SUT Journal of Mathematics
Vol. 35, No. 1 (1999), 1–10

The family of lines which intersect given lines in general position in R^n

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(Received June 7, 1998; Revised October 13, 1998)

Abstract. In this paper are considered the family of straight lines $\{q\}$ in the Euclidean space $R^n(n > 2)$, which intersect given k lines p_i, $(1 \leq i \leq k)$ in general position. We also study the topology of the corresponding family of lines.

AMS 1991 *Mathematics Subject Classification.* Primary 51N10, 51H10.

Key words and phrases. Family of straight lines, topology of a family of straight lines.

§1. INTRODUCTION

It is our duty first to say that until now in the classic literature of geometry [1,2,3,4] as well as in the periodic journals, this topic was not considered. Before we consider the problem, we give some conventions.

a) The straight lines p_i, $(1 \leq i \leq k)$ are said to be in general position, if

(i) for each $r \in \{2, 3, \cdots, \min(n, k)\}$ and for any r lines $p_{i_1}, p_{i_2}, \cdots, p_{i_r}$ there does not exist $(r-1)$-dimensional vector subspace $\sum$ of R^n, such that all lines p_{i_j}, $(1 \leq j \leq r)$ are parallel to $\sum$, and

(ii) each two different lines do not have common points,

b) We will say that two lines intersect if they have a common point or if they are parallel, analogously to the lines in the projective space.

c) We will not consider the degenerated cases. For example, almost always there does not exist a straight line which intersects given $s(s \geq 5)$ straight lines in R^3, although it is possible to find $s, (s \geq 5)$ lines in general positions in R^3, which can be intersected with a line in R^3. Thus we accept that there does not exist a line which intersects $s, (s \geq 5)$ lines in general position in R^3. Also we will assume that each line does not intersect itself. Indeed, otherwise the Theorem 4.1 c) will not hold.

d) The coordinate system is always chosen such that no one of the given k lines is orthogonal to the x_n-axis. So we can use x_n as a parameter for the given lines.

§2. FAMILY OF LINES $\{q\}$ WHICH INTERSECT GIVEN 4 LINES IN R^3

The following theorem will be proven.

Theorem 2.1. *The number s of lines $q_1, q_2, \cdots, q_s$ which intersect 4 given lines in general position in R^3 is less or equal to 2, i.e. $s_{\max} \le 2$.*

Proof. Let the given 4 lines in R^3 be

$$(p_i) \qquad x_1 = a_i + b_i x_3, \qquad x_2 = c_i + d_i x_3, \qquad (a_i, b_i, c_i, d_i \in R; 1 \le i \le 4).$$

There are two possibilities.

1. The line q is not orthogonal to the x_3-axis.

Now let q be given by

$$(q) \qquad x_1 = \alpha + \beta x_3, \qquad x_2 = \gamma + \delta x_3.$$

The necessary and sufficient conditions for the line q to intersect the lines p_i, $(1 \le i \le 4)$, are given by

$$\begin{vmatrix} -1 & 0 & b_i & a_i \\ 0 & -1 & d_i & c_i \\ -1 & 0 & \beta & \alpha \\ 0 & -1 & \delta & \gamma \end{vmatrix} = 0, \qquad (1 \le i \le 4) \tag{2.1}$$

because the system

$$-1 \cdot x_1 + 0 \cdot x_2 + b_i x_3 + a_i x_4 = 0,$$

$$0 \cdot x_1 - 1 \cdot x_2 + d_i x_3 + c_i x_4 = 0,$$

$$-1 \cdot x_1 + 0 \cdot x_2 + \beta x_3 + \alpha x_4 = 0,$$

$$0 \cdot x_1 - 1 \cdot x_2 + \delta x_3 + \gamma x_4 = 0,$$

has non-zero solution x_1, x_1, x_3 and $x_4 = 1$.

