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LOCAL EXISTENCE OF SOLUTIONS TO THE CAUCHY PROBLEM FOR NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. In this paper we consider the local existence to the Cauchy problem for nonlinear Schrödinger equations with power nonlinearities

$$(*)\qquad \begin{cases} i\partial_t u + \frac{1}{2}\Delta u = \mathcal{N}(u, \nabla u, \overline{u}, \nabla\overline{u}), & (t,x) \in \mathbf{R}\times\mathbf{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbf{R}^n, \end{cases}$$

where $n \geq 2$ and

$$\mathcal{N} = \mathcal{N}(u, w, \bar{u}, \bar{w}) = \sum_{l_0 \leq |\alpha|+|\beta|+|\gamma| \leq l_1} \lambda_{\alpha\beta\gamma} u^{\alpha_1} \bar{u}^{\alpha_2} \prod_{j=1}^{n} (w_j)^{\beta_j} \prod_{k=1}^{n} (\bar{w}_k)^{\gamma_k}$$

with $w = (w_j)_{1\leq j\leq n}, \lambda_{\alpha\beta\gamma} \in \mathbf{C}, l_0 \in \mathbf{N}, l_1, l_0 \geq 2$. Classical energy method is useful to show local existence in time of solutions to (*) when $\partial_w \mathcal{N}$ is pure imaginary (see, [10, 14-16]), and in this case it is known that there exists a unique solution if $u_0 \in H^{[\frac{n}{2}]+3,0}$ (see [10]), where $H^{m,s} = \{f \in L^2; \|f\|_{m,s} = \|(1+|x|^2)^{s/2}(1-\Delta)^{m/2} f\|_{L^2} < \infty\}$. However, if $\partial_w \mathcal{N}$ is not pure imaginary, there are only a few results [2,12,13] that require higher order Sobolev spaces compared with [10, 14-16] because the classical energy method does not work for the problem. Our purpose in this paper is to show local existence in time of solutions to (*) in the weighted Sobolev space $H^{[\frac{n}{2}]+6,0} \cap H^{[\frac{n}{2}]+3,2}$ without any size restriction on the data. Our function spaces are more natural than those used in [2,12,13].

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§1. Introduction

In this paper we consider the local existence of solutions to the Cauchy problem for nonlinear Schrödinger equations with power nonlinearities

$$\begin{cases} i\partial_t u + \dfrac{1}{2}\Delta u = \mathcal{N}(u, \nabla u, \overline{u}, \nabla\overline{u}), & (t,x) \in \mathbf{R}\times\mathbf{R}^n, \\ \qquad u(0,x) = u_0(x), & x \in \mathbf{R}^n, \end{cases} \tag{1.1}$$

where $n \geq 2$ and

$$\mathcal{N} = \mathcal{N}(u, w, \bar{u}, \bar{w}) = \sum_{l_0 \leq |\alpha|+|\beta|+|\gamma| \leq l_1} \lambda_{\alpha\beta\gamma} u^{\alpha_1} \bar{u}^{\alpha_2} \prod_{j=1}^{n} (w_j)^{\beta_j} \prod_{k=1}^{n} (\bar{w}_k)^{\gamma_k}$$

with $w = (w_j)_{1 \leq j \leq n}$, $\lambda_{\alpha\beta\gamma} \in \mathbf{C}$, $l_1, l_0 \in \mathbf{N}$, $l_0 \geq 2$.

Our main purpose in this paper is to consider (1.1) in lower order Sobolev spaces compared with the previous works [2, 12, 13]. Our function spaces are similar to ones used in [10, 13-16] in which the condition

$$\text{every component of} \quad \partial_w \mathcal{N} \quad \text{is pure imaginary} \tag{1.2}$$

is assumed. Condition (1.2) is sufficient for application of the classical energy method. When the nonlinear terms do not satisfy the condition (1.2) in order to treat the derivatives of unknown function in nonlinear terms we need some smoothing property of solutions to the linear Schrödinger equation [3, 4, 11, 17, 19] or some multiplication factor associated with nonlinear structure (see [1, 8, 9, 18]). However, the smoothing properties obtained in [3, 11, 17, 19] require some smallness condition on the data (see [12]). An application of the gauge transformation method to the Cauchy problem (1.1) is useful only for the one dimensional case [1, 8] and general space dimensions in the case of some special nonlinearities (see [9, 18]). There are only a few results for the Cauchy problem (1.1) with general nonlinearities in the case of large initial data. In papers [2], [13] the existence of local solutions in higher order Sobolev spaces was proved by using smoothing effects obtained in [4] and are based on the theory on pseudo-differential operators of order 0. More precisely in [2] it is assumed that the initial data $u_0 \in H^{m+l,0} \cap H^{m,1} \cap H^{m-1,2}$, where $m \geq [\frac{n}{2}] + 6$ and l is a sufficiently large integer. Here and below we denote the weighted Sobolev space by

$$H^{m,s} = \{f \in L^2; \|f\|_{m,s} = \|(1+|x|^2)^{s/2}(1-\Delta)^{m/2} f\|_{L^2} < \infty\}.$$

We use the notation $[s]$ denoting the largest integer less than or equals to s. To treat the Cauchy problem (1.1) in lower order Sobolev spaces we avoid in this paper to use the well known results of pseudo-differential operators (L^2 boundedness theorem, Sharp Gårding inequality, see [2]) which need higher order Sobolev spaces. We now state our main result in this paper.

Theorem 1.1. *We assume that $u_0 \in H^{m,0} \cap H^{m-3,2}$ $m \geq [\frac{n}{2}]+6$. Then there exists a unique solution of (1.1) and a positive constant $T > 0$ such that*

$$u \in C([-T,T]; H^{m,0} \cap H^{m-3,2}), \quad T = T(\rho) = O\left(\frac{1}{e^{\rho^{l_1+2}}}\right) \quad \text{as} \quad \rho \to \infty,$$

where $\sqrt{\rho} = \|u_0\|_{m,0} + \|u_0\|_{m-3,2}$, *and* l_1 *is as in* $\mathcal{N}$. *Furthermore* u *satisfies the following smoothing property*

$$\sum_{1\le j\le n}\int_{-T}^{T} \|(1+x_j^2)^{-s/2}(1-\partial_{x_j}^2)^{1/4}u(t)\|_{m,0}^2 dt < \infty,$$

where $s > 1/2$.

To prove Theorem 1.1 we use the following

Notation and function spaces. We introduce the pseudo-differential operator of order 0. We let for $\delta > 0$

$$K_{x_j} = \exp\left(A\int_0^{x_j}\langle x_j'\rangle^{-1-\delta}dx_j'\frac{D_{x_j}}{\langle D_{x_j}\rangle}\right)$$

where the constant $A > 0$ will be chosen below to be sufficiently large, $D_{x_j} = i\partial_{x_j} = i\partial_j$, $\langle x_j\rangle = (1+|x_j|^2)^{1/2}$ and $\langle D_{x_j}\rangle = (1-\partial_j^2)^{1/2}$. We define the operator

$$\begin{aligned} K = \prod_{j=1}^{n} K_{x_j} &= \exp\left[\sum_{j=1}^{n}\left(A\int_0^{x_j}\langle x_j'\rangle^{-1-\delta}dx_j'\frac{D_{x_j}}{\langle D_{x_j}\rangle}\right)\right] \\ &= \sum_{k=0}^{\infty}\frac{A^k}{k!}\left[\sum_{j=1}^{n}\left(\int_0^{x_j}\langle x_j'\rangle^{-1-\delta}dx_j'\frac{D_{x_j}}{\langle D_{x_j}\rangle}\right)\right]^k, \end{aligned}$$

which was first used in [4]. It is easy to see that K is a bounded operator from L^2 into itself and there exists a inverse operator K^{-1} which is also a bounded operator from L^2 into itself. Moreover, we have

$$\|Ku\|, \|K^{-1}u\| \le e^{C_1 nA}\|u\|, \quad C_1 = \int_0^{\infty}\langle x_j\rangle^{-1-\delta}dx_j, \quad K^{-1} = \bar{K},$$

where $\|\cdot\| = \|\cdot\|_{L^2}$ is the norm of the usual L^2 space, L^p is the Lebesgue space with the norm $\|\cdot\|_p = \|\cdot\|_{L^p}$. The operator K is useful to obtain smoothing properties of solutions to (1.1). To prove Theorem 1.1 we use the following function space

$$X_{T,m,\{k,s\}} = \{f \in C([0,T];L^2); \|f\|_{X_{T,m,\{k,s\}}} < \infty\},$$

where

$$\|f\|_{X_{T,m,\{k,s\}}}^2 = \|f\|_{T,m-1,\{k,s\}}^2 + \|f\|_{Y_{T,m}}^2 + \|\partial_t f\|_{T,m-3,\{k-2,s\}}^2 + \|\partial_t f\|_{Y_{T,m-2}}^2$$

and

$$\|f\|_{T,l,\{k,s\}}^2 = \sup_{t\in[0,T]} \left(\|f(t)\|_{l,0}^2 + \|f(t)\|_{k,s}^2\right)$$

$$\|f\|_{Y_{T,m}}^2 = e^{-2C_1 nA} \sum_{|\alpha|=m} \left(\frac{A}{8}\sum_{j=1}^n \int_0^T \|\langle x_j\rangle^{-\frac{1+\delta}{2}} \langle D_{x_j}\rangle^{1/2} KD^\alpha f(\tau)\|^2 d\tau \right.$$
$$\left. + \sup_{t\in[0,T]} \|KD^\alpha f(t)\|^2\right),$$

where $D^\alpha = \prod_{j=1}^n (i\partial_j)^{\alpha_j}, |\alpha| = \sum_{j=1}^n \alpha_j$ and A depends on the size of the data. It is sufficient to choose $A = \rho^{l_1+1}$ for Theorem 1.1. We also define a closed ball in $X_{T,m,\{k,s\}}$ as follows

$$X_{T,m,\{k,s\},\rho} = \left\{ f \in X_{T,m,\{k,s\}}; \quad \|f\|_{T,m-1,\{k,s\}} \le \rho, \right.$$
$$\left. \|\partial_t f\|_{T,m-3,\{k-2,s\}} \le \rho^{l_1+1}, \|f\|_{Y_{T,m}} \le \rho^{l_1+1}, \quad \|\partial_t f\|_{Y_{T,m-2}} \le \rho^{2(l_1+1)} \right\}.$$

In this paper we only treat the case $T > 0$ because the case $T < 0$ can be treated similarly.

§2. Preliminaries

In this section we formulate Lemma 2.1 which states the well-known Sobolev embedding inequality. Then we prepare Lemma 2.2 which is needed to estimate the nonlinear terms. In Lemma 2.3 we give some smoothing effects of solutions to the linear Schrödinger equation.

Lemma 2.1(The Gagliardo- Nirenberg-Sobolev inequality).
Let $1 \le q, r \le \infty$. Let $j, m \in \mathbf{N} \cup \{0\}$ satisfy $0 \le j < m$. Let p and a satisfy $1/p = j/n + a(1/r - m/n) + (1-a)/q$; $j/m \le a < 1$ if $m - j - n/r \in \mathbf{N} \cup \{0\}$, $a = j/m$ otherwise. Then

$$\sum_{|\alpha|=j} \|\partial^\alpha \psi\|_p \le C(n,m,j,q,r) \sum_{|\beta|=m} \|\partial^\beta \psi\|_r^a \|\psi\|_q^{1-a},$$

provided that the right hand side is finite.

For Lemma 2.1, see, e.g., A.Friedman [5].

Lemma 2.2. *We have*

$$\left|\Big(K(f\partial_j g), Kh\Big)\right| \le Ce^{CnA}\|f\|_{[\frac{n}{2}]+2,2}\|g\|\|h\| + C\|f\|_{[\frac{n}{2}]+1,2}\|\langle x_j\rangle^{-s}\langle D_{x_j}\rangle^{1/2}Kg\|\|\langle x_j\rangle^{-s}\langle D_{x_j}\rangle^{1/2}Kh\|$$

provided that the right hand side is finite, where $s \in (1/2, 1]$.

