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A THEOREM OF PICK– BERWALD TYPE FOR A TOTALLY UMBILICAL AFFINE IMMERSION OF GENERAL CODIMENSION

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Abstract. It is known that a totally umbilical affine immersion of general codimension into an affine space is affinely congruent to a graph immersion or a centro-affine immersion. In this paper, we shall investigate a more detailed property of such an immersion.

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Introduction

Throughout this paper, unless otherwise mentioned, we assume that all objects are of class C^∞ and all manifolds are connected ones without boundary. Also, denote by $\Gamma(E)$ the space of all cross sections of a vector bundle E. An affine (or equiaffine) immersion of codimension one into the $(n+1)$-dimensional affine space $(\mathbf{R}^{n+1}, \tilde{\nabla}, \tilde{\omega})$ with the natural equiaffne structure has been studied by some geometricians. In particular, if an equiaffine immersion into $(\mathbf{R}^{n+1}, \tilde{\nabla}, \tilde{\omega})$ satisfies the volume condition, then it is called a Blaschke immersion (see [1],[5]). For a Blaschke immersion, G. Pick and L. Berwald proved the following result (see [5]):

If f is a Blaschke immersion of an $n(\geq 2)$-dimensional manifold (M, ∇, θ) with equiaffine structure into $(\mathbf{R}^{n+1}, \tilde{\nabla}, \tilde{\omega})$ and its cubic form (i.e., the covariant differentiation of its affine fundamental form) vanishes, then $f(M)$ is contained in a hyperquadric in $\mathbf{R}^{n+1}$.

Here we note that such an immersion is totally umbilic (see [5]). On the other hand, in [3], K. Nomizu and U. Pinkall proved the following characterization theorem for a totally umbilical affine immersion of general codimension into an affine space with the natural torsion-free affine connection:

Let f be a totally umbilical affine immersion of an n-dimensional manifold (M, ∇) with torsion-free affine connection into the $(n+r)$-dimensional affine space $(\mathbf{R}^{n+r}, \tilde{\nabla})$ with the natural torsion-free affine connection, where $n \geq 2$

and $r \geq 1$. Then f is affinely congruent to a graph immersion or a centro-affine immersion.

Here a graph immersion is defined as follows. Let F be an $\mathbf{R}^r$-valued function on the n-dimensional affine space $\mathbf{R}^n$ and f an immersion of $\mathbf{R}^n$ into $(\mathbf{R}^{n+r}, \tilde{\nabla})$ defined by $f(x) = (x, F(x)) \in \mathbf{R}^n \times \mathbf{R}^r = \mathbf{R}^{n+r}$ $(x \in \mathbf{R})$. Let N be the transversal bundle along f such that N_x $(x \in \mathbf{R}^n)$ are parallel to the affine subspace $\mathbf{R}^r$ of $\mathbf{R}^{n+r}$. Denote by ∇ the induced connection on $\mathbf{R}^n$ for N. Then f is an affine immersion of $(\mathbf{R}^n, \nabla)$ into $(\mathbf{R}^{n+r}, \tilde{\nabla})$. Such an affine immersion is called a *graph immersion*. Note that its affine shape tensor vanishes identically. Also, a centro-affine immersion is defined as follows. Let f be an immersion of an n-dimensional manifold M into $(\mathbf{R}^{n+r}, \tilde{\nabla})$ admitting an $(r-1)$-dimensional vector subspace $V$of $\mathbf{R}^{n+r}$ such that $f_*(T_xM) \oplus \mathrm{Span}\{f(x)\} \oplus V = \mathbf{R}^{n+r}$ holds for every $x \in M$, where T_xM is the tangent space of M at x, we identify $T_{f(x)}\mathbf{R}^{n+r}$ with $\mathbf{R}^{n+r}$ and $f(x)$ is its position vector. Define a tansversal bundle N along f by $N_x = \mathrm{Span}\{f(x)\} \oplus V$ $(x \in M)$. Denote by ∇ the induced connection on M for N. Then f is an affine immersion of (M, ∇) into $(\mathbf{R}^{n+r}, \tilde{\nabla})$. Such an affine immersion is called the *centro-affine immersion*. Note that its affine shape tensor A does not vanish and that $V = \mathrm{Ker}\,\rho_x$ $(x \in M)$ holds, where ρ is a cross section of N^* with $A = \rho \otimes I$.

In this paper, we shall prove the following result similar to the Pick-Berald theorem for a totally umbilical affine immersion of general codimension.

