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WEIGHTED $L^p - L^q$ INEQUALITIES FOR THE FRACTIONAL INTEGRAL OPERATOR WHEN $1 < q < p < \infty$

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Abstract. We find necessary conditions and sufficient conditions on weights $u(.)$ and $v(.)$ for which the fractional integral operator I_α is bounded from the weighted Lebesgue spaces L^p_v into L^q_u whenever $1 < q < p < \infty$ and $0 < \alpha < n$. Actually such a boundedness is characterized for a large class of weights.

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§1. INTRODUCTION

The fractional integral operator I_α of order α, $0 < \alpha < n$, acts on locally integrable functions of $\mathbb{R}^n$ as

$$(I_\alpha f)(x) = \int_{y\in\mathbb{R}^n} |x-y|^{\alpha-n} f(y)dy.$$

Our purpose in this paper is to derive conditions on weight functions $u(.)$ and $v(.)$ for which there is a constant $C > 0$ such that
(1.1)
$$\left(\int_{\mathbb{R}^n} (I_\alpha f)^q(x)u(x)dx\right)^{\frac{1}{q}} \leq C\left(\int_{\mathbb{R}^n} f^p(x)v(x)dx\right)^{\frac{1}{p}} \qquad \text{for all } f(.) \geq 0$$

and for $1 < q < p < \infty$. The boundedness defined by (1.1) will be also denoted by $I_\alpha : L^p_v \to L^q_u$.

Since inequalities (1.1) have a fundamental role in Analysis (in deriving weighted Poinca ré and Sobolev inequalities, in estimating eigenvalues of some Schrödinger operators,....), they have been studied extensively by many authors for the range $p \leq q$. Recent papers on this topic can be found in [Sa-Wh-Zh] for the American school, in [Ge-Go-Ko] for the Georgian school and in [Ra2] for the author's contribution. Considering (1.1) for the range $q < p$ would enlarge for instance the available results (for $p \leq q$) for weighted Sobolev and Poincaré inequalities.

A significant attempt on a characterization for $I_\alpha : L^p_v \to L^q_u$ with $q < p$, based on a previous work of I. Verbitski [Ve] and E. Sawyer [Sa-Wh-Zh], was done by S. Zhao [Zh]. In the present work we do not investigate on such a question since a necessary and sufficient condition with general weight functions would be useless for practical computations mainly when it is expressed in term of the operator I_α itself and integrations over some set of cubes (see for instance Theorem 1.2, p.98 in [Zh]).

According to a work of I. Verbitski [Ve], a necessary condition for the boundedness $I_\alpha : L^p_v \to L^q_u$ when $q < p$ is

$$\int_{x\in\mathbb{R}^n} \Phi^r(x)u(x)dx < \infty \qquad \text{with } r = \frac{pq}{p-q} \tag{1.2}$$

and

$$\Phi(x) = \sup_{Q\ni x}\Big\{|Q|^{\frac{\alpha}{n}}\Big(\frac{1}{|Q|}\int_Q v^{1-p'}(y)dy\Big)^{\frac{1}{p'}}\Big(\frac{1}{|Q|}\int_Q u(y)dy\Big)^{\frac{1}{p}}\Big\}.$$

Here $p' = \frac{p}{p-1}$ and Q are arbitrary cubes with sides parallel to the coordinates axes. Conversely in [Ra1] (Theorem 2.1, p312), the boundedness $I_\alpha : L^p_v \to L^q_u$ is seen to be held provided that for some $t_1, t_2 > 1$

$$\int_{x\in\mathbb{R}^n} \Phi^r_{t_1,t_2}(x)u(x)dx < \infty \tag{1.3}$$

where

$$\Phi_{t_1,t_2}(x) = \sup_{Q\ni x}\Big\{|Q|^{\frac{\alpha}{n}}\Big(\frac{1}{|Q|}\int_Q v^{(1-p')t_1}(y)dy\Big)^{\frac{1}{p't_1}}\Big(\frac{1}{|Q|}\int_Q u^{t_2}(y)dy\Big)^{\frac{1}{pt_2}}\Big\}.$$

Obviously, by the Hölder inequality, condition (1.3) is stronger than (1.2). The interest on the implication (1.3) $\Longrightarrow$ (1.1) is that the sufficient condition (1.3) is not expressed in term of I_α. However the reader would be aware of the difficulty in checking (1.3). This problem is studied in the remainder of results in [Ra1].

One of the motivations of our present work is the observation that condition (1.3) is not always applicable due to the high integrability required for the weights $u(.)$ and $v^{1-p'}(.)$. Indeed taking $v^{1-p'}(x) = |x|^{-n}\ln^{-p'}(|x|^{-1})$ for $|x| < \frac{1}{2}$ then $\int_{|x|<R} v^{(1-p')t}(x)dx = \infty$, for all $t > 1$ and $R < \frac{1}{2}$, though $\int_{|x|<R} v^{1-p'}(x)dx < \infty$. However for such a weight $v(.)$ (see Corollary 2.4) the boundedness (1.1) can be held. Our second motivation is that a simple characterization for the two-weight inequality (1.1) with $1 < q < p < \infty$ can be derived for a large class of weight functions including those of radial and monotone ones.

