SUT Journal of Mathematics
Vol. 35, No. 1 (1999), 139–148

Extensions of the results on p-hyponormal and log-hyponormal operators by Aluthge and Wang

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(Received April 19, 1999)

Abstract. Aluthge and Wang [4] showed that *"if T is p-hyponormal then T^n is $(\frac{p}{n})$-hyponormal for $p \in (0, 1]$"*. Firstly we obtain precise estimation of this result. Secondly we show that *"if T is* log*-hyponormal then T^n is* log*-hyponormal"* and this result is an extension of Theorem B by Aluthge and Wang [3].

AMS 1991 *Mathematics Subject Classification.* Primary 47B20, 47A63.

Key words and phrases. p-hyponormal operator, log-hyponormal operator, Furuta inequality.

§1. Introduction

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$. Also, an operator T is strictly positive (denoted by $T > 0$) if T is positive and invertible.

An operator T is said to be *p-hyponormal* if $(T^*T)^p \geq (TT^*)^p$ for a positive number p and *log-hyponormal* if T is invertible and $\log T^*T \geq \log TT^*$. p-Hyponormal and log-hyponormal operators were defined as extensions of hyponormal one, i.e., $T^*T \geq TT^*$, and also they have been studied by many authors for instance, [1, 2, 3, 4, 5, 7, 11, 12, 14, 17, 18]. By the celebrated Löwner-Heinz theorem "*$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$*", every p-hyponormal operator is q-hyponormal for $p \geq q > 0$. And every invertible p-hyponormal operator is log-hyponormal since $\log t$ is an operator monotone function.

An operator T is said to be *class A* if $|T^2| \geq |T|^2$ [11]. As an extension of class A operator, we defined *class A(k)* operator if $(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2$ for $k > 0$ [11]. We remark that class A(1) operator is class A operator.

It is well known that there exists a hyponormal operator T such that T^2 is not a hyponormal operator [13, Problem 209]. Very recently, Aluthge and Wang [4] obtained the following theorem.

Theorem A ([4]). *Let T be a p-hyponormal operator for $p \in (0,1]$. The inequalities*

$$(T^{n*}T^n)^{\frac{p}{n}} \geq (T^*T)^p \geq (TT^*)^p \geq (T^nT^{n*})^{\frac{p}{n}}$$

hold for all positive integer n.

Theorem A is a very interesting result, because Theorem A asserts that if T is p-hyponormal for $p \in (0,1]$, then T^n is $(\frac{p}{n})$-hyponormal. Moreover, Aluthge and Wang [3] obtained the following result on log-hyponormal.

Theorem B ([3]). *If T is* log-*hyponormal, then T^{2^n} is* log-*hyponormal for any positive integer n.*

In this paper, we shall show further extensions of Theorem A and Theorem B.

§2. Results

Theorem 1. *Let T be a p-hyponormal operator for $p \in (0,1]$. Then*

(i) $(T^*T) \leq (T^{2^*}T^2)^{\frac{1}{2}} \leq \cdots \leq (T^{n*}T^n)^{\frac{1}{n}}$,

(ii) $(TT^*) \geq (T^2T^{2^*})^{\frac{1}{2}} \geq \cdots \geq (T^nT^{n*})^{\frac{1}{n}}$,

hold for all positive integer n.

In Theorem 1, raising each side of (i) and (ii) to the power $p \in (0,1]$ by Löwner-Heinz theorem and using the p-hyponormality of T, we obtain Theorem A.

Theorem 2. *Let T be a* log-*hyponormal operator. Then*

(i) $(T^*T) \leq (T^{2^*}T^2)^{\frac{1}{2}} \leq \cdots \leq (T^{n*}T^n)^{\frac{1}{n}}$,

(ii) $(TT^*) \geq (T^2T^{2^*})^{\frac{1}{2}} \geq \cdots \geq (T^nT^{n*})^{\frac{1}{n}}$,

hold for all positive integer n.

Corollary 3. *Let T be a* log-*hyponormal operator. Then T^n is also a* log-*hyponormal operator for all positive integer n.*

Corollary 3 is an extension of Theorem B.

§3. Proofs of results

To prove Theorem 1 and Theorem 2, the following Lemma C and Theorem D are important.

Lemma C ([10, 11]). *Let A and B be invertible operators. Then*

$$(BAA^*B^*)^{\lambda} = BA(A^*B^*BA)^{\lambda-1}A^*B^*$$

holds for any real number λ.

Theorem D (Furuta inequality [8]).