Proof. By a simple calculation we get

$$\begin{aligned} K(f\partial_j g) &= K\langle x_j\rangle^{-s}\partial_j(\langle x_j\rangle^s fg) - sK\langle x_j\rangle^{-1}x_j fg - K(\partial_j f)g \\ &= \langle x_j\rangle^{-s}\partial_j K(\langle x_j\rangle^s fg) + F(f,g), \end{aligned} \tag{2.1}$$

where

$$F(f,g) = [K, \langle x_j\rangle^{-s}\partial_j](\langle x_j\rangle^s fg) - sK\langle x_j\rangle^{-1}x_j fg - K(\partial_j f)g.$$

In the same way as in the proof of [Lemma A.4, (a.31), 6] we have

$$\|[K, \langle x_j\rangle^{-s}\partial_j]f\| \le Ce^{CnA}\|f\|.$$

Hence by Lemma 2.1 we obtain

$$\|F(f,g)\| \le Ce^{CnA}\Big(\|f\|_{[\frac{n}{2}]+1,1} + \|f\|_{[\frac{n}{2}]+2,0}\Big)\|g\|. \tag{2.2}$$

By virtue of the Schwarz inequality we get

$$\begin{aligned} &\left|\Big(\langle x_j\rangle^{-s}\partial_j K(\langle x_j\rangle^s fg), Kh\Big)\right| \\ &\quad = \left|\Big(\frac{\partial_j}{\langle D_{x_j}\rangle}\langle D_{x_j}\rangle^{1/2}K(\langle x_j\rangle^s fg), \langle D_{x_j}\rangle^{1/2}\langle x_j\rangle^{-s}Kh\Big)\right| \\ &\quad \le \|\langle D_{x_j}\rangle^{1/2}K(\langle x_j\rangle^s fg)\|\|\langle D_{x_j}\rangle^{1/2}\langle x_j\rangle^{-s}Kh\|. \end{aligned} \tag{2.3}$$

We have via [Lemma A.1, (a.1), 6]

$$\|\langle D_{x_j}\rangle^{1/2}\langle x_j\rangle^{-s}Kh\| \le Ce^{CnA}\|h\| + \|\langle x_j\rangle^{-s}\langle D_{x_j}\rangle^{1/2}Kh\| \tag{2.4}$$

and by [Lemma A.2, 6]

$$\|\langle D_{x_j}\rangle^{1/2}K(\langle x_j\rangle^s fg)\| \le Ce^{CnA}\|\langle x_j\rangle^s f\|_\infty\|g\| + \|K\langle D_{x_j}\rangle^{1/2}(\langle x_j\rangle^s fg)\|. \tag{2.5}$$

We represent the second term of the right hand side of (2.5) as follows

$$\begin{aligned}&K\langle D_{x_j}\rangle^{1/2}(\langle x_j\rangle^s fg)\\&=[K\langle D_{x_j}\rangle^{1/2},\langle x_j\rangle^{2s}f]\langle x_j\rangle^{-s}g+\langle x_j\rangle^{2s}fK\langle D_{x_j}\rangle^{1/2}\langle x_j\rangle^{-s}g\end{aligned}$$

By [Lemma 3.2, 7] we get

$$\|[K\langle D_{x_j}\rangle^{1/2},\langle x_j\rangle^{2s}f]\langle x_j\rangle^{-s}g\|\le Ce^{CnA}\|\langle x_j\rangle^{2s}f\|_{[\frac{n}{2}]+2,0}\|\langle x_j\rangle^{-s}g\| \tag{2.6}$$

and

$$\begin{aligned}&\|\langle x_j\rangle^{2s}fK\langle D_{x_j}\rangle^{1/2}\langle x_j\rangle^{-s}g\|\le\|\langle x_j\rangle^{2s}f[K\langle D_{x_j}\rangle^{1/2},\langle x_j\rangle^{-s}]g\|\\&+\|\langle x_j\rangle^{2s}f\langle x_j\rangle^{-s}[K,\langle D_{x_j}\rangle^{1/2}]g\|+\|\langle x_j\rangle^{2s}f\langle x_j\rangle^{-s}\langle D_{x_j}\rangle^{1/2}Kg\|\\&\le Ce^{CnA}\|f\|_{[\frac{n}{2}]+1,2}\|g\|+\|f\|_{[\frac{n}{2}]+1,2}\|\langle x_j\rangle^{-s}\langle D_{x_j}\rangle^{1/2}Kg\|.\end{aligned} \tag{2.7}$$

Therefore by virtue of estimates (2.1)-(2.7) we have the result of the lemma. □

We next consider the Cauchy problem for the linear Schrödinger equation

$$\begin{cases} i\partial_t u+\Delta u=f, & (t,x)\in\mathbf{R}\times\mathbf{R}^n,\\ u(x,0)=u_0(x), & x\in\mathbf{R}^n.\end{cases} \tag{2.8}$$

Lemma 2.3. *We have the following inequality for the solution u of the Cauchy problem (2.8) with $\delta>0$*

$$\begin{aligned}&\|Ku(t)\|^2+\frac{A}{4}\sum_{j=1}^{n}\int_0^t\|\langle x_j\rangle^{-\frac{1+\delta}{2}}\langle D_{x_j}\rangle^{1/2}Ku(\tau)\|^2d\tau\\&\le e^{2C_1nA}\|u_0\|^2+C(1+A^2)e^{CnA}\int_0^t\|u(\tau)\|^2d\tau\\&+\int_0^t\left|Im\Big(Kf(\tau),Ku(\tau)\Big)\right|d\tau\end{aligned}$$

provided that the right hand side is bounded.

Proof. We put

$$P_{x_j}=\int_0^{x_j}\langle x_j'\rangle^{-2s}dx_j{}'\frac{D_{x_j}}{\langle D_{x_j}\rangle},\quad s=\frac{1+\delta}{4}.$$

Then the operator K_{x_j} defined in Introduction is written as

$$K_{x_j} = \sum_{m=0}^{\infty} \frac{A^m}{m!} P_{x_j}^m \quad A > 0.$$

We have

$$[i\partial_j^2, P_{x_j}] = 2\langle x_j \rangle \frac{D_{x_j}^2}{\langle D_{x_j} \rangle} + i(\partial_j \langle x_j \rangle^{-2s}) \frac{D_{x_j}^2}{\langle D_{x_j} \rangle}$$

therefore by a simple calculation we get for $m \geq 1$

$$\begin{aligned} [i\partial_j^2, P_{x_j}^m] &= \sum_{m_1=0}^{m-1} P_{x_j}^{m-m_1-1} [i\partial_j^2, P_{x_j}] P_{x_j}^{m_1} \\ &= \sum_{m_1=0}^{m-1} P_{x_j}^{m-m_1-1} Q_{x_j} P_{x_j}^{m_1} + \sum_{m_1=0}^{m-1} P_{x_j}^{m-m_1-1} R_{x_j} P_{x_j}^{m_1}, \end{aligned} \tag{2.9}$$

where

$$Q_{x_j} = 2\langle x_j \rangle^{-2s} \frac{D_{x_j}^2}{\langle D_{x_j} \rangle}, \quad R_{x_j} = i(\partial_j \langle x_j \rangle^{-2s}) \frac{D_{x_j}}{\langle D_{x_j} \rangle}.$$

Then using the identity $P_{x_j}^{m-m_1-1} Q_{x_j} P_{x_j}^{m_1} = Q_{x_j} P_{x_j}^{m_1} + [P_{x_j}^{m-m_1-1}, Q_{x_j}]$ and (2.9) we find

$$\begin{aligned} [i\partial_j^2, K] &= [i\partial_j^2, K_{x_j}] \prod_{\substack{1 \leq l \leq n \\ l \neq j}} K_{x_l} \\ &= AQ_{x_j} K + \Bigg(\sum_{m=2}^{\infty} \frac{A^m}{m!} \sum_{m_1=0}^{m-2} [P_{x_j}^{m-m_1-1}, Q_{x_j}] P_{x_j}^{m_1} \\ &\quad + \sum_{m=1}^{\infty} \frac{A^m}{m!} \sum_{m_1=0}^{m-1} P_{x_j}^{m-m_1-1} R_{x_j} P_{x_j}^{m_1} \Bigg) \prod_{\substack{1 \leq l \leq n \\ l \neq j}} K_{x_l}. \end{aligned} \tag{2.10}$$

Applying the operator K to both sides of (2.8), we obtain

$$i\partial_t Ku + \Delta Ku + \sum_{1 \leq j \leq n} i[i\partial_j^2, K]u = Kf. \tag{2.11}$$

Via estimates (2.9) - (2.11) we obtain

$$\begin{aligned} & i\partial_t Ku + \Delta Ku + i \sum_{1 \leq j \leq n} AQ_{x_j} Ku \\ & + \sum_{1 \leq j \leq n} \Bigg(i \sum_{m=2}^{\infty} \frac{A^m}{m!} \sum_{m_1=0}^{m-2} [P_{x_j}^{m-m_1-1}, Q_{x_j}] P_{x_j}^{m_1} u \\ & + i \sum_{m=1}^{\infty} \frac{A^m}{m!} \sum_{m_1=0}^{m-1} P_{x_j}^{m-m_1-1} R_{x_j} P_{x_j}^{m_1} \Bigg) \prod_{\substack{1 \leq l \leq n \\ l \neq j}} K_{x_l} u = Kf. \end{aligned} \tag{2.12}$$

We multiply both sides of (2.12) by $\overline{Ku}$, integrate over $\mathbf{R}^n$ and take the imaginary part to get

$$\frac{1}{2}\frac{d}{dt}\|Ku(t)\|^2 + A\sum_{1\le j\le n}\mathrm{Re}(Q_{x_j}Ku(t),Ku(t)) + \mathrm{Im}(Wu(t),Ku(t)) = \mathrm{Im}(Kf(t),Ku(t)), \tag{2.13}$$

where

$$W = \sum_{1\le j\le n}\Big(i\sum_{m=2}^{\infty}\frac{A^m}{m!}\sum_{m_1=0}^{m-2}[P_{x_j}^{m-m_1-1},Q_{x_j}]P_{x_j}^{m_1}u + i\sum_{m=1}^{\infty}\frac{A^m}{m!}\sum_{m_1=0}^{m-1}P_{x_j}^{m-m_1-1}R_{x_j}P_{x_j}^{m_1}\Big)\prod_{\substack{1\le l\le n\\ l\ne j}}K_{x_l}.$$

We consider the second term of the left hand side of (2.13). Using the representation $D_{x_j}^2/\langle D_{x_j}\rangle = \langle D_{x_j}\rangle - \langle D_{x_j}\rangle^{-1}$ we have

$$\begin{aligned}
&(Q_{x_j}Ku,Ku)\\
&= \big(\langle x_j\rangle^{-s}\langle D_{x_j}\rangle Ku,\langle x_j\rangle^{-s}Ku\big) - \big(\langle x_j\rangle^{-s}\langle D_{x_j}\rangle^{-1}Ku,\langle x_j\rangle^{-s}Ku\big)\\
&= \Big([\langle x_j\rangle^{-s},\langle D_{x_j}\rangle^{1/2}]\langle D_{x_j}\rangle^{1/2}Ku,\langle x_j\rangle^{-s}Ku\Big)\\
&+ \Big(\langle x_j\rangle^{-s}\langle D_{x_j}\rangle^{1/2}Ku,[\langle D_{x_j}\rangle^{1/2},\langle x_j\rangle^{-s}]Ku\Big) + \Big\|\langle x_j\rangle^{-s}\langle D_{x_j}\rangle^{1/2}Ku\Big\|^2\\
&- \big(\langle x_j\rangle^{-s}\langle D_{x_j}\rangle^{-1}Ku,\langle x_j\rangle^{-s}Ku\big)\\
&\ge -Ce^{C_1nA}\Big\|\langle x_j\rangle^{-s}\langle D_{x_j}\rangle^{1/2}Ku\Big\|\,\|u\| + \Big\|\langle x_j\rangle^{-s}\langle D_{x_j}\rangle^{1/2}Ku\Big\|^2\\
&\ge -Ce^{CnA}\|u\|^2 + \frac{1}{2}\Big\|\langle x_j\rangle^{-s}\langle D_{x_j}\rangle^{1/2}Ku\Big\|^2.
\end{aligned}\tag{2.14}$$