Theorem. *Let f be a totally umbilical affine immersion of an n-dimensional manifold (M, ∇) with torsion-free affine connection into the $(n+r)$-dimensional affine space $(\mathbf{R}^{n+r}, \tilde{\nabla})$ with the natural torsion-free affine connection, where $n \geq 2$ and $r \geq 1$. If its affine shape tensor A does not vanish and the covariant differetiation $\nabla\alpha$ of its affine fundamental form vanishes identically, then $f(M)$ is contained in a cylinder over a hyperquadric in an $(n+1)$-dimensional affine subspace of $\mathbf{R}^{n+r}$.*

Fig. 1.

Here we note that $f(M)$ is not necessarily contained in an $(n+1)$-dimensional affine subspace of $\mathbf{R}^{n+r}$ in spite of being totally umbilic and $\nabla\alpha = 0$. In fact, according to the reduction theorem for an affine immersion of K. Nomizu and U. Pinkall (see [3]), the condition $\nabla\alpha = 0$ implies that the dimension of its first normal space N_x^1 at x (i.e., the linear span of the image of α_x) is independent of the choice of $x \in M$ and that $f(M)$ is contained in an $(n+s)$-dimension affine subspace of $\mathbf{R}^{n+r}$, where $s = \dim N_x^1$, but the totally umbilicity of f does not necessarily imply $\dim N_x^1 = 1$. In §3, we shall give an example of a totally umbilical affine immersion as in the statement of Theorem such that the dimension of its first normal space is more than one.

§1. Fundamental formulas and definitions

In this section, we shall recall the fundamental formulas and definitions for an affine immersion. Let (M, ∇) (resp. $(\tilde{M}, \tilde{\nabla})$) be an n (resp. $(n+r)$)-dimensional manifold with torsion-free affine connection. An immersion $f : (M, \nabla) \hookrightarrow (\tilde{M}, \tilde{\nabla})$ is called an *affine immersion* if there is a transversal bundle N along f such that for every tangent vector fields X and Y on M, $\tilde{\nabla}_X f_* Y - f_*(\nabla_X Y)$ is a cross section of N. Note that the choice of such a transversal bundle N in general is not unique. In the sequel, we fix such a bundle N. Set

$$\alpha(X, Y) := \tilde{\nabla}_X f_* Y - f_*(\nabla_X Y).$$

This quantity α becomes an N-valued symmetric tensor field of type $(0, 2)$ on M. This tensor field α is called the *affine fundamental form of* f. For a transversal vector field ξ along f (i.e., $\xi \in \Gamma(N)$), we write

$$\tilde{\nabla}_X \xi = -f_*(A_\xi X) + \nabla^\perp_X \xi,$$

where $A_\xi X \in \Gamma(TM)$ and $\nabla^\perp_X \xi \in \Gamma(N)$. This quantities A becomes a cross section of the tensor product bundle $N^* \otimes T^*M \otimes TM$ and $\nabla^\perp$ becomes a connection on N, where N^* (resp. T^*M) is the dual bundle of N (resp. TM). This tensor field A is called the *affine shape tensor* of f and $\nabla^\perp$ is called the *transversal connection of* f. The covariant differentiation $\nabla\alpha$ of α is defined by

$$(\nabla_X \alpha)(Y, Z) := \nabla^\perp_X(\alpha(Y, Z)) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z)$$

for $X, Y, Z \in \Gamma(TM)$. The affine immersion f is said to be *totally umbilic* if there is $\rho \in \Gamma(N^*)$ with $A = \rho \otimes I$, where I is the identity transformation of TM.

§2. Proof of Theorem

In this section, we shall prove Theorem stated in Introduction.

Proof of Theorem. Let ρ be a cross section of N^* with $A=\rho\otimes I$. Since f is totally umbilic and $A\neq 0$, f is affinely congruent to a centro-affine immersion from the Nomizu-Pinkall theorem. Hence, the transversal space N_x of f at $x\in M$ is decomposed as follows:

$$N_x=\mathrm{Span}\{f(x)\}\oplus\mathrm{Ker}\rho_x.$$

Now we define a hypersurface $\mathfrak{F}_x$ $(x\in M)$ in $\mathbf{R}^{n+r}$ as

$$\begin{aligned}\mathfrak{F}_x:=\{f(x)+f_*(U)+\mu f(x)+\eta\,|\,U\in T_xM,\ \mu\in\mathbf{R},\ \eta\in\mathrm{Ker}\rho_x,\\ \alpha_x(U,U)-(\mu^2+2\mu)f(x)\equiv 0\ (\mathrm{mod}\ \mathrm{Ker}\rho_x)\}.\end{aligned}$$

Fig. 2.