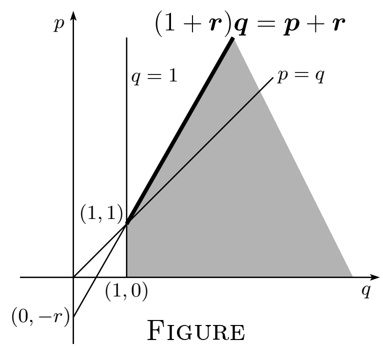
If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.



FIGURE

Tanahashi [16] shows that the domain drawn for p, q and r in the Figure is the best possible one for Theorem D.

Theorem E ([6, 9, 10, 15]). *Let A and B be positive operators satisfying $A^{\alpha} \geq B^{\alpha} \geq 0$ for $\alpha > 0$ or positive invertible operators satisfying $\log A \geq \log B$. Then*

(i) *for each $q \geq 0$ and $t \geq 0$,*
$f_{t,q}(s) = (A^{\frac{t}{2}}B^sA^{\frac{t}{2}})^{\frac{q+t}{s+t}}$ *is decreasing for $s \geq q \geq 0$,*

(ii) *for each $q \geq 0$ and $t \geq 0$,*
$g_{t,q}(s) = (B^{\frac{t}{2}}A^sB^{\frac{t}{2}})^{\frac{q+t}{s+t}}$ *is increasing for $s \geq q \geq 0$.*

Theorem E is obtained by using Theorem D.

Proof of Theorem 1. Let $T = U|T|$ be the polar decomposition of T.

Proof of (i). We will use induction to establish the inequality

$$|T^{n+1}|^{\frac{2n}{n+1}} \geq |T^n|^2 \quad \text{for all positive integer } n. \tag{3.1}$$

In case $n = 1$. Suppose that T is p-hyponormal. T is p-hyponormal if and only if

$$|T|^{2p} \geq |T^*|^{2p}. \tag{3.2}$$

We obtain the following (3.3) by (3.2).

$$T^*|T|^{2p}T \geq T^*|T^*|^{2p}T = T^*(TT^*)^pT = |T|^{2(p+1)}. \tag{3.3}$$

On the other hand, by (3.2) and (ii) of Theorem E, for each $t \geq 0$ and $q \geq 0$, $g_{t,q}(s) = (|T^*|^t|T|^{2s}|T^*|^t)^{\frac{q+t}{s+t}}$ is increasing for $s \geq q \geq 0$. Then we have

$$\begin{aligned}
|T|^2 &\leq (T^*|T|^{2p}T)^{\frac{1}{p+1}} \qquad \text{by (3.3) and Löwner-Heinz theorem} \\
&= (U^*|T^*||T|^{2p}|T^*|U)^{\frac{1}{p+1}} \\
&= U^*(|T^*||T|^{2p}|T^*|)^{\frac{1}{p+1}}U \\
&= U^*g_{1,0}(p)U \\
&\leq U^*g_{1,0}(1)U \\
&= U^*(|T^*||T|^2|T^*|)^{\frac{1}{2}}U \\
&= (U^*|T^*||T|^2|T^*|U)^{\frac{1}{2}} \\
&= (T^*|T|^2T)^{\frac{1}{2}} = |T^2|.
\end{aligned}$$

Hence we obtain (3.1) in case $n = 1$.

Assume that (3.1) holds for $n = 1, 2, \cdots k-1$. Raising each side of (3.1) to the power $\frac{1}{n} \in [0,1]$ by Löwner-Heinz theorem, we have

$$|T^k|^{\frac{2}{k}} \geq |T^{k-1}|^{\frac{2}{k-1}} \geq \cdots \geq |T^2| \geq |T|^2. \tag{3.4}$$

Moreover, by using Löwner-Heinz theorem, the p-hyponormality of T and (3.4) imply the following inequalities.

$$|T^k|^{\frac{2p}{k}} \geq |T^{k-1}|^{\frac{2p}{k-1}} \geq \cdots \geq |T^2|^p \geq |T|^{2p} \geq |T^*|^{2p}. \tag{3.5}$$