Necessary conditions for $I_\alpha : L^p_v \to L^q_u$ with $q < p$ will be stated in Theorem 2.1. These conditions are of two types: the Hardy conditions and the Muckenhoupt condition. In general they are not together sufficient to derive the above boundedness. However in Theorem 2.2, we will see that with a slight strong version of the Muckenhoupt condition then inequality (1.1) can be derived. Consequently, a characterization for (1.1) for many usual weights will be found in Proposition 2.3. Concrete and explicit examples, which cannot be decided from results in [Ra1] and [Zh], will be given in Corollary 2.4.

Our results, stated in §2, are based on the "principle of three parts proof" already used by the author in [Ra2] to tackle the boundedness problem for the case $p \le q$. Two useful basic lemmas are given in §3. And the proofs of all results are performed in the last section.

§2. RESULTS

Throughout this paper it is always assumed that

$$0 < \alpha < n, \qquad 1 < q < p < \infty, \qquad p' = \frac{p}{p-1}, \qquad q' = \frac{q}{q-1},$$

$$r = \frac{pq}{p-q} \qquad \text{or} \qquad \frac{1}{r} = \frac{1}{q} - \frac{1}{p},$$

$$\text{and} \qquad u(.), \qquad v^{1-p'}(.) \qquad \text{are weight functions.}$$

We first give some natural necessary conditions for the boundedness (1.1) to be satisfied.

Theorem 2.1. *Assume the boundedness $I_\alpha : L^p_v \to L^q_u$ does hold. Then*

(2.1)
$$\int_{x\in\mathbb{R}^n} \left[\left(\int_{|x|<|y|} |y|^{(\alpha-n)q} u(y)dy \right)^{\frac{1}{q}} \left(\int_{|z|<|x|} v^{1-p'}(z)dz \right)^{\frac{1}{q'}} \right]^r v^{1-p'}(x)dx < \infty$$

(2.1*)
$$\int_{x\in\mathbb{R}^n} \left[\left(\int_{|x|<|z|} |z|^{(\alpha-n)p'} v^{1-p'}(z)dz \right)^{\frac{1}{p'}} \left(\int_{|y|<|x|} u(y)dy \right)^{\frac{1}{p}} \right]^r u(x)dx < \infty$$

and for each integer $N \ge 1$

$$\sum_{k=-\infty}^{\infty} \left(\mathcal{A}_N(k) \right)^r < \infty \tag{2.2}$$

where

(2.3)
$$\mathcal{A}_N(k) = 2^{k(\alpha-n)} \left(\int_{2^{k-N}<|y|<2^{k+N}} u(y)dy \right)^{\frac{1}{q}} \left(\int_{2^{k-N}<|z|<2^{k+N}} v^{1-p'}(z)dz \right)^{\frac{1}{p'}}.$$

Conditions (2.1) and (2.2) will be often referred as Hardy and Muckenhoupt conditions respectively, and (2.1*) is named as the dual condition of (2.1). Both the Hardy conditions and the Muckenhoupt condition would not be sufficient in general to imply the boundedness (1.1) as it is the case for $p \leq q$.

Our next main result states that this boundedness can be obtained just by using a (sligthly) stronger condition than (2.2). Precisely

Theorem 2.2. *The boundedness $I_\alpha : L^p_v \to L^q_u$ does hold provided that the Hardy conditions* (2.1) *and* (2.1*) *are satisfied and*

$$\sum_{k=-\infty}^{\infty} \big(\widetilde{\mathcal{A}}(k)\big)^r < \infty \tag{2.4}$$

where

$$\widetilde{\mathcal{A}}(k) = 2^{kn[\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}]}\Big(\sup_{2^{k-1}<|x|<2^{k+1}} u(x)\Big)^{\frac{1}{q}}\Big(\sup_{2^{k-1}<|z|<2^{k+2}} v^{1-p'}(z)\Big)^{\frac{1}{p'}}. \tag{2.5}$$

Observe that condition (2.4) is stronger than (2.2) with $N = 1$, since for some fixed constant $c > 0$, which only depends on n, α, p and q:

$$\mathcal{A}(k) = \mathcal{A}_1(k) \leq c\widetilde{\mathcal{A}}(k) \qquad \text{for all integers } k.$$

But (2.4) is not too far from the necessary condition (2.2) since for a large class of weights it turns out that $\widetilde{\mathcal{A}}(k) \leq c_1 \mathcal{A}_N(k)$, for some constant $c_1 > 0$ and integer $N \geq 1$ which only depends on these weights. Precisely, an additional property required for each weight to realize this last inequality is the condition $\mathcal{H}$. That is $w(.) \in \mathcal{H}$ whenever

$$\sup_{4^{-1}R<|y|<4R} w(y) \leq \frac{C}{R^n}\int_{2^{-N}R<|y|<2^N R} w(z)dz \qquad \text{for all } R > 0. \tag{2.6}$$

Here the integer $N \geq 1$ and the constant $C > 0$ depend only on $w(.)$. For a radial and monotone weight $w(.)$, property (2.6) is fulfilled with $N = 3$ and $C > 0$ only depending on n but not on $w(.)$. There exists also non-necessarily monotone weight for which (2.6) is satisfied, as the case of $w(x) = w_1(x)\mathbb{I}_{|x|<1}(x)+w_2(x)\mathbb{I}_{|x|>1}(x)$ with $w_1(.)$ and $w_2(.)$ are radial and monotone (a proof is given in [Ra1]).