By elementary transformations, (2.1) reduces to

$$\begin{vmatrix} a_i - \alpha & b_i - \beta \\ c_i - \gamma & d_i - \delta \end{vmatrix} = 0, \qquad (1 \le i \le 4). \tag{2.2}$$

$$\begin{vmatrix} a_i & b_i \\ c_i & d_i \end{vmatrix} - \begin{vmatrix} \alpha & b_i \\ \gamma & d_i \end{vmatrix} - \begin{vmatrix} a_i & \beta \\ c_i & \delta \end{vmatrix} + \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = 0, \qquad (1 \le i \le 4). \tag{2.3}$$

If the fourth equation of the system (2.3) subtracts from the first three equations $(1 \le i \le 3)$, we obtain three linear equations where α, β and γ can be expressed as linear functions of δ. By substituting of these α, β and γ in the fourth equation (i=4), we obtain the following equation

$$C_1\delta^2 + C_2\delta + C_3 = 0, \tag{2.4}$$

and hence

$$\delta_{1,2} = \frac{-C_2 \pm [C_2^2 - 4C_1C_3]^{1/2}}{2C_1}. \tag{2.5}$$

Thus, in non-degenerated cases:

I. if $C_2^2 - 4C_1C_3 > 0$, then there exist two real solutions δ_1 and δ_2, and hence two lines q_1 and q_2 in R^3;

II. if $C_2^2 - 4C_1C_3 = 0$, then there exists unique real solution $\delta_1 = \delta_2$, and only one line q in R^3;

III. if $C_2^2 - 4C_1C_3 < 0$, then there exist two complex solutions δ_1 and δ_2, and no one line q in R^3.

So we proved that there are at most two real solutions in this case. Moreover, we express C_1, C_2 and C_3 in the following form

$$C_1 = -B_2 - B_1B_2,$$

$$C_2 = d_4B_3 - c_4B_2 + b_4B_1 + a_4 - A_2B_1 + A_1B_2 - A_3,$$

$$C_3 = d_4A_3 - c_4A_2 - b_4A_1 + A_1A_2 - a_4d_4 + b_4c_4,$$

$$A_1 = \frac{j_1}{h_1}, \qquad B_1 = \frac{i_1}{h_1}, \qquad \gamma = A_1 - B_1\delta,$$

$$A_2 = \frac{j_2}{m_2} - \frac{n_2j_1}{m_2h_1}, \qquad B_2 = \frac{n_2i_1}{m_2h_1} - \frac{v_2}{m_2}, \qquad \beta = A_2 + B_2\delta,$$

$$A_3 = \frac{t_3}{p_3} - \frac{q_3f_2}{p_3m_2} + \frac{q_3n_2j_1}{p_3m_2h_1} - \frac{r_3j_1}{p_3h_1},$$

$$B_3 = -\frac{q_3n_2i_1}{p_3m_2h_1} + \frac{q_3v_2}{p_3m_2} + \frac{r_3i_1}{p_3h_1} - \frac{s_3}{p_3}, \qquad \alpha = A_3 + B_3\delta,$$

$$h_1 = n_1m_2 - n_2m_1, \qquad i_1 = v_1m_2 - v_2m_1, \qquad j_1 = f_1m_2 - f_2m_1,$$

$$m_i = q_ip_3 - q_3p_i, \qquad (i = 1, 2)$$

$$n_i = r_ip_3 - r_3p_i, \qquad (i = 1, 2)$$

$$v_i = s_ip_3 - s_3p_i, \qquad (i = 1, 2)$$

$$f_i = t_ip_3 - t_3p_i, \qquad (i = 1, 2)$$

$$p_i = d_4 - d_i, \qquad (i = 1, 2, 3)$$

$$q_i = c_i - c_4, \qquad (i = 1, 2, 3)$$

$$r_i = b_i - b_4, \qquad (i = 1, 2, 3)$$

$$s_i = a_4 - a_i, \qquad (i = 1, 2, 3)$$

$$t_1 = b_1c_1 - a_1d_1 + a_4d_4 - b_4c_4,$$

$$t_2 = b_2c_2 - a_2d_2 + a_4d_4 - b_4c_4,$$

$$t_3 = b_3c_3 - a_3d_3 + a_4d_4 - b_4c_4.$$

2. The line q is orthogonal to the x_3-axis. Then q has the following form

$$x_1 = \alpha + \beta u_1,$$

$$x_2 = \gamma + \beta u_2,$$

$$x_3 = \delta.$$

There exists a line q which intersects the lines p_i, $(1 \le i \le 4)$ (note that q can not be parallel to any of the given lines) if and only if there exist numbers $\alpha, \beta, \gamma, \delta, u_1$ and u_2, $(u_1^2 + u_2^2 \neq 0)$ such that

$$a_i + b_i\delta = \alpha + \beta u_1,$$

$$c_i + d_i\delta = \gamma + \beta u_2,$$

$$(1 \le i \le 4)$$

i.e.