By (3.5), we have $|T^k|^{\frac{2}{k}p} \geq |T^*|^{2p}$. Then for each $t \geq 0$ and $q \geq 0$, $g_{t,q}(s) = (|T^*|^t|T^k|^{\frac{2}{k}s}|T^*|^t)^{\frac{q+t}{s+t}}$ is increasing for $s \geq q \geq 0$ by (ii) in Theorem E. Then we have

$$\begin{aligned}
|T^k|^2 &= T^*|T^{k-1}|^2T \\
&\leq T^*|T^k|^{\frac{2(k-1)}{k}}T \qquad \text{by (3.1) for } n = k-1 \\
&= U^*|T^*||T^k|^{\frac{2(k-1)}{k}}|T^*|U \\
&= U^*(|T^*||T^k|^{\frac{2}{k}(k-1)}|T^*|)^{\frac{k-1+1}{k-1+1}}U \\
&= U^*g_{1,k-1}(k-1)U \\
&\leq U^*g_{1,k-1}(k)U \\
&= U^*(|T^*||T^k|^{\frac{2}{k}k}|T^*|)^{\frac{k-1+1}{k+1}}U \\
&= (U^*|T^*||T^k|^2|T^*|U)^{\frac{k}{k+1}} \\
&= (T^*|T^k|^2T)^{\frac{k}{k+1}} = |T^{k+1}|^{\frac{2k}{k+1}}.
\end{aligned}$$

Hence we obtain (3.1) for all positive integer n.

Then we have $(T^{n+1^*}T^{n+1})^{\frac{1}{n+1}} \geq (T^{n*}T^n)^{\frac{1}{n}}$ for all positive integer n by (3.1) and Löwner-Heinz theorem.

Proof of (ii). We will use induction to establish the inequality

$$|T^{n+1^*}|^{\frac{2n}{n+1}} \leq |T^{n*}|^2 \qquad \text{for all positive integer } n. \tag{3.6}$$

In case $n = 1$. Suppose that T is p-hyponormal. T is p-hyponormal if and only if

$$|T|^{2p} \geq |T^*|^{2p}. \tag{3.2}$$

We obtain the following by (3.2).

$$T|T^*|^{2p}T^* \leq T|T|^{2p}T^* = T(T^*T)^pT^* = |T^*|^{2(p+1)}. \tag{3.7}$$

On the other hand, by (3.2) and (i) of Theorem E, for each $t \geq 0$ and $q \geq 0$, $f_{t,q}(s) = (|T|^t|T^*|^{2s}|T|^t)^{\frac{q+t}{s+t}}$ is decreasing for $s \geq q \geq 0$. Then we have

$$\begin{aligned}
|T^*|^2 &\geq (T|T^*|^{2p}T^*)^{\frac{1}{p+1}} \qquad \text{by (3.7) and Löwner-Heinz theorem} \\
&= (U|T||T^*|^{2p}|T|U^*)^{\frac{1}{p+1}} \\
&= U(|T||T^*|^{2p}|T|)^{\frac{1}{p+1}}U^* \\
&= Uf_{1,0}(p)U^* \\
&\geq Uf_{1,0}(1)U^* \\
&= U(|T||T^*|^2|T|)^{\frac{1}{2}}U^* \\
&= (U|T||T^*|^2|T|U^*)^{\frac{1}{2}} \\
&= (T|T^*|^2T^*)^{\frac{1}{2}} = |T^{2^*}|.
\end{aligned}$$

Hence we obtain (3.6) in case $n = 1$.

Assume that (3.6) holds for $n = 1, 2, \cdots k-1$. Raising each side of (3.6) to the power $\frac{1}{n} \in [0, 1]$ by Löwner-Heinz theorem, we have

$$|T^{k^*}|^{\frac{2}{k}} \leq |T^{k-1^*}|^{\frac{2}{k-1}} \leq \cdots \leq |T^{2^*}| \leq |T^*|^2. \tag{3.8}$$

Moreover, by using Löwner-Heinz theorem, the p-hyponormality of T and (3.8) imply the following inequalities.

$$|T^{k^*}|^{\frac{2p}{k}} \leq |T^{k-1^*}|^{\frac{2p}{k-1}} \leq \cdots \leq |T^{2^*}|^p \leq |T^*|^{2p} \leq |T|^{2p}. \tag{3.9}$$

