \partial_x (\mathcal{J}^k u(t)) \right\| + \left\| \partial_t \mathcal{J}^k u(t) \right\| \leq \frac{1}{2} \kappa + C(\rho, \mu, \nu, \kappa, T).$$

Finally we have obtained the inequalities :

$$\begin{cases} \displaystyle\sum_{k=0}^{l} \left\| \mathcal{J}^k v \right\|^2 \leq \ \rho^2 + C(\rho, \mu, \nu, \kappa, T) \\ \displaystyle\sum_{k=0}^{l} \left\| (\mathcal{J}^k v)_x \right\|^2 \leq \ \mu^2 \\ \displaystyle\sum_{k=0}^{l} \left( \left\| \partial_t \mathcal{J}^k u \right\| + \left\| \partial_t \partial_x (\mathcal{J}^k u) \right\| \right) \leq \frac{1}{2} \kappa + C(\rho, \mu, \nu, \kappa, T) \\ \displaystyle\sum_{k=0}^{l} \left( \|\partial_t \partial_x^{-1} |\mathcal{J}^k u|^2\|_\infty + \|\partial_t \partial_x^{-1} |(\mathcal{J}^k u)_x|^2\|_\infty \right) \leq \frac{1}{2} \nu + C(\rho, \mu, \nu, \kappa, T), \end{cases}$$

where $\lim_{T\to 0} C(\rho,\mu,\nu,\kappa,T) = 0$. Then one infers that there exists a time $T>0$ such that the mapping $\mathfrak{A}$ transforms the ball $\mathfrak{B}$ into itself.
The next step is to show that $\mathfrak{A}$ is a contraction mapping. Let $v^\dagger$ and $\tilde{v}^\dagger$ satisfy $v^\dagger \in \mathfrak{B}$ and $\tilde{v}^\dagger \in \mathfrak{B}$, we define $v = \mathfrak{A}v^\dagger$ and $\tilde{v} = \mathfrak{A}\tilde{v}^\dagger$, $g = \tilde{v} - v$. By definition $g$ satisfies the equation

(3.29)
$$\begin{cases} Lg = \mathcal{N}_1(\tilde{u}^\dagger, \tilde{u}_x^\dagger) g_x + \mathcal{N}_2(\tilde{u}^\dagger, \tilde{u}_x^\dagger)\overline{g_x} + \Big(\mathcal{N}_1(\tilde{u}^\dagger, \tilde{u}_x^\dagger) - \mathcal{N}_1(u^\dagger, u_x^\dagger)\Big) v_x \\ + \Big(\mathcal{N}_2(\tilde{u}^\dagger, \tilde{u}_x^\dagger) - \mathcal{N}_2(u^\dagger, u_x^\dagger)\Big) v_x + \mathcal{R}_0(\tilde{v}^\dagger) - \mathcal{R}_0(v^\dagger). \\ g(x,0) = 0. \end{cases}$$

We apply the operator $\mathcal{J}^k$ for $k \in \{0, \dots, l\}$ to this equation and we want to obtain estimates on all the terms which does not include $g_x$ or $\overline{g_x}$ like

$$\left\| (\nu(u^\dagger, u_x^\dagger) - \nu(\tilde{u}, \tilde{u}_x^\dagger)) v_x \right\| \leq C(\mu) \left\| v^\dagger \tilde{v}^\dagger \right\|.$$

This is the most difficult term to estimate in equation (3.29). One can check that the operator $\mathcal{J}$ satisfy the relations :

$$\begin{aligned} &\mathcal{J}(\phi\psi\overline{w}) = \mathcal{J}(\phi)\psi\overline{w} + \phi\psi\overline{w} - \phi\psi\overline{\mathcal{J}(w)} \\ &\mathcal{J}\left(K(|w|^2)\psi\right) = K(|w|^2)\mathcal{J}\psi + K'(|w|^2)(\overline{w}\mathcal{J}w - w\overline{\mathcal{J}w})\psi, \end{aligned} \tag{3.30}$$

where $K \in \mathcal{C}^\infty([0,+\infty);\mathbb{C})$. It means that $\mathcal{J}$ behaves like the usual derivative $\partial_x$ for nonlinearity of the type $K(|w|^2)\psi\phi\overline{u}$. We want to calculate $\mathcal{J}^k(K(|w^2|\psi)$. The usual calculus rule says (for a proof see for example [15]) : for $g \in \mathcal{C}^\gamma(\mathbb{R},\mathbb{R})$ and $f \in \mathcal{C}^\gamma(\mathbb{R};\mathbb{C})$, $\gamma \in \mathbb{N}$

$$\partial_x^\gamma f(g) = \sum_{s=1}^{\gamma} \frac{1}{s!} f^{(s)}(g) \sum_{\substack{\gamma_1+\ldots+\gamma_s=\gamma \\ \gamma_i \geq 1}} \frac{\gamma!}{\gamma_1 \ldots \gamma_s} \partial_x^{\gamma_1} g \ldots \partial_x^{\gamma_s} g.$$