$$\begin{vmatrix} \alpha - a_i - b_i \cdot \delta & u_1 \\ \gamma - c_i - d_i \cdot \delta & u_2 \end{vmatrix} = 0, \qquad (1 \le i \le 4) \tag{2.6}$$

$$\begin{vmatrix} \alpha & u_1 \\ \gamma & u_2 \end{vmatrix} - \begin{vmatrix} a_i & u_1 \\ c_i & u_2 \end{vmatrix} - \delta \cdot \begin{vmatrix} b_i & u_1 \\ d_i & u_2 \end{vmatrix} = 0, \qquad (1 \le i \le 4). \tag{2.7}$$

If the equations for i=2,3,4 in (2.7) subtract from the first equation (i=1), we obtain three equations

$$\begin{vmatrix} a_1 - a_i & u_1 \\ c_1 - c_i & u_2 \end{vmatrix} + \delta \cdot \begin{vmatrix} b_1 - b_i & u_1 \\ d_1 - d_i & u_2 \end{vmatrix} = 0, \qquad (2 \le i \le 4). \tag{2.8}$$

This homogeneous system of u_1 and u_2 and linear with respect to δ, has unique solution of δ and $u_1 : u_2$, if and only if

$$\frac{a_1 - a_2 + \delta(b_1 - b_2)}{c_1 - c_2 + \delta(d_1 - d_2)} = \frac{a_1 - a_3 + \delta(b_1 - b_3)}{c_1 - c_3 + \delta(d_1 - d_3)} = \frac{a_1 - a_4 + \delta(b_1 - b_4)}{c_1 - c_4 + \delta(d_1 - d_4)} = \frac{u_1}{u_2}$$

has a solution of δ, i.e. if

$$[(A_1C_2 - C_1A_2)(B_1D_3 - B_3D_1) - (A_1C_3 - C_1A_3)(B_1D_2 - B_2D_1)]^2$$
$$= [(A_1C_2 - C_1A_2)(A_1D_3 - D_1A_3 + B_1C_3 - B_3C_1)$$
$$- (A_1C_3 - C_1A_3)(A_1D_2 - D_1A_2 + B_1C_2 - B_2C_1)] \cdot$$
$$\cdot [(A_1D_2 - D_1A_2 + B_1C_2 - B_2C_1)(B_1D_3 - B_3D_1)$$
$$- (A_1D_3 - D_1A_3 + B_1C_3 - B_3C_1)(B_1D_2 - B_2D_1)],$$

where

$$A_i = a_1 - a_{i+1}, \qquad B_i = b_1 - b_{i+1},$$
$$C_i = c_1 - c_{i+1}, \qquad D_i = d_1 - d_{i+1}, \qquad (1 \le i \le 3).$$

If such δ exists, it substitutes in the first equation (i=1) of (2.6) and the linear connection between α and γ can be found, but all such pairs determine the same line. Thus, in this case if such line orthogonal to the x_3-axis which intersects the given lines exists, then it is unique.

Finally, note that the maximal number $s_{\max}$ of intersecting lines together in both cases is 2 but not 3. Indeed, the coordinate system can always be chosen such that there does not exist a line q orthogonal to the x_3-axis which intersects the given lines $p_i, (1 \le i \le 4)$. □

§3. FAMILY OF LINES $\{q\}$ WHICH INTERSECT GIVEN 3 LINES IN R^4

Analogously to the theorem 2.1, now we have the following

Theorem 3.1. *The number s of lines $q_1, q_2, \cdots, q_s$ which intersect 3 given lines in general position in R^4 is less or equal to 4, i.e. $s_{\max} \le 4$.*

Proof. Let the given 3 lines in R^4 are

$$(p_i) \qquad x_1 = a_i + b_i x_4, \qquad x_2 = c_i + d_i x_4, \qquad x_3 = e_i + f_i x_4,$$
$$(a_i, b_i, c_i, d_i, e_i, f_i \in R; \quad (1 \le i \le 3).$$

Analogously as in theorem 2.1, here there are two possibilities.