Therefore a (simple) characterization for $I_\alpha : L^p_v \to L^q_u$ for weights having property (2.6) is now available from our previous results.

Proposition 2.3. *Let $u(.)$, $v^{1-p'}(.) \in \mathcal{H}$. The boundedness $I_\alpha : L^p_v \to L^q_u$ does hold if and only if both the Hardy conditions* (2.1), (2.1*) *and the Muckenhoupt condition* (2.2) *are satisfied.*

Note that in this result, the integer $N \geq 1$ involved in condition (2.2) would depend on properties $\mathcal{H}$ but not directly on the weights.

We will end with explicit examples showing the gain over results in [Ra1] and [Zh].

Corollary 2.4. *Define the weight functions*

$$\begin{aligned} u(x) =& |x|^{\beta-n}1\!\!1_{|x|<\frac{1}{2}}(x) + |x|^{\gamma-n}1\!\!1_{|x|>\frac{1}{2}}(x),\\ v(x) =& |x|^{(p-1)n}\ln^p(|x|^{-1})1\!\!1_{|x|<\frac{1}{2}}(x) + |x|^{\theta-n}1\!\!1_{|x|>\frac{1}{2}}(x). \end{aligned}$$

Suppose $0<\gamma$, $(n-\alpha)q<\beta$ *and* $\theta<np$. *Then* $I_\alpha : L^p_v \to L^q_u$ *if and only if*

$$\text{i) } \alpha p < \theta \quad \text{ii) } \gamma < (n-\alpha)q \quad \text{and} \quad \text{iii) } \alpha + \frac{\gamma}{q} < \frac{\theta}{p}.$$

Also set

$$\begin{aligned} u^*(x) =& |x|^{-n}\ln^{-q}|x|^{-1}1\!\!1_{|x|<\frac{1}{2}}(x) + |x|^{(1-q)(\theta-n)}1\!\!1_{|x|>\frac{1}{2}}(x),\\ v^*(x) =& |x|^{(1-p)(\beta-n)}1\!\!1_{|x|<\frac{1}{2}}(x) + |x|^{(1-p)(\gamma-n)}1\!\!1_{|x|>\frac{1}{2}}(x). \end{aligned}$$

Suppose $0<\gamma$, $(n-\alpha)p'<\beta$ *and* $\theta<nq'$. *Then* $I_\alpha : L^p_{v^*} \to L^q_{u^*}$ *if and only if*

$$\text{iv) } \alpha q' < \theta \quad \text{v) } \gamma < (n-\alpha)p' \quad \text{and} \quad \text{vi) } \alpha + \frac{\gamma}{p'} < \frac{\theta}{q'}.$$

As mentioned in the introduction, for these examples the boundedness $I_\alpha : L^p_v \to L^q_u$ is not obtainable from criterion (1.3) since $\int_{|x|<R} v^{(1-p')t}(x)dx = \infty$ and $\int_{|x|<R} u^{*t}(x)dx = \infty$, for all $t>1$ and $R<\frac{1}{2}$. Also criteria given in [Zh] seem to be difficult to apply for these concrete and explicit examples.

§3. BASIC LEMMAS

First we state a basic Lemma needed for the proofs of Theorem 2.1 and Corollary 2.4.

Lemma 3.1. *Let $0 \leq a < b$ and $0<\gamma$. Then there is a constant $c>0$ such that for all $h(.) \geq 0$:*

$$(3.1) \qquad \Big(\int_{a<|x|<b} h(x)dx\Big)^{1+\gamma} = c\int_{a<|x|<b}\Big[\int_{a<|y|<|x|} h(y)dy\Big]^\gamma h(x)dx.$$

Identity (3.1) can be obtained from the corresponding one-dimensional result after using polar coordinates.

The next result is about the n-dimensional weighted Hardy inequality

(3.2)
$$\Big(\int_{x\in\mathbb{R}^n}\Big[\int_{|y|<|x|} f(y)dy\Big]^q w(x)dx\Big)^{\frac{1}{q}} \le cA\Big(\int_{x\in\mathbb{R}^n} f^p(x)v(x)dx\Big)^{\frac{1}{p}}$$
$$\text{for all } f(.)\ge 0.$$

Lemma 3.2. *Suppose that for some constant $A>0$*

(3.3)
$$\int_{x\in\mathbb{R}^n}\Big[\Big(\int_{|x|<|y|} w(y)dy\Big)^{\frac{1}{q}}\Big(\int_{|z|<|x|} v^{1-p'}(z)dz\Big)^{\frac{1}{q'}}\Big]^r v^{1-p'}(x)dx \le A^r.$$

Then inequality (3.2) *is satisfied for a constant $c>0$ which only depends on n, p and q. Conversely the Hardy condition* (3.3) *is a necessary condition for inequality* (3.2) *to hold.*

A proof of this result was given by P. Drável, H. Heinig and A. Kufner [Dr-He-Ku] (see Theorem 2.2, p7-8).