We show that $\mathfrak{F}_x$ is a cylinder over a hyperquadric in an $(n+1)$-dimensional affine subspace of $\mathbf{R}^{n+r}$. Let p_1 (resp. p_2) be the projection of N_x onto $\mathrm{Ker}\rho_x$ (resp. $\mathrm{Span}\{f(x)\}$). Then it is easy to show that $\alpha_x(U,U)-(\mu^2+2\mu)f(x)\equiv 0$ $(\mathrm{mod}\ \mathrm{Ker}\rho_x)$ holds if and only if

$$p_2(\alpha_x(U,U))=(\mu^2+2\mu)f(x) \tag{2.1}$$

holds. We define a symmetric bilinear form h on T_xM by

$$p_2(\alpha_x(X,Y))=h(X,Y)f(x)$$

for $X,\ Y\in T_xM$. Then (2.1) is equivalent to

$$h(U,U)=\mu^2+2\mu. \tag{2.2}$$

Let $(e_1, \cdots, e_n)$ be a basis of T_xM. We put $U = \sum_{i=1}^{n} U_i e_i$ and $h_{ij} = h(e_i, e_j)$ $(i, j = 1, \cdots, n)$. Then (2.2) is rewritten as

$$\sum_{i,j=1}^{n} h_{ij} U_i U_j = \mu^2 + 2\mu. \tag{2.3}$$

Let $\phi = (y_1, \cdots, y_{n+1})$ be the affine coordinate system of the $(n+1)$-dimensional affine subspace $f_*(T_xM) \oplus \mathrm{Span}\{f(x)\}$ associated with the basis $(f_*e_1, \cdots, f_*e_n, f(x))$, where the origin is the point $f(x)$. Set $v := f(x) + f_*(U) + \mu f(x)$. Then we have

$$\phi(v) = \phi(f(x) + \sum_{i=1}^{n} U_i f_* e_i + \mu f(x)) = (U_1, \cdots, U_n, \mu),$$

that is,

$$y_i(v) = U_i \ (i = 1, \cdots, n), \ y_{n+1}(v) = \mu.$$

Hence (2.3) is rewritten as

$$\sum_{i,j=1}^{n} h_{ij} y_i(v) y_j(v) = y_{n+1}(v)^2 + 2y_{n+1}(v).$$

Therefore, by noticing that $\mathrm{Ker}\,\rho$ is parallel with respect to $\tilde{\nabla}$, we see that $\mathfrak{F}_x$ is a cylinder over a hyperquadric $\sum_{i,j=1}^{n} h_{ij} y_i y_j = y_{n+1}^2 + 2y_{n+1}$ in the $(n+1)$-dimensional affine subspace $f_*(T_xM) \oplus \mathrm{Span}\{f(x)\}$ of $\mathbf{R}^{n+r}$.

Now we shall show that $f(M)$ is contained in $\mathfrak{F}_x$. Fix $x_0 \in M$ and $z_0 \in \mathfrak{F}_{x_0}$. We define a tangent vector field $\tilde{U}$ on M, a function $\tilde{\mu}$ on M and a transversal vector field $\tilde{\eta}$ on M satisfying $\tilde{\eta}_x \in \mathrm{Ker}\rho_x$ $(x \in M)$ by

$$z_0 = f(x) + f_*(\tilde{U}_x) + \tilde{\mu}_x f(x) + \tilde{\eta}_x \quad (x \in M).$$

Fig. 3.

Then we have

$$\tilde{\nabla}_X z_0 = (\tilde{\mu}+1)f_*(X) + f_*(\nabla_X \tilde{U}) + \alpha(X,\tilde{U}) + (X\tilde{\mu})f + \nabla^{\perp}_X \tilde{\eta} = 0$$

for every tangent vector field X on M. By taking notice of the tangent component and the transversal component of this equation, we have

$$\nabla_X \tilde{U} = -(\tilde{\mu}+1)X \tag{2.4}$$

and

$$\alpha(X,\tilde{U}) = -(X\tilde{\mu})f - \nabla^{\perp}_X \tilde{\eta}. \tag{2.5}$$