By virtue of (2.13) and (2.14) we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|Ku(t)\|^2 + \frac{A}{2}\Big\|\langle x_j\rangle^{-s}\langle D_{x_j}\rangle^{1/2}Ku\Big\|^2\\
&\le |\mathrm{Im}(Wu(t),Ku(t))| + |\mathrm{Im}(Kf(t),Ku(t))| + C\|u\|^2e^{CnA}.
\end{aligned}\tag{2.15}$$

We now consider the first and second terms of the right hand side of (2.15). By the estimates

$$\|P_{x_j}u\| \le C\|u\|, \quad \text{and} \quad \|R_{x_j}u\| \le C\|u\|$$

we have

$$\begin{aligned}
&\sum_{1\le j\le n}\left\|\sum_{m=1}^{\infty}\frac{A^m}{m!}\sum_{m_1=0}^{m-1}P_{x_j}^{m-m_1-1}R_{x_j}P_{x_j}^{m_1}\prod_{\substack{1\le l\le n\\ l\ne j}}K_{x_l}u\right\| \\
&\le \sum_{1\le j\le n}\sum_{m=1}^{\infty}\frac{A^m}{m!}\sum_{m_1=0}^{m-1}\|P_{x_j}^{m-m_1-1}R_{x_j}P_{x_j}^{m_1}\prod_{\substack{1\le l\le n\\ l\ne j}}K_{x_l}u\| \\
&\le \sum_{1\le j\le n}\sum_{m=1}^{\infty}\frac{A^m}{m!}\sum_{m_1=0}^{m-1}C^{m-m_1-1}\|R_{x_j}P_{x_j}^{m_1}\prod_{\substack{1\le l\le n\\ l\ne j}}K_{x_l}u\| \\
&\le \sum_{1\le j\le n}\sum_{m=1}^{\infty}\frac{A^m}{m!}\sum_{m_1=0}^{m-1}C^{m}\|\prod_{\substack{1\le l\le n\\ l\ne j}}K_{x_l}u\| \le CnAe^{CnA}\|u\|.
\end{aligned} \tag{2.16}$$

We now consider the term

$$\begin{aligned}
&\sum_{1\le j\le n}\left\|\sum_{m=2}^{\infty}\frac{A^m}{m!}\sum_{m_1=0}^{m-2}[P_{x_j}^{m-m_1-1},Q_{x_j}]P_{x_j}^{m_1}\prod_{\substack{1\le l\le n\\ l\ne j}}K_{x_l}u(t)\right\| \\
&\le \sum_{1\le j\le n}\sum_{m=2}^{\infty}\frac{A^m}{m!}\sum_{m_1=0}^{m-2}\|[P_{x_j}^{m-m_1-1},Q_{x_j}]P_{x_j}^{m_1}\prod_{\substack{1\le l\le n\\ l\ne j}}K_{x_l}u(t)\|.
\end{aligned} \tag{2.17}$$

Since

$$[P_{x_j}^{l},Q_{x_j}]=\sum_{l_1=0}^{l-1}P_{x_j}^{l-l_1-1}[P_{x_j},Q_{x_j}]P_{x_j}^{l_1}$$

the right hand side of (2.17) is bounded from above by

$$\sum_{1\le j\le n}\sum_{m=2}^{\infty}\frac{A^m}{m!}\sum_{m_1=0}^{m-2}\sum_{l_1=0}^{m-m_1-2}C^{m-m_1-l_1-2}\|[P_{x_j},Q_{x_j}]P_{x_j}^{l_1+m_1}\prod_{\substack{1\le l\le n\\ l\ne j}}K_{x_l}u(t)\|. \tag{2.18}$$

We next consider the term $\|[P_{x_j}, Q_{x_j}]f\|$. We easily find that

$$\begin{aligned}
[P_{x_j}, Q_{x_j}] &= \int_0^{x_j} \langle x_j' \rangle^{-2s} dx_j{}' \frac{D_{x_j}}{\langle D_{x_j} \rangle} \langle x_j \rangle^{-2s} \frac{D_{x_j}^2}{\langle D_{x_j} \rangle} \\
&- 2\langle x_j \rangle^{-2s} \frac{D_{x_j}^2}{\langle D_{x_j} \rangle} \int_0^{x_j} \langle x_j' \rangle^{-2s} dx_j{}' \frac{D_{x_j}}{\langle D_{x_j} \rangle} \\
&= [\int_0^{x_j} \langle x_j' \rangle^{-2s} dx_j{}', \frac{D_{x_j}^2}{\langle D_{x_j} \rangle}] \langle x_j \rangle^{-2s} \frac{D_{x_j}}{\langle D_{x_j} \rangle} \\
&- \int_0^{x_j} \langle x_j' \rangle^{-2s} dx_j{}' \frac{D_{x_j}}{\langle D_{x_j} \rangle} (D_{x_j} \langle x_j \rangle^{-2s}) \frac{D_{x_j}}{\langle D_{x_j} \rangle} \\
&- [\langle x_j \rangle^{-2s}, \frac{D_{x_j}^2}{\langle D_{x_j} \rangle}] \int_0^{x_j} \langle x_j' \rangle^{-2s} dx_j{}' \frac{D_{x_j}}{\langle D_{x_j} \rangle}.
\end{aligned} \tag{2.19}$$

Hence by [Lemma A.1, 6] we get

$$\|[P_{x_j}, Q_{x_j}]f\| \le C\|f\|. \tag{2.20}$$

From (2.18), (2.19) and (2.20) it follows that

$$\left\| \sum_{m=2}^{\infty} \frac{A^m}{m!} \sum_{m_1=0}^{m-2} [P_{x_j}^{m-m_1-1}, Q_{x_j}] P_{x_j}^{m_1} u(t) \right\| \le CA^2 e^{CnA} \|u\|. \tag{2.21}$$

We apply (2.16) and (2.21) to (2.15) to obtain the result of Lemma 2.3. □

§3. Proof of Theorem 1.1

We differentiate (1.1) with respect to x_j to obtain

$$i\partial_t u_{x_j} + \frac{1}{2}\Delta u_{x_j} = \partial_u \mathcal{N} \cdot u_{x_j} + \partial_{\bar{u}} \mathcal{N} \cdot \bar{u}_{x_j} + \sum_{l=1}^{n} \left(\partial_{\partial_l u} \mathcal{N} \cdot \partial_l u_{x_j} + \partial_{\overline{\partial_l u}} \mathcal{N} \cdot \partial_l \bar{u}_{x_j} \right).$$

Let us consider the linearized version of the Cauchy problem (1.1)

$$\begin{cases}
i\partial_t u + \frac{1}{2}\Delta u = \mathcal{N}(v, v_1, \cdots, v_n, \bar{v}, \bar{v}_1, \cdots, \bar{v}_n), \quad (t,x) \in \mathbf{R} \times \mathbf{R}^n, \\
i\partial_t u_j + \frac{1}{2}\Delta u_j = \partial_v \mathcal{N} \cdot v_j + \partial_{\bar{v}} \mathcal{N} \cdot \bar{v}_j + \sum_{l=1}^{n} \left(\partial_{v_l} \mathcal{N} \cdot \partial_l u_j + \partial_{\bar{v}_l} \mathcal{N} \cdot \partial_l \bar{u}_j \right), \\
\quad u(0,x) = u_0(x), \quad u_j(0,x) = \partial_j u_0(x), \quad x \in \mathbf{R}^n,
\end{cases} \tag{3.1}$$

where $1 \le j \le n$, $v, v_j \in X_{T,m-1,\{m-4,2\},\rho}$, , $m \ge [\frac{n}{2}] + 6$, $\quad m \in \mathbf{N}$, $\sqrt{\rho} = \|u_0\|_{m,0} + \|u_0\|_{m-3,2}$ and $T = O(1/e^{\rho^{l_1+2}})$. We define the mapping $(u, u_j) =$

$M(v, v_j)$ and show that M is a mapping from $(X_{T,m-1,\{m-4,2\},\rho})^{n+1}$ into itself for some time T if $m \geq [\frac{n}{2}] + 6$. Without loss of generality we can assume that ρ is sufficiently large because we are interested in the case of large initial data. We first consider a priori estimates of solutions in the norms

$$\|u\|_{T,m-2,\{m-4,2\}}, \quad \|u_j\|_{T,m-2,\{m-4,2\}}, \quad \|\partial_t u\|_{T,m-4,\{m-6,2\}},$$

and $\|\partial_t u_j\|_{T,m-4,\{m-6,2\}}$. By Hölder's inequality, Lemma 2.1 and the classical energy method we obtain

$$\sup_{t\in[0,T]} \|u_j(t)\|_{m-2,0} \leq \|u_0\|_{m-1,0} + C\rho^{l_1} \int_0^T \big(\rho + \|u_j(t)\|_{m-2,0} + e^{C_2 A} \sum_{\substack{|\alpha|=m-1 \\ 1\leq j\leq n}} \|KD^\alpha u_j\|\big) dt,$$

here and in what follows C_2 depends on n and C_1 is defined in *Notation and function spaces* . Hence we have

$$\sup_{t\in[0,T]} \|u_j(t)\|_{m-2,0} \leq 2\sqrt{\rho} + C\rho^{l_1} T e^{C_2 A} \sup_{t\in[0,T]} \sum_{\substack{|\alpha|=m-1 \\ 1\leq j\leq n}} \|KD^\alpha u_j\| \tag{3.2}$$

for small time $T = O(1/\rho^{l_1+1})$.

Multiplying both sides of the second equation of (3.1) by x^2, and applying a classical energy method to the resulting equation we get

$$\begin{aligned}
&\sup_{t\in[0,T]} \|x^2 u_j(t)\|_{m-4,0} \leq \sqrt{\rho} + C\int_0^T \big(\|x\nabla u_j(t)\|_{m-4,0} + \|u_j(t)\|_{m-4,0} \\
&+ \left\| x^2\bigg(\partial_v \mathcal{N}\cdot v_j + \partial_{\bar v}\mathcal{N}\cdot \bar v_j + \sum_{l=1}^n \big(\partial_{v_l}\mathcal{N}\cdot \partial_l u_j + \partial_{\bar v_l}\mathcal{N}\cdot \partial_l \bar u_j\big)\bigg)\right\|_{m-4,0} \\
&\leq \sqrt{\rho} + C\int_0^T (\|x^2 u_j(t)\|_{m-4,0} + \|u_j(t)\|_{m-2,0}) dt \\
&+ C\rho^{l_1} T(1 + \|u_j\|_{T,m-2,\{m-4,2\}}) \\
&\leq \sqrt{\rho} + C\rho^{l_1} T(1 + \|u_j\|_{T,m-2,\{m-4,2\}}).
\end{aligned} \tag{3.3}$$

From (3.2) and (3.3) it follows that

$$\|u_j\|_{T,m-2,\{m-4,2\}} \leq C\sqrt{\rho} + C\rho^{l_1} T e^{C_2 A} \sup_{t\in[0,T]} \sum_{\substack{|\alpha|=m-1 \\ 1\leq j\leq n}} \|KD^\alpha u_j\|. \tag{3.4}$$

In the same way as in the proof of (3.4) we get

$$\|\partial_t u_j\|_{T,m-4,\{m-6,2\}} \le C(\sqrt{\rho}+\sqrt{\rho}^{l_1}) + C\rho^{2l_1}Te^{C_2A}\sup_{t\in[0,T]}\sum_{\substack{|\alpha|=m-3\\ 1\le j\le n}}\|KD^\alpha\partial_t u_j\| \tag{3.5}$$

for small time $T = O(1/\rho^{2(l_1+1)})$. We easily see that the following estimates

$$\|u\|_{T,m-2,\{m-4,2\}} \le C\sqrt{\rho} \tag{3.7}$$

and

$$\|\partial_t u\|_{T,m-4,\{m-6,2\}} \le C\left(\sqrt{\rho}+\sqrt{\rho}^{l_1}\right) \tag{3.8}$$

hold.

