SUT Journal of Mathematics
Vol. 34, No. 2 (1998), 105–109

SUPER EDGE-MAGIC GRAPHS

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(Received May 6, 1998)

Abstract. For a graph G, a bijection f from $V(G) \cup E(G)$ to $\{1, 2, \ldots, |V(G)| + |E(G)|\}$ is called an edge-magic labeling of G if $f(u)+f(v)+f(uv)$ is independent on the choice of the edge uv. An edge-magic labeling is called super edge-magic if $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$. A graph G is called edge-magic (resp. super edge-magic) if there exists an edge-magic (resp. super edge-magic) labeling of G. In this paper, we investigate whether several families of graphs are (super) edge-magic or not. We also give several conjectures.

AMS 1991 *Mathematics Subject Classification.* 05C78.

Key words and phrases. graph labelling, edge-magic graphs, super edge-magic graphs.

§1. Introduction

We consider finite undirected graphs without loops and multiple edges. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of a graph G, respectively. The set of vertices adjacent to x in G is denoted by $N_G(x)$, and $\deg_G(x) = |N_G(x)|$ is the degree of x in G. Other terminologies or notation not defined here will be found in [1].

Let G be a graph with p vertices and q edges. A bijection f from $V(G) \cup E(G)$ to $\{1, 2, \cdots, p+q\}$ is called an *edge-magic* labeling of G if there exists a constant s (called the *magic number* of f) such that $f(u) + f(v) + f(uv) = s$ for any edge uv of G. An edge-magic labeling f is called *super edge-magic* if $f(V(G)) = \{1, 2, \cdots, p\}$ and $f(E(G)) = \{p+1, \cdots, p+q\}$. A graph G is called edge-magic (resp. super edge-magic) if there exists an edge-magic (resp. super edge-magic) labeling of G.

In this paper, we investigate whether several families of graphs are (super) edge-magic or not.

[1]This work was supported in part by the Catalan Research Council (CIRIT) under grant #1996BEAI400451

§2. Main results

Lemma 2.1. *If a nontrivial graph G is super edge-magic, then* $|E(G)| \leq 2|V(G)| - 3$.

Proof. Considering the extreme values of the labeling of vertices and edges, the magic number s must satisfy

$$1 + 2 + (|V(G)| + |E(G)|) \leq s \leq |V(G)| + (|V(G)| - 1) + (|V(G)| + 1).$$

□

Kotzig and Rosa [3, 4] proved that cycles and complete bipartite graphs are edge-magic and that a complete graph K_n is edge-magic if and only if $n \in \{1, 2, 3, 5, 6\}$. Using Lemma 2.1, it is easily verified that K_n is super edge-magic if and only if $n \in \{1, 2, 3\}$.

Theorem 2.2. *A cycle C_n is super edge-magic if and only if n is odd.*

Proof. Suppose there exits a super edge-magic labeling f of C_n with the magic number s. Then

$$\begin{aligned} sn &= \sum_{uv \in E(C_n)} \{f(u) + f(v) + f(uv)\} \\ &= 2 \sum_{v \in V(C_n)} f(v) + \sum_{uv \in E(C_n)} f(uv) \\ &= n(n+1) + \frac{(3n+1)n}{2}. \end{aligned}$$

This implies that $\dfrac{3n+1}{2} = s - n - 1$ is an integer. Hence n must be odd.

Let $n = 2m+1$ be an odd integer, $V(C_n) = \{v_0, v_1, \cdots, v_{n-1}\}$, and $E(C_n) = \{v_{n-1}v_0\} \cup \{v_i v_{i+1} | 0 \leq i \leq n-2\}$. Define

$$f(v_i) = \begin{cases} \dfrac{i+2}{2} & \text{if } i \text{ is even}, \\ m + \dfrac{i+3}{2} & \text{if } i \text{ is odd}, \end{cases}$$

$$\begin{aligned} f(v_{n-1}v_0) &= 2n, \\ f(v_i v_{i+1}) &= 2n - 1 - i \quad \text{for } 0 \leq i \leq n-2. \end{aligned}$$

It is easily seen that f is a super edge-magic labeling with the magic number $\dfrac{5n+3}{2}$. (See Figure 1.) □

Note that the edge-magic labeling of odd cycles in [3] is not super edge-magic.