Now we define the transversal vector field Φ on M by

$$\Phi = \alpha(\tilde{U},\tilde{U}) - (\tilde{\mu}^2 + 2\tilde{\mu})f.$$

From $\nabla\alpha = 0$, (2.4) and (2.5), we have

$$\begin{aligned}
\nabla^{\perp}_X \Phi &= \nabla^{\perp}_X(\alpha(\tilde{U},\tilde{U})) - \nabla^{\perp}_X((\tilde{\mu}^2+2\tilde{\mu})f) \\
&= 2\alpha(\nabla_X\tilde{U},\tilde{U}) - 2(\tilde{\mu}+1)(X\tilde{\mu})f - (\tilde{\mu}^2+2\tilde{\mu})\nabla^{\perp}_X f \\
&= -2(\tilde{\mu}+1)\alpha(X,\tilde{U}) - 2(\tilde{\mu}+1)(X\tilde{\mu})f \\
&= 2(\tilde{\mu}+1)(X\tilde{\mu})f + 2(\tilde{\mu}+1)\nabla^{\perp}_X\tilde{\eta} - 2(\tilde{\mu}+1)(X\tilde{\mu})f \\
&= 2(\tilde{\mu}+1)\nabla^{\perp}_X\tilde{\eta}.
\end{aligned}$$

Since $\mathrm{Ker}\rho$ is parallel with respect to $\nabla^{\perp}$, we have $\nabla^{\perp}_X\tilde{\eta} \in \mathrm{Ker}\rho$. Hence we have

$$\nabla^{\perp}_X \Phi \equiv 0 \pmod{\mathrm{Ker}\rho}.$$

It follows from $z_0 \in \mathfrak{F}_{x_0}$ and the definitions of $\tilde{U}$ and $\tilde{\mu}$ that

$$\Phi_{x_0} = \alpha(\tilde{U}_{x_0},\tilde{U}_{x_0}) - (\tilde{\mu}^2_{x_0} + 2\tilde{\mu}_{x_0})f(x_0) \equiv 0 \pmod{\mathrm{Ker}\rho_{x_0}}.$$

Hence we have

$$\Phi_x = \alpha(\tilde{U}_x,\tilde{U}_x) - (\tilde{\mu}^2_x + 2\tilde{\mu}_x)f(x) \equiv 0 \pmod{\mathrm{Ker}\rho_x}$$

for every $x \in M$. Therefore, we can obtain

$$z_0 = f(x) + f_*(\tilde{U}_x) + \tilde{\mu}_x f(x) + \tilde{\eta}_x \in \mathfrak{F}_x \quad (x \in M).$$

This together with the arbitrariness of z_0 deduces $\mathfrak{F}_{x_0} \subset \mathfrak{F}_x$ $(x \in M)$. Furthermore, from the arbitrariness of x_0 and x, we have $\mathfrak{F}_{x_0} = \mathfrak{F}_x$, which implies

$f(x) \in \mathfrak{F}_{x_0}$ because of $f(x) \in \mathfrak{F}_x$. After all, from the arbitrariness of x, we can obtain $f(M) \subset \mathfrak{F}_{x_0}$. This has completed the proof. $\square$

§3. An example

In this section, we shall give an example of a totally umbilical affine immersion as in the statement of Theorem such that the dimension of its first normal space is more than one.

Example. Define a map f from $\mathbf{R}^n$ to $\mathbf{R}^{n+r}$ by

$$f(x_1, \cdots, x_n) := (x_1, \cdots, x_n,\, a_1 x_1^2 + b_1 x_1 + c_1, \cdots, \\ a_{r'} x_{r'}^2 + b_{r'} x_{r'} + c_{r'}, 0, \cdots, 0, c) \\ ((x_1, \cdots, x_n) \in \mathbf{R}^n),$$

where $n \geq 2,\ r \geq 3,\ r' := \min\{n, r-1\},\ a_i\ (i = 1, \cdots, r')$ and c are non-zero constants, and b_i and c_i $(i = 1, \cdots, r')$ are constants. Also, define a map N from $\mathbf{R}^n$ to the Grassmann manifold $G_{r,n+r}$ of all r-dimensional subspaces of $\mathbf{R}^{n+r}$ by

$$N_{(x_1, \cdots, x_n)} := \mathrm{Span}\{\frac{\partial}{\partial y_{n+1}}, \cdots, \frac{\partial}{\partial y_{n+r-1}}, f(x_1, \cdots, x_n)\} \\ ((x_1, \cdots, x_n) \in \mathbf{R}^n),$$