We next consider a priori estimates of solutions to (1.1) in the norms

$$\|u\|_{Y_{T,m-1}},\quad \|u_j\|_{Y_{T,m-1}},\quad \|\partial_t u\|_{Y_{T,m-3}},\quad \|\partial_t u_j\|_{Y_{T,m-3}}.$$

This is our main point in the proof of the result of Theorem 1.1. In order to obtain the desired estimates we use smoothing properties of solutions to the linear Schrödinger equation. We make use of the operator K to get a smoothing effect. However we can not use K directly to the nonlinear Schrödinger equations with nonlinear terms involving $\partial_l \bar{u}$ because $K\partial_l u \ne \overline{K\partial_l \bar{u}}$. We now remove such terms from the original equations by a diagonalization technique (see [2]). We define the pseudo-differential operators

$$\mathcal{A} = \sum_{l=1}^{n} \partial_{v_l}\mathcal{N}\cdot\partial_l,\quad \mathcal{B} = \sum_{l=1}^{n} \partial_{\bar{v}_l}\mathcal{N}\cdot\partial_l.$$

In order to eliminate the term $\sum_{l=1}^{n} \partial_{\bar{v}_l}\mathcal{N}\cdot\partial_l\bar{u}_j$ from (3.1) we rewrite the second equation of system (3.1) in the matrix form

$$i\partial_t\begin{pmatrix} u_j\\ \bar{u}_j\end{pmatrix} + \frac{1}{2}\begin{pmatrix} \Delta-2\mathcal{A} & -2\mathcal{B}\\ 2\bar{\mathcal{B}} & -\Delta+2\bar{\mathcal{A}}\end{pmatrix}\begin{pmatrix} u_j\\ \bar{u}_j\end{pmatrix} = \begin{pmatrix} \partial_v\mathcal{N}\cdot v_j + \partial_{\bar{v}}\mathcal{N}\cdot\bar{v}_j\\ -\overline{\partial_v\mathcal{N}}\cdot\bar{v}_j - \overline{\partial_{\bar{v}}\mathcal{N}}\cdot v_j\end{pmatrix}. \tag{3.9}$$

We define the 2×2 matrix operators Λ and Λ' as follows

$$\Lambda = \begin{pmatrix} 1 & 2\mathcal{B}\langle D\rangle^{-2}\\ 2\bar{\mathcal{B}}\langle D\rangle^{-2} & 1\end{pmatrix},\quad \Lambda' = \begin{pmatrix} 1 & -2\mathcal{B}\langle D\rangle^{-2}\\ -2\bar{\mathcal{B}}\langle D\rangle^{-2} & 1\end{pmatrix},$$

where $\langle D\rangle = (1-\Delta)^{1/2}$. Applying the operator Λ to both sides of (3.9), we obtain

$$\begin{aligned} &i\partial_t\Lambda\begin{pmatrix} u_j\\ \bar{u}_j\end{pmatrix} + \frac{1}{2}\Lambda\begin{pmatrix} \Delta-2\mathcal{A} & -2\mathcal{B}\\ 2\bar{\mathcal{B}} & -\Delta+2\bar{\mathcal{A}}\end{pmatrix}\begin{pmatrix} u_j\\ \bar{u}_j\end{pmatrix}\\ &= \Lambda\begin{pmatrix} \partial_v\mathcal{N}\cdot v_j + \partial_{\bar{v}}\mathcal{N}\cdot\bar{v}_j\\ -\overline{\partial_v\mathcal{N}}\cdot v_j - \overline{\partial_{\bar{v}}\mathcal{N}}\cdot\bar{v}_j\end{pmatrix} + [i\partial_t,\Lambda]\begin{pmatrix} u_j\\ \bar{u}_j\end{pmatrix}. \end{aligned} \tag{3.10}$$

By a direct calculation we find

$$\begin{aligned}\Lambda'\Lambda &= \begin{pmatrix} 1 & -\mathcal{B}\langle D\rangle^{-2} \\ -\bar{\mathcal{B}}\langle D\rangle^{-2} & 1 \end{pmatrix}\begin{pmatrix} 1 & \mathcal{B}\langle D\rangle^{-2} \\ \bar{\mathcal{B}}\langle D\rangle^{-2} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1-\mathcal{B}\langle D\rangle^{-2}\bar{\mathcal{B}}\langle D\rangle^{-2} & 0 \\ 0 & 1-\bar{\mathcal{B}}\langle D\rangle^{-2}\mathcal{B}\langle D\rangle^{-2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \mathcal{B}\langle D\rangle^{-2}\bar{\mathcal{B}}\langle D\rangle^{-2} & 0 \\ 0 & \bar{\mathcal{B}}\langle D\rangle^{-2}\mathcal{B}\langle D\rangle^{-2} \end{pmatrix}.\end{aligned}$$

We substitute the identity into (3.10) to get

$$\begin{aligned}&i\partial_t\Lambda\begin{pmatrix} u_j \\ \bar{u}_j \end{pmatrix} + \frac{1}{2}\Lambda\begin{pmatrix} \Delta-2\mathcal{A} & -2\mathcal{B} \\ 2\bar{\mathcal{B}} & -\Delta+2\bar{\mathcal{A}} \end{pmatrix}\Lambda'\Lambda\begin{pmatrix} u_j \\ \bar{u}_j \end{pmatrix} \\ &= \Lambda\begin{pmatrix} \partial_v\mathcal{N}\cdot v_j + \partial_{\bar{v}}\mathcal{N}\cdot\bar{v}_j \\ -\overline{\partial_v\mathcal{N}}\cdot\bar{v}_j - \overline{\partial_{\bar{v}}\mathcal{N}}\cdot v_j \end{pmatrix} + [i\partial_t,\Lambda]\begin{pmatrix} u_j \\ \bar{u}_j \end{pmatrix} \\ &- \frac{1}{2}\Lambda\begin{pmatrix} \Delta-2\mathcal{A} & -2\mathcal{B} \\ 2\bar{\mathcal{B}} & -\Delta+2\bar{\mathcal{A}} \end{pmatrix}\begin{pmatrix} \mathcal{B}\langle D\rangle^{-2}\bar{\mathcal{B}}\langle D\rangle^{-2} & 0 \\ 0 & \bar{\mathcal{B}}\langle D\rangle^{-2}\mathcal{B}\langle D\rangle^{-2} \end{pmatrix}\begin{pmatrix} u_j \\ \bar{u}_j \end{pmatrix}.\end{aligned} \tag{3.11}$$

It is easy to see that

$$\begin{aligned}&\Lambda\begin{pmatrix} \Delta-2\mathcal{A} & -2\mathcal{B} \\ 2\bar{\mathcal{B}} & -\Delta+2\bar{\mathcal{A}} \end{pmatrix}\Lambda' \\ &= \begin{pmatrix} 1 & \mathcal{B}\langle D\rangle^{-2} \\ \bar{\mathcal{B}}\langle D\rangle^{-2} & 1 \end{pmatrix}\begin{pmatrix} \Delta-2\mathcal{A} & -2\mathcal{B} \\ 2\bar{\mathcal{B}} & -\Delta+2\bar{\mathcal{A}} \end{pmatrix}\begin{pmatrix} 1 & -\mathcal{B}\langle D\rangle^{-2} \\ -\bar{\mathcal{B}}\langle D\rangle^{-2} & 1 \end{pmatrix} \\ &= \begin{pmatrix} \Delta-2\mathcal{A}+2\mathcal{B}\langle D\rangle^{-2}\bar{\mathcal{B}} & -2\mathcal{B}+\mathcal{B}\langle D\rangle^{-2}(-\Delta+2\bar{\mathcal{A}}) \\ 2\bar{\mathcal{B}}+\bar{\mathcal{B}}\langle D\rangle^{-2}(\Delta-2\mathcal{A}) & -\Delta+2\bar{\mathcal{A}}-2\bar{\mathcal{B}}\langle D\rangle^{-2}\mathcal{B} \end{pmatrix}\begin{pmatrix} 1 & -\mathcal{B}\langle D\rangle^{-2} \\ -\bar{\mathcal{B}}\langle D\rangle^{-2} & 1 \end{pmatrix} \\ &= \begin{pmatrix} \Delta-2\mathcal{A} & 0 \\ 0 & -\Delta+2\bar{\mathcal{A}} \end{pmatrix} \\ &+ \begin{pmatrix} 0 & -\Delta\mathcal{B}\langle D\rangle^{-2}-\mathcal{B}\langle D\rangle^{-2}\Delta-2\mathcal{B} \\ \Delta\bar{\mathcal{B}}\langle D\rangle^{-2}+\bar{\mathcal{B}}\langle D\rangle^{-2}\Delta+2\mathcal{B} & 0 \end{pmatrix} \\ &+ \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ -\bar{\mathcal{A}}_{12} & -\bar{\mathcal{A}}_{11} \end{pmatrix},\end{aligned}$$

where

$$\mathcal{A}_{11} = 2\mathcal{B}\langle D\rangle^{-2}\bar{\mathcal{B}} + \Big(-2\mathcal{B}+\mathcal{B}\langle D\rangle^{-2}(-\Delta+2\bar{\mathcal{A}})\Big)(-\bar{\mathcal{B}}\langle D\rangle^{-2}), \tag{3.12}$$

$$\mathcal{A}_{12} = 2\mathcal{B}\langle D\rangle^{-2}\bar{\mathcal{A}} + (2\mathcal{A}-2\mathcal{B}\langle D\rangle^{-2}\bar{\mathcal{B}})\mathcal{B}\langle D\rangle^{-2}.$$

Since

$$-\Delta\mathcal{B}\langle D\rangle^{-2}-\mathcal{B}\langle D\rangle^{-2}\Delta-2\mathcal{B}=-[\Delta,\mathcal{B}]\langle D\rangle^{-2}-2\mathcal{B}\langle D\rangle^{-2}\Delta-2\mathcal{B}$$
$$=-[\Delta,\mathcal{B}]\langle D\rangle^{-2}-2\mathcal{B}\langle D\rangle^{-2}$$

we have

$$\Lambda\begin{pmatrix}\Delta-2\mathcal{A} & -2\mathcal{B}\\ 2\bar{\mathcal{B}} & -\Delta+2\bar{\mathcal{A}}\end{pmatrix}\Lambda'=\begin{pmatrix}\Delta-2\mathcal{A} & 0\\ 0 & -\Delta+2\bar{\mathcal{A}}\end{pmatrix}+\begin{pmatrix}\mathcal{A}_{11} & \mathcal{B}_{12}\\ -\bar{\mathcal{B}}_{12} & -\bar{\mathcal{A}}_{11}\end{pmatrix},$$

where

$$\mathcal{B}_{12}=2\mathcal{B}\langle D\rangle^{-2}\bar{\mathcal{A}}+(2\mathcal{A}-2\mathcal{B}\langle D\rangle^{-2}\bar{\mathcal{B}})\mathcal{B}\langle D\rangle^{-2}-[\Delta,\mathcal{B}]\langle D\rangle^{-2}-2\mathcal{B}\langle D\rangle^{-2} \quad (3.13)$$

and

$$[\Delta,\mathcal{B}]\langle D\rangle^{-2}=2\sum_{l=1}^{n}(\nabla\partial_{\bar{v}_l}\mathcal{N})\cdot\nabla\partial_l\langle D\rangle^{-2}+\sum_{l=1}^{n}(\Delta\partial_{\bar{v}_l}\mathcal{N})\partial_l\langle D\rangle^{-2}.$$