1. The line q is not orthogonal to the x_4-axis.

Now, let q be given by

$$(q) \qquad x_1 = \alpha + \beta x_4, \qquad x_2 = \gamma + \delta x_4, \qquad x_3 = \eta + \varphi x_4.$$

The necessary and sufficient condition for the line q to intersect the lines p_i, $(1 \le i \le 3)$ is to exists x_4, such that

$$a_i + b_i x_4 = \alpha + \beta x_4, \qquad c_i + d_i x_4 = \gamma + \delta x_4,$$
$$e_i + f_i x_4 = \eta + \varphi x_4, \qquad (1 \le i \le 3),$$

i.e.

$$(a_i-\alpha)+x_4(b_i-\beta)=0, \qquad (c_i-\gamma)+x_4(d_i-\delta)=0,$$
$$(e_i-\eta)+x_4(f_i-\varphi)=0, \qquad (1\le i\le 3).$$

Changing the coordinate system, it may be obtained that $(a_i,b_i)\neq(\alpha,\beta)$ for each i, $(1\le i\le 3)$. Thus, the line q intersects the given three lines p_1, p_2 and p_3, if and only if

$$\begin{vmatrix} a_i-\alpha & b_i-\beta \\ c_i-\gamma & d_i-\delta \end{vmatrix}=0 \qquad \text{and} \qquad \begin{vmatrix} a_i-\alpha & b_i-\beta \\ e_i-\eta & f_i-\varphi \end{vmatrix}=0, \qquad (1\le i\le 3)$$

i.e.

$$\begin{vmatrix} a_i & b_i \\ c_i & d_i \end{vmatrix}-\begin{vmatrix} \alpha & b_i \\ \gamma & d_i \end{vmatrix}-\begin{vmatrix} a_i & \beta \\ c_i & \delta \end{vmatrix}+\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}=0, \qquad (1\le i\le 3) \tag{3.1}$$

$$\begin{vmatrix} a_i & b_i \\ e_i & f_i \end{vmatrix}-\begin{vmatrix} \alpha & b_i \\ \eta & f_i \end{vmatrix}-\begin{vmatrix} a_i & \beta \\ e_i & \varphi \end{vmatrix}+\begin{vmatrix} \alpha & \beta \\ \eta & \varphi \end{vmatrix}=0, \qquad (1\le i\le 3). \tag{3.2}$$

Thus, we obtain 6 equations with unknowns $\alpha,\beta,\gamma,\delta,\eta$ and φ. If from (3.1) the firsts equation (i=1) subtracts from the other two equations, then γ and δ can be expressed as linear functions of α and β, and further they can be substituted in the first equation in (3.1). Hence, the first equation becomes a quadratic equation of α and β. Similarly, from (3.2) we can obtain another quadratic equation of α and β. This system of two quadratic equations of α and β is equivalent to one equation of fourth degree, and hence has at most 4 solutions, i.e. there are at most four lines q_i, $(1\le i\le 4)$ which intersect the given 3 lines in R^4.

2. The line q is orthogonal to the x_4-axis. Then q has the following form

$$(q) \qquad x_1=\alpha+\beta u_1, \qquad x_2=\gamma+\beta u_2, \qquad x_3=\delta+\beta u_3, \qquad x_4=\eta.$$

Similarly, as in the proof of the theorem 2.1, here it verifies that there exists at most one line q intersecting the given three lines. Moreover, analogously as in that proof, the total number of intersecting lines in both cases is at most 4. □

§4. THE TOPOLOGY OF p-PARAMETRIC FAMILIES OF INTERSECTING LINES

Now we will prove that the cases of theorems 2.1 and 3.1 are unique when there exists a discrete set of intersecting lines $\{q\}$. Let be given k lines p_i, $(1\le i\le k)$ in general position in R^n. Each of them is given by $n-1$ equations

$$x_1=a_{1i}+b_{1i}x_n,$$

$$x_2 = a_{2i} + b_{2i} x_n,$$

$$\cdots\cdots$$

$$x_{n-1} = a_{(n-1)i} + b_{(n-1)i} x_n.$$

Analogously as in the proofs of the theorems 2.1 and 3.1, the case when q is orthogonal to the x_n-axis can be neglected and it does not influence to the final conclusion. Hence we suppose that q is given by the following $n-1$ equations