§4. PROOFS OF RESULTS

Proof of Theorem 2.1.

The implication (1.1) $\Longrightarrow$ (2.1). Observe that

$$|x|^{\alpha-n}\int_{|y|<|x|} f(y)dy \le 2^{n-\alpha}\int_{|x-y|<2|x|} |x-y|^{\alpha-n}f(y)dy \le 2^{n-\alpha}(I_\alpha f)(x)$$

for all $f(.)\ge 0$. So the boundedness $I_\alpha : L^p_v \to L^q_u$ implies the Hardy inequality (3.2) with $w(x)=|x|^{q(\alpha-n)}u(x)$. Consequently, condition (3.3) arises because of the second part of Lemma 3.2. The Hardy condition (2.1) is nothing else than (3.3) due to this choice of $w(.)$.

The implication (1.1) $\Longrightarrow$ (2.1*). By a duality argument, inequality (1.1) is equivalent to $I_\alpha : L^{q'}_{u^{1-q'}} \to L^{p'}_{v^{1-p'}}$. By analogue arguments as used for the implication (1.1) $\Longrightarrow$ (2.1), this last boundedness implies condition (2.1*), since $p'<q'$.

The implication (1.1) $\Longrightarrow$ (2.2). Let us fix nonnegative integers N, $M\ge 1$ and define the function

$$g_{N,M}(x)$$
$$=\sum_{k=-M}^{N} 2^{k(\alpha-n)\frac{r}{p}}\Big(\int_{2^{k-N_0}<|z|<2^{k+N_0}} u(z)dz\Big)^{\frac{r}{pq}}\Big(\int_{2^{k-N_0}<|z|<|x|} v^{1-p'}(z)dz\Big)^{\frac{r}{pq'}}$$

$$\times\, v^{1-p'}(x)\,\mathbb{I}_{2^{k-N_0}<|x|<2^{k+N_0}}(x)$$

Here $N_0 \geq 1$ is the integer (N) involved in condition (2.2). Obviously

(4.1)
$$B(N,M) = \sum_{k=-M}^{N}\Big[2^{k(\alpha-n)}\Big(\int_{2^{k-N_0}<|z|<2^{k+N_0}} u(z)dz\Big)^{\frac{1}{q}}\times \Big(\int_{2^{k-N_0}<|z|<2^{k+N_0}} v^{1-p'}(z)dz\Big)^{\frac{1}{p'}}\Big]^r < \infty$$

and it can be assumed that $B(N,M) > 0$. The points keys for obtaining (2.2) are

$$\int_{x\in\mathbb{R}^n} g^p_{N,M}(x)v(x)dx \leq c_0 B(N,M) \tag{4.2}$$

and

$$\int_{x\in\mathbb{R}^n} (I_\alpha g_{N,M})^q(x)u(x)dx \geq cB(N,M) \tag{4.3}$$

for some constants c_0, $c > 0$ which do not depend on the integers N and M. Indeed with (4.3) and (4.2), the boundedness $I_\alpha : L^p_v \to L^q_u$ yields $\Big(B(N,M)\Big)^{\frac{1}{q}} \leq c_1\Big(B(N,M)\Big)^{\frac{1}{p}}$. This last inequality, point (4.1) and $\frac{1}{r} = \frac{1}{q} - \frac{1}{p} > 0$ lead to

$$B(N,M) \leq c_1^r.$$

The Muckenhoupt condition (2.2) arises from this last estimate by letting $N, M \to \infty$. At this point, the proof of (1.1) $\Longrightarrow$ (2.2) is now reduced to that of (4.2) and (4.3).

Inequality (4.2) follows after using the definition of $g_{N,M}(.)$, the identity $p(1-p')+1 = (1-p')$, the identity (3.1) (with $h(.) = v^{1-p'}(.)$ and $\gamma = \frac{r}{q'}$) and the fact that $\mathbb{I}_{2^{k-N_0}<|.|<2^{k+N_0}}(.) = \sum_{l=-N_0}^{N_0-1} \mathbb{I}_{2^{k+l}<|.|<2^{k+l+1}}(.)$ almost everywhere. Indeed

$$\begin{aligned} n\int_{\mathbb{R}^n} & g^p_{N,M}(x)v(x)dx \\ &\leq c(N_0)\sum_{l=-N_0}^{N_0-1}\sum_{k=-M}^{N} 2^{k(\alpha-n)r}\Big(\int_{2^{k-N_0}<|z|<2^{k+N_0}} u(z)dz\Big)^{\frac{r}{q}}\times \\ &\qquad \int_{2^{k+l}<|x|<2^{k+l+1}}\Big[\int_{2^{k-N_0}<|z|<|x|} v^{1-p'}(z)dz\Big]^{\frac{r}{q'}} v^{1-p'}(x)dx \end{aligned}$$