where $(y_1, \cdots, y_{n+r})$ is the natural coordinate system of $\mathbf{R}^{n+r}$ and $f(x_1, \cdots, x_n)$ is its position vector. Easily we have

$$f_*(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i} + (2a_i x_i + b_i)\frac{\partial}{\partial y_{n+i}} \quad (i = 1, \cdots, r'),$$
$$f_*(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial y_j} \quad (j = r'+1, \cdots, n),$$
$$f(x_1, \cdots, x_n) = \sum_{j=1}^{n} x_j \frac{\partial}{\partial y_j} + \sum_{j=1}^{r'} (a_j x_j^2 + b_j x_j + c_j)\frac{\partial}{\partial y_{n+j}} + c\frac{\partial}{\partial y_{n+r}}.$$

From these relations, we can show the linearly independence of $f_*(\frac{\partial}{\partial x_1}), \cdots, f_*(\frac{\partial}{\partial x_n}), \frac{\partial}{\partial y_{n+1}}, \cdots, \frac{\partial}{\partial y_{n+r-1}},\ f(x_1, \cdots, x_n)$. That is, f is an immersion and N is regarded as a transversal bundle along f. Let $\tilde{\nabla}$ be the natural torsion-free affine connection of $\mathbf{R}^{n+r}$ and ∇ the induced connection on $\mathbf{R}^n$ for N.

Since $N = \mathrm{Span}\{\frac{\partial}{\partial y_{n+1}}, \cdots, \frac{\partial}{\partial y_{n+r-1}}\} \oplus \mathrm{Span}\{f(x_1, \cdots, x_n)\}$ and $\mathrm{Span}\{\frac{\partial}{\partial y_{n+1}}, \cdots, \frac{\partial}{\partial y_{n+r-1}}\}$ is parallel with respect to $\tilde{\nabla}$, the immersion f is a centro-affine immersion of $(\mathbf{R}^n, \nabla)$ into $(\mathbf{R}^{n+r}, \tilde{\nabla})$. That is, the immersion f is totally umbilic and its affine shape tensor does not vanish. Concretely its affine shape tensor A is given by $A = \rho \otimes I$, where ρ is the cross section of N^* defined by $\rho(f(x_1, \cdots, x_n)) = -1$ and $\rho(\frac{\partial}{\partial y_{n+i}}) = 0$ $(i = 1, \cdots, r-1)$. Also, we have

$$\tilde{\nabla}_{\frac{\partial}{\partial x_i}} f_*(\frac{\partial}{\partial x_i}) = 2a_i \frac{\partial}{\partial y_{n+i}} \ (i = 1, \cdots, r'),$$

$$\tilde{\nabla}_{\frac{\partial}{\partial x_i}} f_*(\frac{\partial}{\partial x_i}) = 0 \ (i = r'+1, \cdots, n), \quad \tilde{\nabla}_{\frac{\partial}{\partial x_i}} f_*(\frac{\partial}{\partial x_j}) = 0 \ (1 \leq i \neq j \leq n)$$

and hence

$$\begin{gathered} \alpha(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i}) = 2a_i \frac{\partial}{\partial y_{n+i}} \ (i = 1, \cdots, r'), \\ \alpha(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i}) = 0 \ (i = r'+1, \cdots, n), \\ \alpha(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = 0 \ (1 \leq i \neq j \leq n), \quad \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = 0 \ (i, j = 1, \cdots, n). \end{gathered}$$

Thus its first normal space is spanned by $\frac{\partial}{\partial y_{n+1}}, \cdots, \frac{\partial}{\partial y_{n+r'}}$ at each point of M, that is, its dimension is equal to r' (≥ 2). Also, we have

$$\begin{aligned} (\nabla_{\frac{\partial}{\partial x_i}} \alpha)(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}) &= \nabla^{\perp}_{\frac{\partial}{\partial x_i}} (\alpha(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k})) - \alpha(\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}) \\ &\quad - \alpha(\frac{\partial}{\partial x_j}, \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_k}) \\ &= 0 \qquad\qquad (i, j, k = 1, \cdots, n), \end{aligned}$$

which implies $\nabla \alpha = 0$. Thus this affine immersion $f : (\mathbf{R}^n, \nabla) \hookrightarrow (\mathbf{R}^{n+r}, \tilde{\nabla})$ is a desired totally umbilical affine immersion.

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