In our situation, $g = |w|^2$ and $f = K$. Thanks of (3.29), we can apply this relation to obtain $\mathcal{J}^k(K(|w^2|\psi)$, because it tells us that one can use the Leibnitz formula with $\partial g = \overline{w}\mathcal{J}w - w\overline{\mathcal{J}w}$. Thus for any $\gamma \in \mathbb{N}$

$$\partial^\gamma g = \sum_{\gamma_1+\gamma_2=\gamma} (-1)^{\gamma_1} \frac{\gamma!}{\gamma_1!\gamma_2!} \overline{\mathcal{J}^{\gamma_1} w} \mathcal{J}^{\gamma_2} w.$$

We define $Z^p$ by

$$Z^p(w) = \sum_{q=0}^{p} (-1)^{p-q} \binom{p}{q} \overline{\mathcal{J}^{p-q} w} \mathcal{J}^q w.$$

It implies for $m \in \mathbb{N}$

$$\mathcal{J}^m\left(K(|w|^2)\psi\right) = \sum_{n=0}^{m} \frac{m!}{(m-n)!}\mathcal{J}^{m-n}\psi$$
$$\times \sum_{s=1}^{n} K^{(s)}(|w|^2) \sum_{\substack{\gamma_1+\ldots+\gamma_s=n \\ \gamma_i \geq 1}} \frac{n!}{s!\gamma_1!\ldots\gamma_s!} Z^{(\gamma_1)}(w)\ldots Z^{(\gamma_s)}(w).$$

Then by definition of $\mathcal{N}_1$ and $\mathcal{N}_2$ it is possible to make an explicit calculation of $\mathcal{J}^k\mathcal{N}_1$. For example

$$\mathcal{J}^k\left((K_6(|\tilde{u^\dagger}|^2)\tilde{u_x^\dagger})\overline{\tilde{u_x^\dagger}}v_x\right) = \sum_{k_1=0}^{k}\sum_{k_2=0}^{k-k_1}\sum_{s=0}^{k_2}$$
$$\sum_{n=1}^{s} \sum_{\substack{\gamma_1+\ldots+\gamma_s=\gamma \\ \gamma_i \geq 1}} \frac{k!}{k_1!(k-(k_1+k_2))!(k_2-s)!n!}\frac{s!}{\gamma_1!\ldots\gamma_s!}$$
$$\times \mathcal{J}^{k-(k_1+k_2)}v_x \cdot \overline{\mathcal{J}^{k_1}u_x^\dagger} \cdot \mathcal{J}^{k_2-s}u \cdot Z^{\gamma_1}(u^\dagger)\cdots Z^{\gamma_s}(u^\dagger).$$

Then there exists a positive constant $C(\mu)$ such that

$$\left\|\mathcal{J}^k\left((K_6(|\tilde{u^\dagger}|^2)\tilde{u_x^\dagger})\overline{\tilde{u_x^\dagger}}v_x\right) - \mathcal{J}^k\left((K_6(|u^\dagger|^2)u_x^\dagger\overline{u_x^\dagger}v_x\right)\right\| \leq$$
$$C(\mu)\sum_{s=0}^{k}\|\mathcal{J}^s\tilde{g}\| + C(\mu)\|K_6(|\tilde{u^\dagger}|^2) - K_6(|u^\dagger|^2)\|_\infty.$$

$$\|K_6(|\tilde{u^\dagger}|^2) - K_6(|u^\dagger|^2)\|_\infty \leq \sup_{z\in[0,\max(\|\tilde{u^\dagger}\|_\infty^2,\|u^\dagger\|_\infty^2)]} |K_6'(z)| \cdot \||\tilde{u^\dagger}|^2 - |u^\dagger|^2\|_\infty.$$

As $\tilde{u^\dagger} \in \mathfrak{B}$ and $u^\dagger \in \mathfrak{B}$, one gets by denoting $g^\dagger = \tilde{v}^\dagger - v^\dagger$

$$\|K_6(|\tilde{u^\dagger}|^2) - K_6(|u^\dagger|^2)\|_\infty \leq C(\rho)\left\|g^\dagger\right\|.$$