Therefore we obtain

$$\begin{aligned}&i\partial_t\Lambda\begin{pmatrix}u_j\\ \bar{u}_j\end{pmatrix}+\frac{1}{2}\begin{pmatrix}\Delta-2\mathcal{A} & 0\\ 0 & -\Delta+2\bar{\mathcal{A}}\end{pmatrix}\Lambda\begin{pmatrix}u_j\\ \bar{u}_j\end{pmatrix}\\ &=\Lambda\begin{pmatrix}\partial_v\mathcal{N}\cdot v_j+\partial_{\bar{v}}\mathcal{N}\cdot\bar{v}_j\\ -\overline{\partial_v\mathcal{N}}\cdot\bar{v}_j-\overline{\partial_{\bar{v}}\mathcal{N}}\cdot v_j\end{pmatrix}+[i\partial_t,\Lambda]\begin{pmatrix}u_j\\ \bar{u}_j\end{pmatrix}-\frac{1}{2}\begin{pmatrix}\mathcal{A}_{11} & \mathcal{B}_{12}\\ -\bar{\mathcal{B}}_{12} & -\bar{\mathcal{A}}_{11}\end{pmatrix}\Lambda\begin{pmatrix}u_j\\ \bar{u}_j\end{pmatrix}\\ &-\frac{1}{2}\Lambda\begin{pmatrix}\Delta-2\mathcal{A} & -2\mathcal{B}\\ 2\bar{\mathcal{B}} & -\Delta+2\bar{\mathcal{A}}\end{pmatrix}\begin{pmatrix}\mathcal{B}\langle D\rangle^{-2}\bar{\mathcal{B}}\langle D\rangle^{-2} & 0\\ 0 & \bar{\mathcal{B}}\langle D\rangle^{-2}\mathcal{B}\langle D\rangle^{-2}\end{pmatrix}\begin{pmatrix}u_j\\ \bar{u}_j\end{pmatrix},\end{aligned} \tag{3.14}$$

where

$$[i\partial_t,\Lambda]=\begin{pmatrix}0 & 2(i\partial_t\mathcal{B})\langle D\rangle^{-2}\\ 2(i\partial_t\bar{\mathcal{B}})\langle D\rangle^{-2} & 0\end{pmatrix}.$$

Now differentiating (3.9) with respect to x we get analogously to (3.14)

$$\begin{aligned}&i\partial_t\Lambda\begin{pmatrix}D^\alpha u_j\\ D^\alpha\bar{u}_j\end{pmatrix}+\frac{1}{2}\begin{pmatrix}\Delta-2\mathcal{A} & 0\\ 0 & -\Delta+2\bar{\mathcal{A}}\end{pmatrix}\Lambda\begin{pmatrix}D^\alpha u_j\\ D^\alpha\bar{u}_j\end{pmatrix}\\ &=\Lambda\begin{pmatrix}D^\alpha(\partial_v\mathcal{N}\cdot v_j+\partial_{\bar{v}}\mathcal{N}\cdot\bar{v}_j)\\ D^\alpha(-\overline{\partial_v\mathcal{N}}\cdot\bar{v}_j-\overline{\partial_{\bar{v}}\mathcal{N}}\cdot v_j)\end{pmatrix}\\ &+\Lambda\begin{pmatrix}D^\alpha\sum_{l=1}^n(\partial_v\mathcal{N}\cdot\partial_l u_j+\partial_{\bar{v}}\mathcal{N}\cdot\partial_l\bar{u}_j)-\sum_{l=1}^n(\partial_v\mathcal{N}\cdot\partial_l D^\alpha u_j+\partial_{\bar{v}}\mathcal{N}\cdot\partial_l D^\alpha\bar{u}_j)\\ -D^\alpha\sum_{l=1}^n(\overline{\partial_v\mathcal{N}}\cdot\partial_l\bar{u}_j+\overline{\partial_{\bar{v}}\mathcal{N}}\cdot\partial_l u_j)+\sum_{l=1}^n(\overline{\partial_v\mathcal{N}}\cdot\partial_l D^\alpha\bar{u}_j+\overline{\partial_{\bar{v}}\mathcal{N}}\cdot\partial_l D^\alpha u_j)\end{pmatrix}\\ &+\begin{pmatrix}0 & 2(i\partial_t\mathcal{B})\langle D\rangle^{-2}\\ 2(i\partial_t\bar{\mathcal{B}})\langle D\rangle^{-2} & 0.\end{pmatrix}\begin{pmatrix}D^\alpha u_j\\ D^\alpha\bar{u}_j\end{pmatrix}-\frac{1}{2}\begin{pmatrix}\mathcal{A}_{11} & \mathcal{B}_{12}\\ -\bar{\mathcal{B}}_{12} & -\bar{\mathcal{A}}_{11}\end{pmatrix}\Lambda\begin{pmatrix}D^\alpha u_j\\ D^\alpha\bar{u}_j\end{pmatrix}\\ &-\frac{1}{2}\Lambda\begin{pmatrix}\Delta-2\mathcal{A} & -2\mathcal{B}\\ 2\bar{\mathcal{B}} & -\Delta+2\bar{\mathcal{A}}\end{pmatrix}\begin{pmatrix}\mathcal{B}\langle D\rangle^{-2}\bar{\mathcal{B}}\langle D\rangle^{-2} & 0\\ 0 & \bar{\mathcal{B}}\langle D\rangle^{-2}\mathcal{B}\langle D\rangle^{-2}\end{pmatrix}\begin{pmatrix}D^\alpha u_j\\ D^\alpha\bar{u}_j\end{pmatrix},\end{aligned} \tag{3.15}$$

where $|\alpha| = m-1$. From the definitions of the operators $\Lambda, \mathcal{B}, \mathcal{A}_{11}, \mathcal{A}_{12}, \mathcal{B}_{12}$, we see that every term of the right hand side of (3.15) is harmless in our problem. We will prove that the L^2 norm of every term of the right hand side of (3.15) is estimated from above by the Sobolev spaces of order $m-1$.

Using Lemma 2.1 and Hölder inequality we will prove the following inequality

$$\left\| D^\alpha \prod_{j=1}^{l} f_j \right\| \le C \prod_{j=1}^{l} \|f_j\|_\infty^{1-\beta_j} \|D^\alpha f_j\|^{\beta_j},$$

where $\beta_j \in [0,1]$ are such that $\sum_{j=1}^{l} \beta_j = 1$. First let us consider the case $l = 2$. By the Leibniz rule we get $\|D^\alpha f_1 f_2\| \le C \sum_{|\alpha'|=0}^{|\alpha|} \|D^{\alpha-\alpha'} f_1 \cdot D^{\alpha'} f_2\|$ and then by the Hölder inequality we have $\|D^{\alpha-\alpha'} f_1 \cdot D^{\alpha'} f_2\| \le C \|D^{\alpha-\alpha'} f_1\|_p \|D^{\alpha'} f_2\|_q$, with $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Then applying Lemma 2.1 we obtain

$$\|D^{\alpha-\alpha'} f_1\|_p \le C \|f_1\|_\infty^{1-\beta_1} \|D^\alpha f_1\|^{\beta_1}$$

if $\frac{1}{p} = \frac{|\alpha-\alpha'|}{n} + \beta_1 \left(\frac{1}{2} - \frac{|\alpha|}{n}\right)$ and in the same way

$$\|D^{\alpha'} f_2\|_q \le C \|f_2\|_\infty^{1-\beta_2} \|D^\alpha f_2\|^{\beta_2}$$

if $\frac{1}{q} = \frac{|\alpha'|}{n} + \beta_2 \left(\frac{1}{2} - \frac{|\alpha|}{n}\right)$. Now the condition $\beta_1 + \beta_2 = 1$ follows from $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Therefore we get

$$\left\| D^\alpha \prod_{j=1}^{2} f_j \right\| \le C \prod_{j=1}^{2} \|f_j\|_\infty^{1-\beta_j} \|D^\alpha f_j\|^{\beta_j},$$

where $\beta_j \in [0,1]$ are such that $\sum_{j=1}^{2} \beta_j = 1$. Then using this particular result and arguing by induction with respect to $l \ge 2$ we have with $g = \prod_{j=1}^{l} f_j$

$$\begin{aligned}
&\|D^\alpha \prod_{j=1}^{l+1} f_j\| = \|D^\alpha g f_{l+1}\| \\
&\le C \left\| \prod_{j=1}^{l} f_j \right\|_\infty^{1-\beta_0} \left\| D^\alpha \prod_{j=1}^{l} f_j \right\|^{\beta_0} \|f_{l+1}\|_\infty^{1-\beta_{l+1}} \|D^\alpha f_{l+1}\|^{\beta_{l+1}} \\
&\le C \prod_{j=1}^{l} \|f_j\|_\infty^{1-\beta_0} \|f_j\|_\infty^{(1-\beta_j)\beta_0} \|D^\alpha f_j\|^{\beta_j \beta_0} \|f_{l+1}\|_\infty^{1-\beta_{l+1}} \|D^\alpha f_{l+1}\|^{\beta_{l+1}},
\end{aligned}$$

where $\beta_0+\beta_{l+1}=1$. By virtue of the equality $\sum_{j=1}^{l}\beta_0\beta_j+\beta_{l+1}=1$ we get the desired estimate. Using this estimate, we consider estimates of nonlinear terms. We may assume that $\mathcal{N}$ is a polynomial of degree $l_1\geq 2$ since we consider the large solutions. Therefore we get with $\partial_v\mathcal{N}=v^{\alpha_1-1}\bar{v}^{\alpha_2}\prod_{k=1}^{n}v_k^{\beta_k}\prod_{k=1}^{n}\bar{v}_k^{\gamma_k}$, $\alpha_1+\alpha_2+\sum_{1\leq k\leq n}\beta_k+\sum_{1\leq k\leq n}\gamma_k=l_1$

$$\begin{aligned}
&\sum_{|\delta|\leq m-1}\|D^\delta\partial_v\mathcal{N}\|\leq C\sum_{|\delta|\leq m-1}\|D^\delta v\|^{\xi_1}\|v\|_\infty^{\alpha_1-\xi_1-1}\|D^\delta\bar{v}\|^{\xi_2}\|\bar{v}\|_\infty^{\alpha_2-\xi_2}\\
&\times\prod_{k=1}^{n}\|v_k\|_\infty^{\beta_k-\eta_k}\|D^\delta v_k\|^{\eta_k}\prod_{k=1}^{n}\|\bar{v}_k\|_\infty^{\gamma_k-\phi_k}\|D^\delta\bar{v}_k\|^{\phi_k}\\
&\leq C\|v\|_{m-1,0}^{\alpha_1-1}\|\bar{v}\|_{m-1,0}^{\alpha_2}\prod_{k=1}^{n}\|v_k\|_{m-1,0}^{\beta_k}\|v_k\|_{m-1,0}^{\gamma_k}\leq C\rho^{l_1-1}.
\end{aligned}$$

In the same way we obtain for $|\alpha|\leq m-1$

$$\|D^\alpha\partial_{\bar{v}}\mathcal{N}\|+\|D^\alpha\partial_{v_j}\mathcal{N}\|+\|D^\alpha\partial_{\bar{v}_j}\mathcal{N}\|+\|\partial_v\mathcal{N}\|_\infty+\|\partial_{\bar{v}}\mathcal{N}\|_\infty\leq C\rho^{l_1-1}.$$

Hence we have with $\beta_1+\beta_2=1$ and $\beta_3+\beta_4=1$

$$\begin{aligned}
&\left\|\Lambda\begin{pmatrix}D^\alpha(\partial_v\mathcal{N}\cdot v_j+\partial_{\bar{v}}\mathcal{N}\cdot\bar{v}_j)\\ D^\alpha(-\overline{\partial_v\mathcal{N}}\cdot\bar{v}_j-\overline{\partial_{\bar{v}}\mathcal{N}}\cdot v_j)\end{pmatrix}\right\|_{L^2\times L^2}\\
&\leq C\|D^\alpha(\partial_v\mathcal{N}\cdot v_j-\partial_{\bar{v}}\mathcal{N}\cdot\bar{v}_j)\|+C\|B\langle D\rangle^{-2}D^\alpha(-\overline{\partial_v\mathcal{N}}\cdot\bar{v}_j-\overline{\partial_{\bar{v}}\mathcal{N}}\cdot v_j)\|\\
&\leq C\|\partial_v\mathcal{N}\|_\infty^{1-\beta_1}\|D^\alpha\partial_v\mathcal{N}\|^{\beta_1}\|v_j\|_\infty^{1-\beta_2}\|D^\alpha v_j\|^{\beta_2}\\
&+C\|\partial_v\mathcal{N}\|_\infty^{1-\beta_3}\|D^{\alpha-1}\partial_v\mathcal{N}\|^{\beta_3}\|v_j\|_\infty^{1-\beta_4}\|D^\alpha v_j\|^{\beta_4}\\
&\leq C\rho^{l_1-1}\|v_j\|_{m-1,0}+C\rho^{l_1-1}\rho^{l_1-1}\|v_j\|_{m-1,0}\\
&\leq C(1+\rho^{l_1})\rho^{l_1}\quad\text{for}\quad 0\leq t\leq T,
\end{aligned}\tag{3.16}$$