$$\leq c(N_0)\sum_{k=-M}^{N}2^{k(\alpha-n)r}\Big(\int_{2^{k-N_0}<|z|<2^{k+N_0}}u(z)dz\Big)^{\frac{r}{q}}\times$$
$$\int_{2^{k-N_0}<|x|<2^{k+N_0}}\Big[\int_{2^{k-N_0}<|z|<|x|}v^{1-p'}(z)dz\Big]^{\frac{r}{q'}}v^{1-p'}(x)dx$$
$$= c_1(N_0)\sum_{k=-M}^{N}2^{k(\alpha-n)r}\Big(\int_{2^{k-N_0}<|z|<2^{k+N_0}}u(z)dz\Big)^{\frac{r}{q}}\times$$
$$\Big(\int_{2^{k-N_0}<|z|<2^{k+N_0}}v^{1-p'}(z)dz\Big)^{\frac{r}{p'}}.$$

Estimate (4.3) is based on

$$\int_{2^{k-N_0}<|y|<2^{k+N_0}}g_{N,M}(y)dy\geq c2^{k(\alpha-n)\frac{r}{p}}\times \tag{4.4}$$
$$\Big(\int_{2^{k-N_0}<|z|<2^{k+N_0}}u(z)dz\Big)^{\frac{r}{pq}}\Big(\int_{2^{k-N_0}<|z|<2^{k+N_0}}v^{1-p'}(z)dz\Big)^{\frac{r}{pq'}+1}.$$

Indeed from this last inequality it follows that

$$\int_{\mathbb{R}^n}(I_\alpha g_{N,M})^q(x)u(x)dx$$
$$\geq c(N_0)\sum_{k=-\infty}^{\infty}\int_{2^{k-N_0}<|x|<2^{k+N_0}}(I_\alpha g_{N,M})^q(x)u(x)dx$$
$$\geq c(N_0)\sum_{k=-M}^{N}\int_{2^{k-N_0}<|x|<2^{k+N_0}}\Big[\int_{2^{k-N_0}<|y|<2^{k+N_0}}|x-y|^{\alpha-n}g_{N,M}(y)dy\Big]^q u(x)dx$$
$$\geq c_1\sum_{k=-M}^{N}2^{k(\alpha-n)q}\Big(\int_{2^{k-N_0}<|y|<2^{k+N_0}}g_{N,M}(y)dy\Big)^q\Big(\int_{2^{k-N_0}<|z|<2^{k+N_0}}u(z)dz\Big)$$
$$\geq c_2\sum_{k=-M}^{N}2^{k(\alpha-n)q(1+\frac{r}{p})}\Big(\int_{2^{k-N_0}<|z|<2^{k+N_0}}u(z)dz\Big)^{\frac{r}{p}+1}\times$$
$$\Big(\int_{2^{k-N_0}<|z|<2^{k+N_0}}v^{1-p'}(z)dz\Big)^{q(\frac{r}{pq'}+1)}$$
$$= c_2\sum_{k=-M}^{N}\Big[2^{k(\alpha-n)}\Big(\int_{2^{k-N_0}<|z|<2^{k+N_0}}u(z)dz\Big)^{\frac{1}{q}}\Big(\int_{2^{k-N_0}<|z|<2^{k+N_0}}v^{1-p'}(z)dz\Big)^{\frac{1}{p'}}\Big]^r$$
$$= c_2B(N,M).$$

To derive (4.4), the point is the identity (3.1). Indeed, for each integer $k\in\{-M,\ldots,N\}$,

$$
\begin{aligned}
\int_{2^{k-N_0}<|y|<2^{k+N_0}} g_{N,M}(y)dy &\geq 2^{k(\alpha-n)\frac{r}{p}}\Big(\int_{2^{k-N_0}<|z|<2^{k+N_0}} u(z)dz\Big)^{\frac{r}{pq}} \times \\
&\qquad \int_{2^{k-N_0}<|y|<2^{k+N_0}} \Big[\int_{2^{k-N_0}<|z|<|y|} v^{1-p'}(z)dz\Big]^{\frac{r}{pq'}} v^{1-p'}(y)dy \\
&\geq c2^{k(\alpha-n)\frac{r}{p}}\Big(\int_{2^{k-N_0}<|z|<2^{k+N_0}} u(z)dz\Big)^{\frac{r}{pq}} \times \\
&\qquad \Big(\int_{2^{k-N_0}<|z|<2^{k+N_0}} v^{1-p'}(z)dz\Big)^{\frac{r}{pq'}+1}.
\end{aligned}
$$

Proof of Theorem 2.2.

Since

$$
(I_\alpha f)(x) = A_1(x) + (\widetilde{I}_\alpha f)(x) + A_3(x) \qquad \text{for all } f(.) \geq 0,
$$

with

$$
\begin{aligned}
A_1(x) &= \int_{|y|\leq\frac{1}{2}|x|} |x-y|^{\alpha-n} f(y)dy \\
(\widetilde{I}_\alpha f)(x) = A_2(x) &= \int_{\frac{1}{2}|x|<|y|<2|x|} |x-y|^{\alpha-n} f(y)dy \\
A_3(x) &= \int_{2|x|\leq|y|} |x-y|^{\alpha-n} f(y)dy,
\end{aligned}
$$

then to get the boundedness $I_\alpha : L^p_v \to L^q_u$ it is sufficient to estimate each of $\int_{\mathbb{R}^n} A_i^q(x)u(x)dx$, $i \in \{1,2,3\}$, by $C\Big(\int_{\mathbb{R}^n} f^p(x)v(x)dx\Big)^{\frac{q}{p}}$, with C a nonnegative and fixed constant.