$$x_1 = \alpha_1 + \beta_1 x_n,$$

$$x_2 = \alpha_2 + \beta_2 x_n,$$

$$\cdots\cdots$$

$$x_{n-1} = \alpha_{n-1} + \beta_{n-1} x_n.$$

The intersection of q and p_i reduces to solve the following system

$$a_{1i} - \alpha_1 + (b_{1i} - \beta_1) x_n = 0,$$

$$a_{2i} - \alpha_2 + (b_{2i} - \beta_2) x_n = 0,$$

$$\cdots\cdots$$

$$a_{(n-1)i} - \alpha_{n-1} + (b_{(n-1)i} - \beta_{n-1}) x_n = 0,$$

which reduces to $n-2$ equations by elimination of x_n. Thus, the total number of equations becomes $k(n-2)$, and there are $2(n-1)$ unknowns: $\alpha_1, \cdots, \alpha_{n-1}, \beta_1, \cdots, \beta_{n-1}$. The number of free parameters is

$$p = 2(n-1) - k(n-2).$$

It means, that for given k and n if $p > 0$, then the set of lines $\{q\}$ which intersect given k lines in general position in R^n is a p-parametric family of lines.

We note that we have discrete set of lines if $p = 0$, i.e.

$$k = \frac{2(n-1)}{n-2} = 2 + \frac{2}{n-2}.$$

Thus $n-2 = 1$ or $n-2 = 2$, and hence unique two cases are just those from the theorems 2.1 and 3.1. Note that

(i) $k = 1$ implies $p = n$,
(ii) $k = 2$ implies $p = 2$,
(iii) $k = 3$ implies $p = 4 - n$ and $p > 0$ only for $n = 3$ and then $p = 1$,

(iv) $k > 3$ implies that $p \leq 0$ (assuming that $n > 2$).

Our further aim is to consider the topology of p-parametric families of lines for $p > 0$. First note that the set of all lines in R^n can be endowed with a topology as follows. The set of all pairs (q, Q) where q is a line and $Q \in q$, is equivalent to $R^n \times RP^{n-1}$, because for arbitrary point Q in R^n and the direction of q can arbitrary be chosen in the space RP^{n-1}. Thus, the set of all such pairs can be endowed with the topology of $R^n \times RP^{n-1}$. In this topological space we define a relation of equivalence ρ as follows. The pairs (p_1, Q_1) and (p_2, Q_2) are in relation ρ if the lines p_1 and p_2 coincide. Thus, the set of all lines in R^n is endowed with the topology of the factor space $R^n \times RP^{n-1}/\rho$. This topological space is a differentiable manifold with dimension $2(n-1)$. Indeed in the previous discussion we considered one coordinate neighborhood where each line is parameterized by x_n as parameter. Similarly we have n such coordinate neighborhoods if each line is parameterized by x_i, $(1 \leq i \leq n)$ and these n coordinate neighborhoods cover the whole topological space of lines in R^n, and the elements of the corresponding matrices are analytical functions, and hence it is an analytical differentiable manifold. Thus, each set of the lines in R^n has the induced topology from the topology of $R^n \times RP^{n-1}/\rho$. Now we will consider the topologies in the p-parametric families of lines in (i), (ii) and (iii).

Theorem 4.1. *(a) If $k = 2$, then the n-parametric family of lines $\{q\}$ intersecting given 2 lines in general position in R^n, is homeomorphic to a 2-dimensional tore with one point thrown on.*

(b) If $k = 3$ and $n = 3$, then the 1-parametric family of lines $\{q\}$ intersecting given 3 lines in general position in R^3, is homeomorphic to S^1.

(c) If $k = 1$, then the n-parametric family of lines $\{q\}$ intersecting given line p in R^n, is analytical manifold.

Proof. (a) Let be $k = 2$ and p_1 and p_2 do not intersect. For each pair (A, B) $A \in p_1$ and $B \in p_2$ except A and B simultaneously are infinite points of the projective lines p_1 and p_2, there exist unique line q such that $p_1 \cap q = \{A\}$ and $p_2 \cap q = \{B\}$. Since each projective line is homeomorphic to S^1, we obtain that in this case the 2-parametric family of intersecting lines is homeomorphic to 2-dimensional tore with one point thrown on.