By (3.9), we have $|T^{k^*}|^{\frac{2}{k}p} \leq |T|^{2p}$. Then for each $t \geq 0$ and $q \geq 0$, $f_{t,q}(s) = (|T|^t|T^{k^*}|^{\frac{2}{k}s}|T|^t)^{\frac{q+t}{s+t}}$ is decreasing for $s \geq q \geq 0$, by (i) of Theorem E. Then we have

$$\begin{aligned}
|T^{k^*}|^2 &= T|T^{k-1^*}|^2T^* \\
&\geq T|T^{k^*}|^{\frac{2(k-1)}{k}}T^* \qquad \text{by (3.6) for } n = k-1 \\
&= U|T||T^{k^*}|^{\frac{2(k-1)}{k}}|T|U^* \\
&= U(|T||T^{k^*}|^{\frac{2}{k}(k-1)}|T|)^{\frac{k-1+1}{k-1+1}}U^* \\
&= Uf_{1,k-1}(k-1)U^* \\
&\geq Uf_{1,k-1}(k)U^* \\
&= U(|T||T^{k^*}|^{\frac{2}{k}k}|T|)^{\frac{k-1+1}{k+1}}U^* \\
&= (U|T||T^{k^*}|^2|T|U^*)^{\frac{k}{k+1}} \\
&= (T|T^{k^*}|^2T^*)^{\frac{k}{k+1}} = |T^{k+1}|^{\frac{2k}{k+1}}.
\end{aligned}$$

Hence we obtain (3.6) for all positive integer n.

Then we have $(T^{n+1}T^{n+1^*})^{\frac{1}{n+1}} \leq (T^nT^{n^*})^{\frac{1}{n}}$ for all positive integer n by (3.6) and Löwner-Heinz theorem.

Whence the proof of Theorem 1 is complete. □

We need the following Theorem F in order to give a proof of Theorem 2.

Theorem F ([11]). *Every* log*-hyponormal operator is class $A(k)$ for $k > 0$.*

Proof of Theorem 2. Let $T = U|T|$ be the polar decomposition of T.

Proof of (i). We will use induction to establish the inequality

(3.1) $$|T^{n+1}|^{\frac{2n}{n+1}} \geq |T^n|^2 \qquad \text{for all positive integer } n.$$

In case $n = 1$. Suppose that T is log-hyponormal. Then T is class A by Theorem F. T is class A if and only if $|T^2| \geq |T|^2$. Hence we obtain (3.1) in case $n = 1$.

Assume that (3.1) holds for $n = 1, 2, \cdots k-1$. Raising each side of (3.1) to the power $\frac{1}{n} \in [0,1]$ by Löwner-Heinz theorem, we have

(3.10) $$|T^k|^{\frac{2}{k}} \geq |T^{k-1}|^{\frac{2}{k-1}} \geq \cdots \geq |T^2| \geq |T|^2.$$

Moreover, by the log-hyponormality of T and (3.10), we obtain the following inequalities.

$$\log |T^k|^{\frac{2}{k}} \geq \cdots \geq \log |T^2| \geq \log |T|^2 \geq \log |T^*|^2. \tag{3.11}$$

By (3.11), we have $\log |T^k|^{\frac{2}{k}} \geq \log |T^*|^2$. Then for each $t \geq 0$ and $q \geq 0$, $g_{t,q}(s) = (|T^*|^t |T^k|^{\frac{2}{k}s} |T^*|^t)^{\frac{q+t}{s+t}}$ is increasing for $s \geq q \geq 0$ by (ii) in Theorem E. Then we have

$$\begin{aligned}
|T^k|^2 &= T^*|T^{k-1}|^2 T \\
&\leq T^*|T^k|^{\frac{2(k-1)}{k}} T \qquad \text{by (3.1) for } n = k-1 \\
&= U^*|T^*||T^k|^{\frac{2(k-1)}{k}}|T^*|U \\
&= U^*(|T^*||T^k|^{\frac{2}{k}(k-1)}|T^*|)^{\frac{k-1+1}{k-1+1}} U \\
&= U^* g_{1,k-1}(k-1) U \\
&\leq U^* g_{1,k-1}(k) U \\
&= U^*(|T^*||T^k|^{\frac{2}{k}k}|T^*|)^{\frac{k-1+1}{k+1}} U \\
&= (U^*|T^*||T^k|^2|T^*|U)^{\frac{k}{k+1}} \\
&= (T^*|T^k|^2 T)^{\frac{k}{k+1}} = |T^{k+1}|^{\frac{2k}{k+1}}.
\end{aligned}$$

Hence we obtain (3.1) for all positive integer n.

Then we have $(T^{n+1*}T^{n+1})^{\frac{1}{n+1}} \geq (T^{n*}T^n)^{\frac{1}{n}}$ for all positive integer n by (3.1) and Löwner-Heinz theorem.