Observe that $A_1(x) \leq c|x|^{\alpha-n}\int_{|y|<|x|} f(y)dy$, since $\frac{1}{2}|x| < |x-y|$ whenever $|y| \leq \frac{1}{2}|x|$. By the Hardy condition (2.1) [which is (3.3) with $w(x) = |x|^{(\alpha-n)q}u(x)$] and by Lemma 3.2, the conclusion arises since

$$
\begin{aligned}
&\int_{\mathbb{R}^n} A_1^q(x)u(x)dx \\
&\leq c^q \int_{\mathbb{R}^n} \Big[\int_{|y|<|x|} f(y)dy\Big]^q |x|^{(\alpha-n)q}u(x)dx \\
&\leq C\Big(\int_{\mathbb{R}^n} f^p(x)v(x)dx\Big)^{\frac{q}{p}}.
\end{aligned}
$$

Note that $A_3(x) \leq c\int_{|x|<|y|} |y|^{\alpha-n} f(y)dy$, since $\frac{1}{2}|y| < |x-y|$ whenever $2|x| < |y|$. At this stage the conclusion also follows since

$$\begin{aligned}
&\int_{\mathbb{R}^n} A_3^q(x)u(x)dx \\
&\leq c^q \int_{\mathbb{R}^n} \Big[\int_{|x|<|y|} |y|^{\alpha-n} f(y)dy\Big]^q u(x)dx \\
&\leq C\Big(\int_{\mathbb{R}^n} f^p(x)v(x)dx\Big)^{\frac{q}{p}}.
\end{aligned}$$

This last Hardy inequality can be also obtained from Lemma 3.2 by using a duality argument. Indeed the problem is reduced to inequality (3.2) with q, p, $w(.)$ and $v(.)$ respectively replaced by p', q', $|.|^{(\alpha-n)p'}v^{1-p'}(.)$ and $u^{1-q'}(.)$. Therefore (2.1^*) yields the corresponding condition (3.3).

The real task is now to prove that

$$\int_{\mathbb{R}^n} (\widetilde{I}_\alpha f)^q(x)u(x)dx \leq C\Big(\int_{\mathbb{R}^n} f^p(x)v(x)dx\Big)^{\frac{q}{p}}$$

for all $f(.) \geq 0$. For this purpose, it is convenient to introduce the following notations:

$$E_k = \{z \in \mathbb{R}^n;\ 2^k < |x| \leq 2^{k+1}\}, \quad F_k = \{y \in \mathbb{R}^n;\ 2^{k-1} < |y| \leq 2^{k+2}\},$$

$$\mathcal{C}(x) = \{z \in \mathbb{R}^n;\ \frac{1}{2}|x| < |z| < 2|x|\}, \quad \mathcal{U}_k = \sup_{x\in E_k} u(x), \quad \mathcal{W}_k = \sup_{z\in F_k} v^{1-p'}(z).$$

Then

$$v^{1-p'}(z) \leq \mathcal{W}_k \qquad \text{whenever } z \in \mathcal{C}(x) \text{ and } x \in E_k.$$

By the Hölder inequality and this observation, for each $x \in E_k$

$$\begin{aligned}
&(\widetilde{I}_\alpha f)(x) \\
&\leq \Big(\int_{z\in\mathcal{C}(x)} |x-z|^{\alpha-n} v^{1-p'}(z)dz\Big)^{\frac{1}{p'}} \Big(\int_{y\in\mathcal{C}(x)} |x-y|^{\alpha-n} f^p(y)v(y)dy\Big)^{\frac{1}{p}} \\
&\leq c_1\Big(|x|^\alpha \mathcal{W}_k\Big)^{\frac{1}{p'}} \Big(\int_{y\in\mathcal{C}(x)} |x-y|^{\alpha-n} f^p(y)v(y)dy\Big)^{\frac{1}{p}} \\
&\leq c_2\Big(2^{k\alpha} \mathcal{W}_k\Big)^{\frac{1}{p'}} \Big(\int_{y\in\mathcal{C}(x)} |x-y|^{\alpha-n} f^p(y)v(y)dy\Big)^{\frac{1}{p}}.
\end{aligned}$$