(b) Let p_1, p_2 and p_3 be 3 lines in general position in R^3. We will prove that from each point of the projective line p_1 there exists unique line q intersecting the lines p_2 and p_3. Let $\sum$ be a plane through p_2 which is parallel to p_3. Then p_1 is not parallel to $\sum$ because the lines are in general position, and let $p_1 \cap \sum = \{M\}$. For each point $P \in p_1$, ($P \neq M$ and P may be the infinity point) there exists unique plane Π passing through P and p_2. Since p_1, p_2 and p_3 are in general position and $P \neq M, \Pi$ intersects the line p_3 and let $\Pi \cap p_3 = \{Q\}$. Then PQ is the required line. We note that PQ intersects p_2, and note also that the line q continuously depends on the points of p_1.

If P tends to M, then q tends to the line passing through P and parallel to p_3. This line intersects all three lines. Hence we obtain that the 1-parametric family of lines q intersecting p_1, p_2 and p_3 is homeomorphic to RP^1, i.e. S^1.

(c) Let $k = 1$ and p be a line in R^n. Without loss of generality we assume that p be the x_1-axis. If the line q is not orthogonal to the x_i-axis, then it can be written as

$$(q) \quad \frac{x_1 - x_1^0}{l_1} = \frac{x_2 - x_2^0}{l_2} = \cdots = \frac{x_{i-1} - x_{i-1}^0}{l_{i-1}} = \frac{x_i}{1}$$
$$= \frac{x_{i+1} - x_{i+1}^0}{l_{i+1}} = \cdots = \frac{x_n - x_n^0}{l_n}$$

and hence we obtain a coordinate neighborhood with $2(n-1)$ coordinates: $x_1^0, \cdots, x_{i-1}^0, x_{i+1}^0, \cdots, x_n^0, l_1, \cdots, l_{i-1}, l_{i+1}, \cdots, l_n$. Then p and q intersect, i.e. they have one common point or they are parallel if and only if

$$\operatorname{rank}\begin{bmatrix} x_1^0 & x_2^0 & \cdots & x_{i-1}^0 & 0 & x_{i+1}^0 & \cdots & x_n^0 \\ l_1 & l_2 & \cdots & l_{i-1} & 1 & l_{i+1} & \cdots & l_n \end{bmatrix} = 2.$$

Hence two cases are possible:

1^o. If $i > 1$, then it means that $x_2^0 = \cdots = x_{i-1}^0 = x_{i+1}^0 = \cdots = x_n^0 = 0$ and there are n independent coordinates: $x_1^0, l_1, \cdots, l_{i-1}, l_{i+1}, \cdots, l_n$.

2^o. If $i = 1$, then it means that

$$A_{uv} \equiv \begin{vmatrix} x_u^0 & x_v^0 \\ l_u & l_v \end{vmatrix} = 0, \qquad (1 < u < v \le n) \qquad \text{i.e.} \quad \operatorname{rank} \le 2,$$

and $x_2^0 \ne 0 \vee \cdots \vee x_n^0 \ne 0 \vee l_2 \ne 0 \vee \cdots \vee l_n \ne 0$, i.e. rank ≥ 2. These two conditions are equivalent to

$$\operatorname{rank}\begin{bmatrix} x_2^0 & x_3^0 & \cdots & x_n^0 \\ l_2 & l_3 & \cdots & l_n \end{bmatrix} = 1.$$

This neighborhood is n-dimensional manifold which is covered by two charts

i) $(x_2^0, \cdots, x_n^0) \ne (0, 0, \cdots, 0)$, $(l_2, \cdots, l_n) = \alpha(x_2^0, \cdots, x_n^0)$ and hence $x_2^0, \cdots, x_n^0, \alpha$ are coordinates,

ii) $(l_2, \cdots, l_n) \ne (0, 0, \cdots, 0)$, $(x_2^0, \cdots, x_n^0) = \beta(l_2, \cdots, l_n)$ and hence $l_2, \cdots, l_n, \beta$ are coordinates.

Since any line q which intersects p is not orthogonal at least to one of the axes $x_1, \cdots, x_n$, we obtain that the set of lines $\{q\}$ intersecting the given line p in R^n is a manifold. Moreover it is analytical because the elements of the Jacobi matrix between two of these $n+1$ coordinate neighborhoods, has analytical elements, which are rational algebraic functions. □

At the end we can conclude that under the assumptions in the section 1, we obtain that the set of lines $\{q\}$ intersecting given lines in general position is empty set or it is an analytical manifold. Note that if the set of such lines $\{q\}$ is finite, then we obtain a manifold with dimension 0.

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