Proof of (ii). We will use induction to establish the inequality

$$|T^{n+1*}|^{\frac{2n}{n+1}} \leq |T^{n*}|^2 \qquad \text{for all positive integer } n. \tag{3.6}$$

In case $n = 1$. Suppose that T is log-hyponormal. By Theorem F, T is class A if and only if

$$(T^*|T|^2 T)^{\frac{1}{2}} = |T^2| \geq |T|^2. \tag{3.12}$$

By Lemma C, then (3.12) is equivalent to the following.

$$T^*|T|(|T|TT^*|T|)^{\frac{-1}{2}}|T|T \geq T^*T.$$

Then we have

$$|T|^2 \geq (|T||T^*|^2|T|)^{\frac{1}{2}}. \tag{3.13}$$

By (3.13), we have

$$\begin{aligned}|T^*|^2 &= U|T|^2U^* \\ &\geq U(|T||T^*|^2|T|)^{\frac{1}{2}}U^* \\ &= (U|T||T^*|^2|T|U^*)^{\frac{1}{2}} \\ &= (T|T^*|^2T^*)^{\frac{1}{2}} = |T^{2^*}|.\end{aligned}$$

Hence we obtain (3.6) in case $n = 1$.

Assume that (3.6) holds for $n = 1, 2, \cdots k-1$. Raising each side of (3.6) to the power $\frac{1}{n} \in [0,1]$ by Löwner-Heinz theorem, we have

$$|T^{k^*}|^{\frac{2}{k}} \leq |T^{k-1^*}|^{\frac{2}{k-1}} \leq \cdots \leq |T^{2^*}| \leq |T^*|^2. \tag{3.14}$$

Moreover, by the log-hyponormality of T and (3.14), we obtain the following inequalities.

$$\log |T^{k^*}|^{\frac{2}{k}} \leq \cdots \leq \log |T^{2^*}| \leq \log |T^*|^2 \leq \log |T|^2. \tag{3.15}$$

By (3.15), we have $\log |T^{k^*}|^{\frac{2}{k}} \leq \log |T|^2$. Then for each $t \geq 0$ and $q \geq 0$, $f_{t,q}(s) = (|T|^t|T^{k^*}|^{\frac{2}{k}s}|T|^t)^{\frac{q+t}{s+t}}$ is decreasing for $s \geq q \geq 0$ by (i) in Theorem E. Then we have

$$\begin{aligned}|T^{k^*}|^2 &= T|T^{k-1^*}|^2T^* \\ &\geq T|T^{k^*}|^{\frac{2(k-1)}{k}}T^* \qquad \text{by (3.6) for } n = k-1 \\ &= U|T||T^{k^*}|^{\frac{2(k-1)}{k}}|T|U^* \\ &= U(|T||T^{k^*}|^{\frac{2}{k}(k-1)}|T|)^{\frac{k-1+1}{k-1+1}}U^* \\ &= Uf_{1,k-1}(k-1)U^* \\ &\geq Uf_{1,k-1}(k)U^* \\ &= U(|T||T^{k^*}|^{\frac{2}{k}k}|T|)^{\frac{k-1+1}{k+1}}U^* \\ &= (U|T||T^{k^*}|^2|T|U^*)^{\frac{k}{k+1}} \\ &= (T|T^{k^*}|^2T^*)^{\frac{k}{k+1}} = |T^{k+1^*}|^{\frac{2k}{k+1}}.\end{aligned}$$

Hence we obtain (3.6) for all positive integer n.

Then we have $(T^{n+1}T^{n+1^*})^{\frac{1}{n+1}} \leq (T^nT^{n^*})^{\frac{1}{n}}$ for all positive integer n by (3.6) and Löwner-Heinz theorem.

Whence the proof of Theorem 2 is complete. □

Proof of Corollary 3. By Theorem 2,

(i) $(T^*T) \le (T^{2*}T^2)^{\frac{1}{2}} \le \cdots \le (T^{n*}T^n)^{\frac{1}{n}}$

and

(ii) $(TT^*) \ge (T^2T^{2*})^{\frac{1}{2}} \ge \cdots \ge (T^nT^{n*})^{\frac{1}{n}}$

hold for all positive integer n. Since T is log-hyponormal, we have

$$\log(T^{n*}T^n)^{\frac{1}{n}} \ge \log(T^*T) \ge \log(TT^*) \ge \log(T^nT^{n*})^{\frac{1}{n}}.$$

Hence $\log(T^{n*}T^n) \ge \log(T^nT^{n*})$ holds for all positive integer n, i.e., T^n is log-hyponormal. □

Acknowledgement. I would like to express my cordial thanks to Professor Takayuki Furuta for his guidance and encouragement.

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