Note that we do not have the derivatives of the highest order m in the form

$D^\alpha(\partial_v\mathcal{N}\cdot\partial_l u_j) - \partial_v\mathcal{N}\cdot\partial_l D^\alpha u_j$ when $|\alpha| = m-1$. Therefore we have

$$\begin{aligned}
&\left\| \Lambda \begin{pmatrix} \sum_{l=1}^n \left(D^\alpha(\partial_v\mathcal{N}\cdot\partial_l u_j + \partial_{\bar v}\mathcal{N}\cdot\partial_l\bar u_j) - \partial_v\mathcal{N}\cdot\partial_l D^\alpha u_j - \partial_{\bar v}\mathcal{N}\cdot\partial_l D^\alpha \bar u_j\right) \\ \sum_{l=1}^n \left(\overline{\partial_v\mathcal{N}}\cdot\partial_l D^\alpha\bar u_j + \overline{\partial_{\bar v}\mathcal{N}}\cdot\partial_l D^\alpha u_j - D^\alpha(\overline{\partial_v\mathcal{N}}\cdot\partial_l\bar u_j + \overline{\partial_{\bar v}\mathcal{N}}\cdot\partial_l u_j)\right)\end{pmatrix}\right\|_{L^2\times L^2} \\
&\le C\sum_{|\alpha|=m-2}\|\partial_v\mathcal{N}\|_\infty^{1-\beta_1}\|D^\alpha\partial_v\mathcal{N}\|^{\beta_1}\|\partial_l u_j\|_\infty^{1-\beta_2}\|D^\alpha\partial_l u_j\|^{\beta_2} + C\|\partial_v\mathcal{N}\|\|D^{\alpha+1}u_j\| \\
&+ C\|\partial_v\mathcal{N}\|_\infty\|D^{\alpha-1}\partial_v\mathcal{N}\|^{\beta_3}\|\partial_v\mathcal{N}\|^{1-\beta_3}\|\partial_l u_j\|^{1-\beta_4}\|D^{\alpha-1}\partial_l u_j\|^{\beta_4} \\
&\le C\rho^{l_1-1}\|u_j\|_{m-2,0} + C\rho^{l_1-1}e^{C_2A}\sum_{|\alpha|=m-1}\|KD^\alpha u_j\| \\
&+ C\rho^{2l_1}\left(\|u_j\|_{m-2,0} + e^{C_2A}\sum_{|\alpha|=m-1}\|KD^\alpha u_j\|\right) \\
&\le C(1+\rho^{l_1})\rho^{l_1}e^{C_2A}\left(\|u_j(t)\|_{m-2,0} + \sum_{|\alpha|=m-1}\|KD^\alpha u_j(t)\|\right) \qquad (3.17)
\end{aligned}$$

for all $0 \le t \le T$. Since by Lemma 2.1 $\|\partial_t\partial_{\bar v}\mathcal{N}\|_\infty \le C\rho^{l_1}$ we have

$$\begin{aligned}
&\left\| \begin{pmatrix} 0 & 2(i\partial_t\mathcal{B})\langle D\rangle^{-2} \\ 2(i\partial_t\bar{\mathcal{B}})\langle D\rangle^{-2} & 0\end{pmatrix}\begin{pmatrix} D^\alpha u_j \\ D^\alpha\bar u_j\end{pmatrix}\right\|_{L^2\times L^2} \\
&\le C\|\partial_t\partial_{\bar v}\mathcal{N}\|_\infty\|D^{\alpha-1}\bar u_j\| \le C\rho^{l_1}\|u_j\|_{m-2,0}. \qquad (3.18)
\end{aligned}$$

In the same manner we get the following estimates

$$\begin{aligned}
&\left\| \frac{1}{2}\begin{pmatrix} \mathcal{A}_{11} & \mathcal{B}_{12} \\ -\bar{\mathcal{B}}_{12} & -\bar{\mathcal{A}}_{11}\end{pmatrix}\Lambda\begin{pmatrix} D^\alpha u_j \\ D^\alpha\bar u_j\end{pmatrix}\right\|_{L^2\times L^2} \\
&\le C\rho^{l_1}(1+\rho^{3l_1})e^{C_2A}\left(\|u_j(t)\|_{m-2,0} + \sum_{|\alpha|=m-1}\|KD^\alpha u_j(t)\|\right) \qquad (3.19)
\end{aligned}$$

and

$$\begin{aligned}
&\left\| \frac{1}{2}\Lambda\begin{pmatrix} \Delta - 2\mathcal{A} & -2\mathcal{B} \\ 2\bar{\mathcal{B}} & -\Delta + 2\bar{\mathcal{A}}\end{pmatrix}\begin{pmatrix} \mathcal{B}\langle D\rangle^{-2}\bar{\mathcal{B}}\langle D\rangle^{-2} & 0 \\ 0 & \bar{\mathcal{B}}\langle D\rangle^{-2}\mathcal{B}\langle D\rangle^{-2}\end{pmatrix}\begin{pmatrix} D^\alpha u_j \\ D^\alpha\bar u_j\end{pmatrix}\right\|_{L^2\times L^2} \\
&\le C\rho^{l_1}(1+\rho^{3l_1})e^{C_2A}\left(\|u_j(t)\|_{m-2,0} + \sum_{|\alpha|=m-1}\|KD^\alpha u_j(t)\|\right) \qquad (3.20)
\end{aligned}$$

for all $0 \le t \le T$. Thus estimates (3.16)-(3.20) show that the right hand side of (3.15) is not difficult to treat. We now prove a priori estimates of solutions to (3.15) by using (3.16)-(3.20). In order to get the desired a priori estimates we rewrite (3.15) as a single equation

$$\begin{cases} i\partial_t w_j^\alpha + \frac{1}{2}\Delta w_j^\alpha = \sum_{l=1}^n \partial_{v_l}\mathcal{N}\cdot\partial_l w_j^\alpha + F_1, \\ w_j^\alpha = D^\alpha u_j + 2\mathcal{B}\langle D\rangle^{-2}D^\alpha \bar{u}_j, \end{cases} \tag{3.21}$$

where $|\alpha| = m-1$. From (3.16)-(3.20) we see that

$$\|F_1\| \le C\rho^{l_1}(1+\rho^{3(l_1+1)})e^{C_2A}\Big(\|u_j(t)\|_{m-2,0} + \sum_{|\alpha|=m-1}\|KD^\alpha u_j(t)\|\Big)$$
$$+ C(1+\rho^{l_1})\rho^{l_1+1}e^{C_2A} \quad \text{for} \quad 0 \le t \le T. \tag{3.22}$$

In the same way as in the derivation of (3.21) we have the equation connected with $D^\delta \partial_t u_j$

$$\begin{cases} i\partial_t \tilde{w}_j^\delta + \frac{1}{2}\Delta \tilde{w}_j^\delta = \sum_{l=1}^n \partial_{v_l}\mathcal{N}\cdot\partial_l \tilde{w}_j^\delta + F_2, \\ \tilde{w}_j^\delta = D^\delta\partial_t u_j + 2\mathcal{B}\langle D\rangle^{-2}D^\delta\partial_t \bar{u}_j, \end{cases} \tag{3.23}$$

where $|\delta| = m-3$. Here F_2 has the estimate

$$\|F_2\| \le C\rho^{2l_1}\Big((1+\rho^{6l_1})e^{C_2A}(\|\partial_t u_j(t)\|_{m-4,0} + \sum_{|\alpha|=m-3}\|KD^\alpha\partial_t u_j(t)\|$$
$$+ \|u_j(t)\|_{m-2,0} + \sum_{|\alpha|=m-1}\|KD^\alpha u_j(t)\|\Big) + C(1+\rho^{2l_1})\rho^{2l_1}e^{C_2A} \tag{3.24}$$

for all $0 \le t \le T$. We apply Lemma 2.3 to the equation (3.21) and use (3.22) to obtain

$$\|Kw_j^\alpha(t)\|^2 + \frac{A}{4}\sum_{k=1}^n\int_0^t \|\langle x_k\rangle^{-\frac{1+\delta}{2}}\langle D_{x_k}\rangle^{1/2}Kw_j^\alpha(\tau)\|^2 d\tau$$
$$\le e^{2C_1nA}\|w_j(0)\|^2 + C(1+A^2)e^{C_2A}\int_0^t \|w_j^\alpha(\tau)\|^2 d\tau$$
$$+ 2\int_0^t \|F_1(\tau)\|\|Kw_j^\alpha(\tau)\|d\tau + 4\int_0^t \Big|\mathrm{Im}\Big(K\sum_{l=1}^n \partial_{v_l}\mathcal{N}\cdot\partial_l w_j^\alpha(\tau), Kw_j^\alpha(\tau)\Big)\Big|d\tau$$
$$\le e^{2C_1nA}(\rho^{1/2}+\rho^{l_1/2}) + e^{C_2A}T\sup_{t\in[0,T]}\|w_j^\alpha(t)\|^2$$
$$+ C\rho^{l_1}(1+\rho^{3l_1})e^{C_2A}\int_0^T (\|u_j(t)\|_{m-2,0} + \sum_{|\alpha|=m-1}\|KD^\alpha u_j(t)\|)^2 dt$$
$$+ C(1+\rho^{l_1})\rho^{l_1}e^{C_2A}\int_0^T \sum_{|\alpha|=m-1}\|KD^\alpha u_j(t)\|dt$$
$$+ 4\int_0^T \Big|\mathrm{Im}\Big(K\sum_{l=1}^n \partial_{v_l}\mathcal{N}\cdot\partial_l w_j^\alpha(\tau), Kw_j^\alpha(\tau)\Big)\Big|d\tau. \tag{3.25}$$

Since $|\alpha| = m-1$ we have by (3.2)

$$\|KD^\alpha u_j(t)\| \le \|Kw_j^\alpha(t)\| + 2\sum_{|\alpha|=m-1} \|K\mathcal{B}\langle D\rangle^{-2}D^\alpha u_j(t)\|$$
$$\le \|Kw_j^\alpha(t)\| + Ce^{C_2A}\rho^{l_1} \sup_{t\in[0,T]} \|u_j(t)\|_{m-2,0}$$
$$\le \|Kw_j^\alpha(t)\| + Ce^{C_2A}\rho^{l_1}\left(2\sqrt{\rho} + C\rho^{l_1}Te^{C_2A} \sup_{t\in[0,T]} \sum_{\substack{|\alpha|=m-1\\1\le j\le n}} \|KD^\alpha u_j(t)\|\right).$$

Hence

$$\sup_{t\in[0,T]} \sum_{\substack{|\alpha|=m-1\\1\le j\le n}} \|KD^\alpha u_j(t)\| \le 2\sup_{t\in[0,T]} \sum_{\substack{|\alpha|=m-1\\1\le j\le n}} \|Kw_j^\alpha(t)\| + Ce^{C_2A}\rho^{l_1}\sqrt{\rho} \tag{3.26}$$

for small time $T = O(1/e^{\rho^{l_1+2}})$, if we take $A = \rho^{l_1+1}$. Similarly, we get

$$\sup_{t\in[0,T]} \sum_{\substack{|\alpha|=m-1\\1\le j\le n}} \|Kw_j^\alpha(t)\| \le 2\sup_{t\in[0,T]} \sum_{\substack{|\alpha|=m-1\\1\le j\le n}} \|KD^\alpha u_j(t)\| + Ce^{C_2A}\rho^{l_1}\sqrt{\rho}. \tag{3.27}$$