It implies

$$\left\|\mathcal{J}\left(K_6(|\tilde{u^\dagger}|^2)\tilde{u_x^\dagger})\overline{\tilde{u_x^\dagger}}v_x - (K_6(|u^\dagger|^2)u_x^\dagger\overline{u_x^\dagger}v_x\right)\right\| \leq C(\mu)(1+C(\rho))\sum_{s=0}^{1}\left\|\mathcal{J}^s g^\dagger\right\|.$$

One can obtain samely

$$\left\|\mathcal{J}^k\left(K_6(|\tilde{u^\dagger}|^2)\tilde{u_x^\dagger})\overline{\tilde{u_x^\dagger}}v_x - (K_6(|u^\dagger|^2)u_x^\dagger\overline{u_x^\dagger}v_x\right)\right\| \leq$$
$$(C(\mu) + C(\mu)C(\rho))\sum_{s=0}^{k}\left\|\mathcal{J}^s g^\dagger\right\|.$$

The following inequalities are obtained in the same way :

$$\sum_{j=1}^{2}\left\|\mathcal{J}^k\left(\mathcal{N}_j(\tilde{u^\dagger},\tilde{u_x^\dagger})-\mathcal{N}_j(u^\dagger,u_x^\dagger)\right)v_x\right\|\leq C(\rho)C(\mu)\sum_{s=0}^{k}\|\mathcal{J}^s g\|,$$

and

$$\left\|\mathcal{J}^k(\mathcal{R}_0(\tilde{v}^\dagger)-\mathcal{R}_0(v^\dagger))\right\|\leq C(\mu)\sum_{j=0}^{l}\left\|\mathcal{J}^j\tilde{g}\right\|.$$

Considering the function $g$ similarly to the function $h$, one obtains from (3.29) as above by using the diagonalization technique

$$\sum_{k=0}^{l}\frac{d}{dt}\left\|\mathcal{S}_0\mathcal{J}^k g\right\|^2+(1-C\delta)\left\|\omega\mathcal{S}_0\sqrt{|\partial_x|}\mathcal{J}^k g\right\|^2\leq$$
$$(C(\mu)+C(\nu))\sum_{k=0}^{l}\left(\|\mathcal{J}^k\tilde{g}\|+\|\mathcal{J}^k g\|\right).$$

Therefore integrating the above inequality with respect to time $t$ and choosing a suitable $\delta$, on a sufficiently small interval $T>0$, we get the desired estimate

$$\sup_{t\in[0,T]}\sum_{k=0}^{l}\|\mathcal{J}^k g\|\leq C(\rho)T\sup_{t\in[0,T]}\sum_{k=0}^{l}\|\mathcal{J}^k\tilde{g}\|\leq\frac{1}{2}\sup_{t\in[0,T]}\sum_{k=0}^{l}\|\mathcal{J}^k\tilde{g}\|.$$

Thus the transformation $\mathcal{A}$ is a contraction mapping. Therefore there exists a unique solution $u\in\mathcal{C}\left([0,T];\mathcal{H}^{2,0}\right)$ of the Cauchy problem (A) such that $\mathcal{J}^k u\in\mathcal{L}^\infty\left(0,T;\mathcal{H}^{3,0}\right)$ for any $0\leq k\leq l$. This completes the proof of Lemma 3.1. □

*Proof of Theorem 1.1* Using the identity

$$\begin{aligned}\mathcal{J}^l u&=M(2it\partial_x)^l\overline{M}u\\&=\sum_{k=0}^{l-1}\frac{l!}{k!(l-k)!}\left(\sum_{s=[\frac{l-k}{2}]}^{l-k}(-\imath)^s t^{l-k-s}P_s(x)\right)(2it\partial_x)^k u+(2it\partial_x)^l u,\end{aligned}$$

where $P_s(X)$ is a polynomial of degree s with time independent positive coefficients, we get the estimate

$$\begin{aligned}|2t|^l\left\|\langle x\rangle^{-l}\partial_x^l u\right\|\leq&\left\|\langle x\rangle^{-l}\mathcal{J}^l u\right\|+C\sum_{k=0}^{l-1}(1+|2t|^k)\left\|\langle x\rangle^{-k}\partial_x^k u\right\|\\\leq&C(T)\sum_{k=1}^{l}\left(\left\|\langle x\rangle^{-k}\mathcal{J}^k u\right\|+\|u\|\right).\end{aligned}$$

Hence the result of Theorem 1.1 follows. □

The author would like to thank the referee for his comments and Pr. Nakao Hayashi for his advices.

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