Using this last inequality, the Hölder inequality, the Fubini theorem, conditions (2.4) and (2.5) and finally another Hölder inequality, the conclusion arises as follows

$$\int_{\mathbb{R}^n} (\widetilde{I}_\alpha f)^q u(x)dx = \sum_{k=-\infty}^{\infty} \int_{x\in E_k} (\widetilde{I}_\alpha f)^q u(x)dx$$

$$\le c_3 \sum_{k=-\infty}^{\infty} \Big(2^{k\alpha}\mathcal{W}_k\Big)^{\frac{q}{p'}} \mathcal{U}_k \int_{x\in E_k} \Big[\int_{y\in\mathcal{C}(x)} |x-y|^{\alpha-n} f^p(y)v(y)dy\Big]^{\frac{q}{p}} dx$$

$$\le c_3 \sum_{k=-\infty}^{\infty} 2^{k\alpha\frac{q}{p'}+kn(1-\frac{q}{p})}\mathcal{U}_k\mathcal{W}_k^{\frac{q}{p'}} \times$$

$$\Big[\int_{x\in E_k}\Big(\int_{y\in\mathcal{C}(x)} |x-y|^{\alpha-n} f^p(y)v(y)dy\Big)dx\Big]^{\frac{q}{p}}$$

$$\le c_3 \sum_{k=-\infty}^{\infty} 2^{k\alpha\frac{q}{p'}+kn(1-\frac{q}{p})}\mathcal{U}_k\mathcal{W}_k^{\frac{q}{p'}} \times$$

$$\Big[\int_{y\in F_k} f^p(y)v(y)\Big(\int_{x\in E_k} |x-y|^{\alpha-n} dx\Big)dy\Big]^{\frac{q}{p}}$$

$$\le c_4 \sum_{k=-\infty}^{\infty} 2^{k\alpha\frac{q}{p'}+kn(1-\frac{q}{p})+k\alpha\frac{q}{p}}\mathcal{U}_k\mathcal{W}_k^{\frac{q}{p'}}\Big(\int_{y\in F_k} f^p(y)v(y)dy\Big)^{\frac{q}{p}}$$

$$= c_4 \sum_{k=-\infty}^{\infty} \Big[2^{kn[\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}]}\mathcal{U}_k^{\frac{1}{q}}\mathcal{W}_k^{\frac{1}{p'}}\Big]^q\Big(\int_{y\in F_k} f^p(y)v(y)dy\Big)^{\frac{q}{p}}$$

$$\le c_4 \sum_{k=-\infty}^{\infty} \widetilde{\mathcal{A}}^q(k)\Big(\int_{y\in F_k} f^p(y)v(y)dy\Big)^{\frac{q}{p}}$$

$$\le c_4 \Big(\sum_{j=-\infty}^{\infty} \widetilde{\mathcal{A}}^{\frac{qp}{p-q}}(j)\Big)^{1-\frac{q}{p}}\Big(\sum_{k=-\infty}^{\infty}\int_{y\in F_k} f^p(y)v(y)dy\Big)^{\frac{q}{p}}$$

$$\le c_4 A^{r(1-\frac{q}{p})}\Big(\sum_{k=-\infty}^{\infty}\int_{2^{k-1}<|y|<2^k} + \int_{2^k<|y|<2^{k+1}} +$$

$$\int_{2^{k+1}<|y|<2^{k+2}} f^p(y)v(y)dy\Big)^{\frac{q}{p}}$$

$$= c_5 A^q\Big(\int_{\mathbb{R}^n} f^p(y)v(y)dy\Big)^{\frac{q}{p}}.$$

Proof of Proposition 2.3.

For the necessary part, by Theorem 2.1, the Hardy conditions (2.1) and (2.1*) and also the Muckenhoupt condition (2.2) are implied by the boundedness $I_\alpha : L^p_v \to L^q_u$. Here, the integer $N \ge 1$ involved in condition (2.2) and (2.3) can be chosen as a common constant resulting from the assumptions $u(.)$, $v^{1-p'}(.) \in \mathcal{H}$ (see definition and (2.6)).

For the sufficient part, by Theorem 2.2, the boundedness $I_\alpha : L^p_v \to L^q_u$ will hold whenever the Hardy conditions (2.1), (2.1*) and the slight reinforcement

Muckenhoupt condition (2.4) are satisfied. Using $u(.)$, $v^{1-p'}(.) \in \mathcal{H}$ then

$$\widetilde{\mathcal{A}}(k) \leq c\mathcal{A}_N(k) \qquad \text{for all integers } k.$$

Here $c > 0$ just depends on n, α, q, p and on a common constant involved in assumptions $u(.)$, $v^{1-p'}(.) \in \mathcal{H}$. This last inequality means that the Muckenhoupt condition (2.2) implies the stronger one (2.4), and consequently the conclusion arises immediately.

Proof of Corollary 2.4.

Proof for the first example. Necessity of conditions i), ii), iii). Suppose that $I_\alpha : L^p_v \to L^q_u$. Then by Theorem 2.1, the Hardy and Muckenhoupt conditions (2.1), (2.1*) and (2.2) are satisfied.

If $\theta \leq \alpha p$, then the Hardy condition (2.1*) does not hold since:

$$\int_{|z|>100} |z|^{(\alpha-n)p'} v^{1-p'}(z)dz = \int_{|z|>100} |z|^{(\alpha-\frac{\theta}{p})p'} |z|^{-n} dz = \infty.$$

Similarly the Hardy condition (2.1) is not satisfied if $(n-\alpha)q \leq \gamma$. And the fact that $\alpha + \frac{\gamma}{q} - \frac{\theta}{p} < 0$ is an immediate consequence of the Muckenhoupt condition (2.2) since

$$\infty > \sum_{k=-\infty}^{\infty} (\mathcal{A}_2(k))^r > c \sum_{k=100}^{\infty} 2^{k[\alpha+\frac{\gamma}{q}-\frac{\theta}{p}]r} \text{ for some fixed constant } c > 0.$$

Sufficiency of conditions i), ii) and iii). In view of Theorem 2.2, to get the boundedness $I_\alpha : L^p_v \to L^q_u$, the task is to check conditions (2.1), (2.1*) and (2.4).