From estimates (3.25) - (3.27) it follows that

$$(1 - Ce^{C_2\rho^{l_1+1}}T) \sup_{t\in[0,T]} \sum_{\substack{|\alpha|=m-1\\1\le j\le n}} \|Kw_j^\alpha(t)\|^2$$
$$+ \frac{A}{4} \sum_{\substack{|\alpha|=m-1\\1\le j\le n}} \sum_{k=1}^n \int_0^t \|\langle x_k\rangle^{-\frac{1+\delta}{2}}\langle D_{x_k}\rangle^{1/2} Kw_j^\alpha(\tau)\|^2 d\tau$$
$$\le e^{2C_1nA}\rho^{l_1} + 4\sum_{\substack{|\alpha|=m-1\\1\le j\le n}} \int_0^t \left|\mathrm{Im}\left(K\sum_{l=1}^n \partial_{v_l}\mathcal{N}\cdot\partial_l w_j^\alpha(\tau), Kw_j^\alpha(\tau)\right)\right| d\tau.$$

We apply Lemma 2.2 to the second term of the right hand side of the above inequality to get

$$(1 - Ce^{C_1nA}\rho^{4(l_1+1)}T) \sup_{t\in[0,T]} \sum_{\substack{|\alpha|=m-1\\1\le j\le n}} \|Kw_j^\alpha(t)\|^2$$
$$+ \left(\frac{A}{4} - C\rho^{l_1}\right) \sum_{\substack{|\alpha|=m-1\\1\le j\le n}} \sum_{k=1}^n \int_0^T \|\langle x_k\rangle^{-\frac{1+\delta}{2}}\langle D_{x_k}\rangle^{1/2} Kw_j^\alpha(\tau)\|^2 d\tau$$
$$\le Ce^{2C_1nA}\rho^{l_1}.$$

If we take $A = \rho^{l_1+1}$ and $T = O(1/e^{\rho^{l_1+2}})$ we obtain

$$\sup_{t\in[0,T]} \sum_{\substack{|\alpha|=m-1\\ 1\le j\le n}} \|Kw_j^{\alpha}(t)\|^2 + \frac{A}{6}\sum_{\substack{|\alpha|=m-1\\ 1\le j\le n}}\sum_{k=1}^{n}\int_0^T \|\langle x_k\rangle^{-\frac{1+\delta}{2}}\langle D_{x_k}\rangle^{1/2} Kw_j^{\alpha}(t)\|^2 dt \le Ce^{2C_1 nA}\rho^{l_1} \le e^{2C_1 nA}\rho^{l_1+1}. \tag{3.28}$$

By (3.26) and (3.28) we find that

$$\sup_{t\in[0,T]} \sum_{\substack{|\alpha|=m-1\\ 1\le j\le n}} \|KD^{\alpha}u_j(t)\| + \frac{A}{8}\sum_{\substack{|\alpha|=m-1\\ 1\le j\le n}}\sum_{k=1}^{n}\int_0^T \|\langle x_k\rangle^{-\frac{1+\delta}{2}}\langle D_{x_k}\rangle^{1/2} KD^{\alpha}u_j(t)\|^2 dt\Big) \le e^{2C_1 nA}\rho^{l_1+1}. \tag{3.29}$$

In the same way as in the proof of (3.29) we have

$$\sup_{t\in[0,T]} \sum_{\substack{|\alpha|=m-3\\ 1\le j\le n}} \|KD^{\alpha}\partial_t u_j(t)\| + \frac{A}{8}\sum_{\substack{|\alpha|=m-3\\ 1\le j\le n}}\sum_{k=1}^{n}\int_0^T \|\langle x_k\rangle^{-\frac{1+\delta}{2}}\langle D_{x_k}\rangle^{1/2} KD^{\alpha}\partial_t u_j(t)\|^2 dt \le e^{2C_1 nA}\rho^{2(l_1+1)}. \tag{3.30}$$

Thus we see that there exists a time $T = O(1/e^{\rho^{l_1+2}})$ such that

$$\{u, u_1, \cdots, u_n\} \in (X_{T,m-1,\{m-4,2\},\rho})^{n+1}. \tag{3.31}$$

We now let a sequence $\{u^{(k)}, u_1^{(k)}, \cdots, u_n^{(k)}\}$ satisfy the equation (3.1) with $u = u^{(k)}$, $u_j = u_j^{(k)}$, $v = u^{(k-1)}$,$v_j = u_j^{(k-1)}$ and with the same initial data, where $\{u^{(0)}, u_1^{(0)}, \cdots, u_n^{(0)}\}$ is a solution of the linear Schrödinger equation. We prove that the sequence
$\{u^{(k)}, u_1^{(k)}, \cdots, u_n^{(k)}\}$ is a Cauchy sequence in $(X_{T,m-1,\{m-4,2\}})^{n+1}$. From (3.31) we already know that

$$\{u^{(k)}, u_1^{(k)}, \cdots, u_n^{(k)}\} \in (X_{T,m-1,\{m-4,2\},\rho})^{n+1} \quad \text{for any} \quad k \in \mathbf{N}.$$

In the same way as in the proof of (3.4)-(3.8) we see that

$$\begin{aligned}
&\|U^{(k)}\|_{T,m-2,\{m-4,2\}} + \|U_j^{(k)}\|_{T,m-2,\{m-4,2\}} \\
&+ \|\partial_t U^{(k)}\|_{T,m-4,\{m-6,2\}} + \|\partial_t U_j^{(k)}\|_{T,m-4,\{m-6,2\}} \\
&\le C\rho^{2l_1} T e^{C_2 A}\Big(\|U^{(k-1)}\|_{T,m-2,\{m-4,2\}} + \|U_j^{(k-1)}\|_{T,m-2,\{m-4,2\}} \\
&+ \|\partial_t U^{(k-1)}\|_{T,m-4,\{m-6,2\}} + \|\partial_t U_j^{(k-1)}\|_{T,m-4,\{m-6,2\}} \\
&+ \sum_{\substack{|\alpha|=m-1\\ |\beta|=m-3\\ 1\le j\le n}} \big(\|KD^\alpha U^{(k-1)}\| + \|KD^\alpha U_j^{(k-1)}\| \\
&\qquad\qquad + \|KD^\beta \partial_t U^{(k-1)}\| + \|KD^\beta \partial_t U_j^{(k-1)}\|\big)\Big),
\end{aligned} \tag{3.32}$$

where $U^{(k)} = u^{(k)} - u^{(k-1)}$, $U_j^{(k)} = u_j^{(k)} - u_j^{(k-1)}$. As in the proof of (3.21) we get

$$\begin{aligned}
&i\partial_t W_j^{(k)} + \Delta W_j^{(k)} \\
&\quad = 2\sum_{l=0}^{n} \Big(\partial_{v_l}\mathcal{N}(u^{(k-1)},\cdots)\partial_l w_j^{(k)} - \partial_{v_l}\mathcal{N}(u^{(k-2)},\cdots)\partial_l w_j^{(k-1)} \Big) \\
&\quad + F_1(u^{(k-1)},\cdots) - F_1(u^{(k-2)},\cdots) \equiv \tilde{\mathcal{N}},
\end{aligned} \tag{3.33}$$

where $W_j^{(k)} = w_j^{(k)} - w_j^{(k-1)}$, $w_j^{(k)} = D^\beta u_j^{(k)} + 2\mathcal{B}(u^{(k-1)},\cdots)\langle D\rangle^{-2} D^\beta \bar{u}_j^{(k)}$. We apply Lemma 2.3 to (3.33) to obtain

$$\begin{aligned}
&\|KW_j^{(k)}(t)\|^2 + \frac{A}{4}\sum_{1\le l\le n}\int_0^t \|\langle x_l\rangle^{-\frac{1+\delta}{2}}\langle D_{x_l}\rangle^{1/2} KW_j^{(k)}(\tau)\|^2 d\tau \\
&\le CA^2 e^{C_2 A}\int_0^t \|W_j^{(k)}(\tau)\|^2 d\tau + \int_0^t \Big| \mathrm{Im}\Big(K\tilde{\mathcal{N}}(\tau), KW_j^{(k)}(\tau)\Big)\Big| d\tau.
\end{aligned} \tag{3.34}$$

By Hölder's inequality and Lemma 2.1 we get

$$\begin{aligned}
&\int_0^T \|F_1(u^{(k-1)},\cdots) - F_1(u^{(k-2)},\cdots)\| \|KW_j^{(k)}(t)\| dt \\
&\le C\rho^{l_1} e^{2C_2 A} T\Big(\|U^{(k-1)}\|_{T,m-2,\{m-4,2\}} + \|U_j^{(k-1)}\|_{T,m-2,\{m-4,2\}} \\
&+ \sum_{|\beta|=m-1} \big(\|KD^\beta U^{(k-1)}\| + \|KD^\beta U_j^{(k-1)}\|\big)\Big) \sup_{t\in[0,T]} \|KW_j^{(k)}(t)\|.
\end{aligned} \tag{3.35}$$

We also have by Lemma 2.2 , the Schwarz inequality and (3.31)

$$\int_0^T \left|\mathrm{Im}\Big(K\sum_{l=1}^n \partial_{v_l}\mathcal{N}(u^{(k-1)},\cdots)\cdot\partial_l W_j^{(k)}(t)\right.$$
$$\left.+\sum_{l=1}^n(\partial_{v_l}\mathcal{N}(u^{(k-1)},\cdots)-\partial_{v_l}\mathcal{N}(u^{(k-2)},\cdots))\partial_l w_j^{(k-1)}(t), KW_j^{(k)}(t)\Big)\right|dt$$
$$\leq C\sum_{j=1}^n \rho^{l_1}\int_0^T \|\langle x_j\rangle^{-\frac{1+\delta}{2}}\langle D_{x_j}\rangle^{1/2}KW_j^{(k)}(t)\|^2 dt$$
$$+C\rho^{l_1-1}\Big(\sup_{t\in[0,T]}\|U^{(k-1)}(t)\|_{m-4,2}+\sup_{t\in[0,T]}\|U_j^{(k-1)}(t)\|_{m-4,2}\Big)$$
$$\times\sum_{l=1}^n\int_0^T \|\langle x_l\rangle^{-\frac{1+\delta}{2}}\langle D_{x_l}\rangle^{1/2}KW_j^{(k)}(t)\|\|\langle x_j\rangle^{-\frac{1+\delta}{2}}\langle D_{x_j}\rangle^{1/2}Kw_j^{(k-1)}(t)\|dt.$$

We again apply Lemma 2.2 , the Schwarz inequality and (3.31) to see that the right hand side of the above inequality is estimated by the value

$$C\sum_{j=1}^n \rho^{l_1}\int_0^T \|\langle x_j\rangle^{-\frac{1+\delta}{2}}\langle D_{x_j}\rangle^{1/2}KW_j^{(k)}(t)\|^2 dt$$
$$+C\rho^{l_1-1}e^{C_2A}\Big(\sup_{t\in[0,T]}\|U^{(k-1)}(t)\|_{m-4,2}+\sup_{t\in[0,T]}\|U_j^{(k-1)}(t)\|_{m-4,2}\Big)$$
$$\times\sum_{l=1}^n\int_0^T \|\langle x_l\rangle^{-\frac{1+\delta}{2}}\langle D_{x_l}\rangle^{1/2}KW_j^{(k)}(t)\|dt$$
$$\leq C\sum_{1\leq l\leq n}\rho^{l_1}(1+C\rho^{l_1}e^{C_2A}T)\int_0^T \|\langle x_l\rangle^{-\frac{1+\delta}{2}}\langle D_{x_l}\rangle^{1/2}KW_j^{(k)}(t)\|^2 dt$$
$$+\frac{1}{8}\Big(\sup_{t\in[0,T]}\|U^{(k-1)}(t)\|_{m-4,2}+\sup_{t\in[0,T]}\|U_j^{(k-1)}(t)\|_{m-4,2}\Big)^2. \tag{3.36}$$