Figure 1: Super edge-magic labeling of C_9

Theorem 2.3. *A complete bipartite graph $K_{m,n}$ is super edge-magic if and only if $m = 1$ or $n = 1$.*

Proof. It is easily verified that $K_{1,n}$ is super edge-magic. Suppose $K_{m,n}$ with $2 \leq m \leq n$ is super edge-magic. Lemma 2.1 implies that $mn \leq 2(m+n) - 3$, that is, $(m-2)(n-2) \leq 1$. Hence $m = 2$ or $m = n = 3$. It is straightforward to check that $K_{3,3}$ does not admit a super edge-magic labeling. Let f be a super edge-magic labeling of $K_{2,n}$ with the magic number s. Since $K_{2,2} = C_4$ is not super edge-magic by Theorem 2.2, we may assume that $n \geq 3$. By the proof of Lemma 2.1, $s = 3n + 5$ or $s = 3n + 6$. Suppose $s = 3n + 5$. In this case, define $g(x) = n + 3 - f(x)$ for $x \in V(K_{2,n})$ and $g(xy) = 4n + 5 - f(xy)$ for $xy \in E(K_{2,n})$. Then

$$\begin{aligned} g(x) + g(y) + g(xy) &= 6n + 11 - (f(x) + f(y) + f(xy)) \\ &= 3n + 6, \end{aligned}$$

that is, g is a super edge-magic labeling with the magic number $3n+6$. Hence we may assume that $s = 3n + 6$. Let x_i be the vertex of $K_{2,n}$ with $f(x_i) = i$, $1 \leq i \leq n + 2$. Since x_1x_2 is not an edge, x_1 and x_2 belong to the same partite class. Let A be the class that contains x_1 and x_2, and let B be the other class. Considering the label $3n + 2$, we see that x_1x_3 is an edge. Hence

$x_3 \in B$. If $x_4 \in B$, both x_1x_4 and x_2x_3 are edges. This is impossible since $f(x_1) + f(x_4) = 5 = f(x_2) + f(x_3)$. This implies that $x_4 \in A$. The only possibility of the edge with the label $3n$ is x_1x_5. However, this is not possible since $f(x_2) + f(x_5) = 7 = f(x_4) + f(x_3)$. □

Theorem 2.4. *A wheel graph W_n of order n is not super edge-magic. Moreover, W_n is not edge-magic if $n \equiv 0 \bmod 4$.*

Proof. Since $|E(W_n)| = 2n - 2 > 2|V(W_n)| - 3 = 2n - 3$, W_n is not super edge-magic by Lemma 2.1.

Let $V(W_n) = \{v_0, v_1, \cdots, v_{n-1}\}$ with $\deg(v_0) = n - 1$, and suppose f is an edge-magic labeling of W_n with the magic number s. Then

$$2(n-1)s = \sum_{i=1}^{3n-2} i + (n-2)f(v_0) + 2\sum_{j=1}^{n-1} f(v_j).$$

Suppose $n \equiv 0 \bmod 4$. Then

$$\sum_{i=1}^{3n-2} i = \frac{1}{2}(3n-2)(3n-1)$$

is an odd number. On the other hand,

$$2(n-1)s - (n-2)f(v_0) - 2\sum_{j=1}^{n-1} f(v_j)$$

is an even number. This is a contradiction. □

§3. Discussions

The following conjecture given in [3] is still open (see also [5]).

Conjecture 3.1. *Every tree is edge-magic.*

In this paper, we introduced the notion of super edge-magic graphs. We have checked that every tree of order up to 15 is super edge-magic. So, the following stronger conjecture may be true.

Conjecture 3.2. *Every tree is super edge-magic.*

Theorem 2.4 states that W_n is not edge-magic when $n \equiv 0 \bmod 4$. How about the other cases?

Conjecture 3.3. *W_n is edge-magic if $n \not\equiv 0 \bmod 4$.*

We have checked that Conjecture 3.3 is true when $n \leq 30$.

Finally, we conjecture that a nearly complete graph is not edge-magic.

Conjecture 3.4. *Suppose a graph G of order $n + m$ contains a complete graph of order n. If $n \gg m$, then G is not edge-magic.*

It is tempting to conjecture that if a graph contains a large complete graph, then it is not edge-magic. However, this is not true, since any graph can be embedded in a connected super edge-magic graph as an induced subgraph ([2]).

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