To deal with the Hardy condition (2.1) the idea is to divide the integral with respect to x into ones on the regions $|x| < \frac{1}{2}$ and $|x| > \frac{1}{2}$. This division is required because of the nature of the weights u and v, and actually each region is associated to two integrals. For example, corresponding to $|x| < \frac{1}{2}$ we have to evaluate

$$I_{11} = \int_{|x|<\frac{1}{2}} \left[\left(\int_{|x|<|y|<\frac{1}{2}} |y|^{(\alpha-n)q} u_1(y)dy \right)^{\frac{1}{q}} \left(\int_{|z|<|x|} \sigma_1(z)dz \right)^{\frac{1}{q'}} \right]^r \sigma_1(x)dx$$

and

$$I_{12} = \int_{|x|<\frac{1}{2}} \left[\int_{|z|<|x|} \sigma_1(z)dz \right]^{\frac{r}{q'}} \sigma_1(x)dx$$

where $u_1(z) = u(z) = |z|^{\beta-n}$ and $\sigma_1(z) = v^{1-p'}(z) = |z|^{-n} \ln^{-p'}(|z|^{-1})$ for $|z| < \frac{1}{2}$.

The integral I_{11} can be bounded just by using $(\alpha-n)q+\beta > 0$ since

$$I_{11} \leq c_1 \int_{|x|<\frac{1}{2}} \ln^{-\frac{p'r}{pq'}}(|x|^{-1}) \times \sigma_1(z)dz \leq c_2 \int_{|x|<\frac{1}{2}} \sigma_1(z)dz = c_3.$$

And I_{12} is carried by using (3.1) in Lemma 3.1 as

$$I_{12} \approx \Big(\int_{|x|<\frac{1}{2}} \sigma_1(z)dz\Big)^{\frac{r}{q'}+1} = c_4.$$

The estimates for the two integrals corresponding to the region $|x| > \frac{1}{2}$ require the use of *ii*), *iii*) and $\theta < np$.

The dual Hardy condition (2.1*) can be checked by using the same arguments as for (2.1). Here the estimates for the two integrals corresponding to the region $|x| < \frac{1}{2}$ require the use of (3.1) in Lemma 3.1 and $0 < (n-\alpha)q < \beta$. And for the integrals corresponding to the region $|x| > \frac{1}{2}$, assumptions *i*), *iii*) and $\gamma > 0$ are needed.

To check the condition (2.4) observe that by the definition of the weights then for some fixed constant $c > 0$: $\widetilde{\mathcal{A}}(k) \leq 2^{-k\frac{n}{p'}} = c2^{k[\alpha+\frac{\beta}{q}-n]}$ for all $k \leq -4$ and $\widetilde{\mathcal{A}}(k) \leq c2^{k[\alpha+\frac{\gamma}{q}-\frac{\theta}{p}]}$ for all $k \geq 1$. Therefore condition (2.4) follows from assumptions *ii*) and *iii*).

Finally we will now end with the

Proof for the second example. By duality the boundedness $I_\alpha : L^p_{v^*} \to L^q_{u^*}$ holds if and only if $I_\alpha : L^{\overline{p}}_{\overline{v}} \to L^{\overline{q}}_{\overline{u}}$ with $\overline{p} = q'$, $\overline{q} = p'$, $\overline{v} = (u^*)^{1-q'}$ and $\overline{u} = (v^*)^{1-p'}$. Observe that $\overline{u}(y) = \begin{cases} |y|^{\beta-n} & \text{for } |y| < \frac{1}{2} \\ |y|^{\gamma-n} & \text{for } |y| > \frac{1}{2} \end{cases}$ and $\overline{v}(z) = \begin{cases} |z|^{n(\overline{p}-1)}\ln^{-\overline{p}}(|z|^{-1}) & \text{for } |z| < \frac{1}{2} \\ |z|^{\theta-n} & \text{for } |z| > \frac{1}{2}. \end{cases}$ So using the first example, when $\gamma > 0$, $(n-\alpha)p' = (n-\alpha)\overline{q} < \beta$, $\theta < nq' = n\overline{p}$ then $I_\alpha : L^p_{v^*} \to L^q_{u^*}$ (or equivalently $I_\alpha : L^{\overline{p}}_{\overline{v}} \to L^{\overline{q}}_{\overline{u}}$) if and only if $\alpha q' = \alpha\overline{p} < \theta$, $\gamma < (n-\alpha)p' = (n-\alpha)\overline{q}$ and $\alpha + \frac{\gamma}{p'} = \alpha + \frac{\gamma}{\overline{q}} < \frac{\theta}{\overline{p}} = \frac{\theta}{q'}$.

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