From (3.34), (3.35) and (3.36) it follows that there exists a time $T=O(1/e^{\rho^{l_1+2}})$

such that

$$
\begin{aligned}
&\sup_{t\in[0,T]} \sum_{|\beta|=m-1} \|KW_j^{(k)}(t)\|^2 \\
&+ \frac{A}{8} \sum_{\substack{|\beta|=m-1\\ 1\le l\le n}} \int_0^T \|\langle x_l\rangle^{-\frac{1+\delta}{2}} \langle D_{x_l}\rangle^{1/2} KW_j^{(k)}(\tau)\|^2 d\tau \\
&\le \frac{1}{4}\Bigg(\|U^{(k-1)}\|_{T,m-2,\{m-4,2\}} + \|U_j^{(k-1)}\|_{T,m-2,\{m-4,2\}} \\
&+ \sum_{|\beta|=m-1} \big(\|KD^\beta U^{(k-1)}\| + \|KD^\beta U_j^{(k-1)}\| \big) \Bigg)^2. \qquad (3.37)
\end{aligned}
$$

By the definition of $W_j^{(k)}$ we have

$$
\begin{aligned}
W_j^{(k)} &= D^\beta u_j^{(k)} + 2\mathcal{B}(u^{(k-1)},\cdots)\langle D\rangle^{-2} D^\beta \bar{u}_j^{(k)} - D^\beta u_j^{(k-1)} \\
&- 2\mathcal{B}(u^{(k-2)},\cdots)\langle D\rangle^{-2} D^\beta \bar{u}_j^{(k-1)} \\
&= D^\beta U_j^{(k)} + 2\mathcal{B}(u^{(k-1)},\cdots)\langle D\rangle^{-2} D^\beta \bar{U}_j^{(k)} \\
&+ 2\Big(\mathcal{B}(u^{(k-1)},\cdots) - \mathcal{B}(u^{(k-2)},\cdots) \Big) \langle D\rangle^{-2} D^\beta \bar{u}_j^{(k-1)}.
\end{aligned}
$$

Hence by (3.31) and Lemma 2.1 we obtain

$$
\begin{aligned}
&\sum_{|\beta|=m-1} \|KW_j^{(k)}(t)\|^2 + \frac{A}{8} \sum_{\substack{|\beta|=m-1\\ 1\le l\le n}} \int_0^T \|\langle x_l\rangle^{-\frac{1+\delta}{2}} \langle D_{x_l}\rangle^{1/2} KW_j^{(k)}(\tau)\|^2 d\tau \\
&\le \sum_{|\beta|=m-1} \|KD^\beta U_j^{(k)}(t)\|^2 \\
&+ \frac{A}{8} \sum_{\substack{|\beta|=m-1\\ 1\le l\le n}} \int_0^T \|\langle x_l\rangle^{-\frac{1+\delta}{2}} \langle D_{x_l}\rangle^{1/2} KD^\beta U_j^{(k)}(\tau)\|^2 d\tau \\
&+ Ce^{C_2 A}\rho^{l_1} \Big(\|U_j^{(k-1)}\|_{T,m-2,\{m-4,2\}} + \|U_j^{(k-2)}\|_{T,m-2,\{m-4,2\}} \Big)^2 \qquad (3.38)
\end{aligned}
$$

and

$$\sum_{|\beta|=m-1} \|KD^\beta U_j^{(k)}(t)\|^2 + \frac{A}{8} \sum_{\substack{|\beta|=m-1\\1\le l\le n}} \int_0^T \|\langle x_l\rangle^{-\frac{1+\delta}{2}} \langle D_{x_l}\rangle^{1/2} KD^\beta U_j^{(k)}(\tau)\|^2 d\tau$$
$$\le \sum_{|\beta|=m-1} \|KW_j^{(k)}(t)\|^2 + \frac{A}{8} \sum_{\substack{|\beta|=m-1\\1\le l\le n}} \int_0^T \|\langle x_l\rangle^{-\frac{1+\delta}{2}} \langle D_{x_l}\rangle^{1/2} KW_j^{(k)}(\tau)\|^2 d\tau$$
$$+ Ce^{C_2 A}\rho^{l_1} \left(\|U_j^{(k-1)}\|_{T,m-2,\{m-4,2\}} + \|U_j^{(k-2)}\|_{T,m-2,\{m-4,2\}} \right)^2. \tag{3.39}$$

In the same way as in the proofs of (3.37), (3.38) and (3.39) we have

$$\sup_{t\in[0,T]} \sum_{|\beta|=m-3} \|K\partial_t W_j^{(k)}(t)\|^2 + \frac{A}{8} \sum_{\substack{|\beta|=m-3\\1\le l\le n}} \int_0^T \|\langle x_l\rangle^{-\frac{1+\delta}{2}} \langle D_{x_l}\rangle^{1/2} K\partial_t W_j^{(k)}(\tau)\|^2 d\tau$$
$$\le \frac{1}{4}\Bigg(\|U^{(k-1)}\|_{T,m-2,\{m-4,2\}} + \|U_j^{(k-1)}\|_{T,m-2,\{m-4,2\}} + \sum_{|\beta|=m-1} \left(\|KD^\beta U^{(k-1)}\| + \|KD^\beta U_j^{(k-1)}\|\right)\Bigg)^2, \tag{3.40}$$

$$\sum_{|\beta|=m-3} \|K\partial_t W_j^{(k)}(t)\|^2 + \frac{A}{8} \sum_{\substack{|\beta|=m-3\\1\le l\le n}} \int_0^T \|\langle x_l\rangle^{-\frac{1+\delta}{2}} \langle D_{x_l}\rangle^{1/2} K\partial_t W_j^{(k)}(\tau)\|^2 d\tau$$
$$\le \sum_{|\beta|=m-3} \|KD^\beta \partial_t U_j^{(k)}(t)\|^2 + \frac{A}{8} \sum_{\substack{|\beta|=m-3\\1\le l\le n}} \int_0^T \|\langle x_l\rangle^{-\frac{1+\delta}{2}} \langle D_{x_l}\rangle^{1/2} KD^\beta \partial_t U_j^{(k)}(\tau)\|^2 d\tau$$
$$+ Ce^{C_2 A}\rho^{l_1} \left(\|U_j^{(k-1)}\|_{T,m-2,\{m-4,2\}} + \|U_j^{(k-2)}\|_{T,m-2,\{m-4,2\}} \right) \tag{3.41}$$

and

$$\sum_{|\beta|=m-1}\|KD^\beta U_j^{(k)}(t)\|^2+\frac{A}{8}\sum_{\substack{|\beta|=m-1\\1\le l\le n}}\int_0^T\|\langle x_l\rangle^{-\frac{1+\delta}{2}}\langle D_{x_l}\rangle^{1/2}KD^\beta U_j^{(k)}(\tau)\|^2d\tau$$
$$\le\sum_{|\beta|=m-1}\|KW_j^{(k)}(t)\|^2+\frac{A}{8}\sum_{\substack{|\beta|=m-1\\1\le l\le n}}\int_0^T\|\langle x_l\rangle^{-\frac{1+\delta}{2}}\langle D_{x_l}\rangle^{1/2}KW_j^{(k)}(\tau)\|^2d\tau$$
$$+Ce^{C_2A}\rho^{l_1}\Big(\|U_j^{(k-1)}\|_{T,m-2,\{m-4,2\}}+\|U_j^{(k-2)}\|_{T,m-2,\{m-4,2\}}\Big). \tag{3.42}$$

From the estimates (3.32), (3.37) - (3.42) it follows that there exists a time $T=O(1/e^{\rho^{l_1+2}})$ such that

$$\|U^{(k)}\|_{X_{T,m-1,\{m-4,2\}}}+\|U_j^{(k)}\|_{X_{T,m-1,\{m-4,2\}}}$$
$$\le\frac{1}{8}\sum_{1\le l\le 3}\Big(\|U^{(k-l)}\|_{X_{T,m-1,\{m-4,2\}}}+\|U_j^{(k-l)}\|_{X_{T,m-1,\{m-4,2\}}}\Big). \tag{3.43}$$

We define

$$L_k=\sum_{1\le l\le 3}\Big(\|U^{(k+1-l)}\|_{X_{T,m-1,\{m-4,2\}}}+\|U_j^{(k+1-l)}\|_{X_{T,m-1,\{m-4,2\}}}\Big).$$

Then we have by (3.43)

$$L_k\le\frac{1}{2}L_{k-3}$$

which implies that $\{u^{(k)},u_1^{(k)},\cdots,u_n^{(k)}\}$ is a Cauchy sequence in $(X_{T,m-1,\{m-4,2\}})^{n+1}$. Hence there exists a unique solution $\{u,u_1,\cdots,u_n\}$ satisfying

$$\begin{cases} i\partial_t u+\frac{1}{2}\Delta u=\mathcal{N}(u,u_1,\cdots,u_n,\bar u,\bar u_1,\cdots,\bar u_n), \quad (t,x)\in\mathbf{R}\times\mathbf{R}^n,\\ i\partial_t u_j+\frac{1}{2}\Delta u_j=\partial_u\mathcal{N}\cdot u_j+\partial_{\bar u}\mathcal{N}\cdot\bar u_j+\sum_{l=1}^n\Big(\partial_{u_l}\mathcal{N}\cdot\partial_l u_j+\partial_{\bar u_l}\mathcal{N}\cdot\partial_l\bar u_j\Big),\\ u(0,x)=u_0(x),\quad u_j(0,x)=\partial_j u_0(x),\quad x\in\mathbf{R}^n,\quad 1\le j\le n.\end{cases} \tag{3.45}$$

By the uniqueness of solutions we see that $u_j=\partial_j u$. Therefore $u\in C([0,T];H^{m,0})\cap C([0,T];H^{m-3,2})$ since K^{-1} is a bounded operator in L^2. Finally we prove a smoothing effect for the solutions. We already know

$$\sup_{t\in[0,T]}\|u(t)\|_{m,0}^2+\sum_{\substack{|\alpha|=m\\1\le j\le n}}\int_0^T\|\langle x_j\rangle^{-s}\langle D_{x_j}\rangle^{1/2}KD^\alpha u(t)\|^2dt<\infty. \tag{3.46}$$

By a direct calculation we get

$$\begin{aligned}&\sum_{|\alpha|=m}\|D^\alpha\langle x_j\rangle^{-s}\langle D_{x_j}\rangle^{1/2}u(t)\|\\&\le C\left(\|u(t)\|_{m,0}+\sum_{|\alpha|=m}\|\langle x_j\rangle^{-s}\langle D_{x_j}\rangle^{1/2}D^\alpha u(t)\|\right)\\&\le Ce^{C_2A}\left(\|u(t)\|_{m,0}+\sum_{|\alpha|=m}\|K\langle x_j\rangle^{-s}\langle D_{x_j}\rangle^{1/2}D^\alpha u(t)\|\right)\quad\text{for}\quad s>1/2.\end{aligned}\tag{3.47}$$

We also have

$$\begin{aligned}\|K\langle x_j\rangle^{-s}\langle D_{x_j}\rangle^{1/2}f\|&\le\|K[\langle x_j\rangle^{-s},\langle D_{x_j}\rangle^{1/2}]f\|+\|[K\langle D_{x_j}\rangle^{1/2},\langle x_j\rangle^{-s}]f\|\\&+\|\langle x_j\rangle^{-s}[K,\langle D_{x_j}\rangle^{1/2}]f\|+\|\langle x_j\rangle^{-s}\langle D_{x_j}\rangle^{1/2}Kf\|.\end{aligned}\tag{3.48}$$

We apply [Lemma A.1, 6], [Lemma 3.2, 6] and [Lemma A.2, 6] to the first three terms of the right hand side of (3.48), respectively. Then we get

$$\|K\langle x_j\rangle^{-s}\langle D_{x_j}\rangle^{1/2}f\|\le Ce^{C_2A}\left(\|f\|+\|\langle x_j\rangle^{-s}\langle D_{x_j}\rangle^{1/2}Kf\|\right).\tag{3.49}$$

By inequalities (3.46), (3.47) and (3.49) we get the last estimate of Theorem 1.1. This completes the proof of Theorem 1.1. □

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