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WEIGHTED INEQUALITIES FOR THE LAPLACE TRANSFORM

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Abstract. Necessary conditions and sufficient conditions are given in order that the Laplace Transform is bounded between two Lebesgue spaces with weights. Such a boundedness is characterized for a large class of weights.

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§1 Introduction and Results

The Laplace Transform is defined as

$$(\mathcal{L}f)(x) = \int_0^\infty f(y)\exp[-xy]dy, \qquad 0 < x < \infty.$$

Throughout this paper it is assumed that

$$1 < p, q < \infty, \qquad p' = \frac{p}{p-1}, \qquad q' = \frac{q}{q-1}, \qquad r = \frac{qp}{p-q}$$

and

$$u(.), \qquad v(.), \qquad v^{1-p'}(.)$$

are weight functions, i.e. nonnegative and locally integrable functions.

Our purpose is to derive necessary conditions and sufficient conditions on $u(.)$ and $v(.)$ for which $\mathcal{L}$ is bounded from the Lebesgue space $L^p_v = L^p(]0,\infty[, v(x)dx)$ into $L^q_u = L^q(]0,\infty[, u(x)dx)$. That is for some constant $C > 0$

$$(1.1) \quad \Big(\int_0^\infty (\mathcal{L}f)^q(x)u(x)dx\Big)^{\frac{1}{q}} \leq C\Big(\int_0^\infty f^p(x)v(x)dx\Big)^{\frac{1}{p}} \qquad \text{for all } f(.) \geq 0.$$

For convenience this boundedness is also denoted by $\mathcal{L} : L^p_v \to L^q_u$.

This problem has been investigated by many authors. Indeed a sufficient condition ensuring (1.1) was given by K. Andersen and H. Heinig [An-Hg],

[Hg] (see also [Hz2], [An]). And a sufficient condition *close* to be necessary was found by S. Bloom [Bm].

Our present contribution is first to provide variant sufficient conditions for $\mathcal{L}: L^p_v \to L^q_u$ improving the result of S. Bloom [Bm] whenever $q > 4$. The technic used in [Bm] is to reduce the problem (1.1) to weighted estimates involving Hardy operator and Hardy antidifferentiation operator. As in [An-Hg] and [Hg], our approach is based on the weighted inequality

$$(1.2)\quad \Big(\int_0^\infty (Tf)^q(x)u(x)dx\Big)^{\frac{1}{q}} \leq C\Big(\int_0^\infty f^p(x)v(x)dx\Big)^{\frac{1}{p}} \qquad \text{for all } f(.) \geq 0$$

where

$$(Tf)(x) = \int_0^x f(y)\exp[-xy^{-1}]dy, \qquad 0 < x < \infty.$$

Although a sufficient condition for (1.2) is available in [An-Hg], [Hg] and [Hz2], here we give a new one which is not too far to be necessary for $T: L^p_v \to L^q_u$. The second contribution in this work is to provide another approach for $\mathcal{L}: L^p_v \to L^q_u$ which can be extended to treat boundednesses problems for many integral operators in higher dimension.

Our method for dealing with (1.2) is first to break the operator T into small pieces and next to do summations just by using Hölder inequalities and the fast increase of the exponential function.

Our first result reads as

Theorem 1. *Suppose that $\mathcal{L}: L^p_v \to L^q_u$. Then for some constant $A > 0$*

$$(1.3)\qquad \Big(\int_0^{R^{-1}} u(x)dx\Big)^{\frac{1}{q}}\Big(\int_0^R v^{1-p'}(y)dy\Big)^{\frac{1}{p'}} \leq A \qquad \text{for all } R > 0$$

and

$$(1.4)\quad \Big(\int_{R^{-1}}^\infty u(x)\exp[-qRx]dx\Big)^{\frac{1}{q}}\Big(\int_0^R v^{1-p'}(y)dy\Big)^{\frac{1}{p'}} \leq A \qquad \text{for all } R > 0.$$

Conversely let $p \leq q$. Then $\mathcal{L}: L^p_v \to L^q_u$ whenever for some $A > 0$ and $0 < \varepsilon \leq 1$

$$(1.5)\quad \Big(\int_{R^{-1}}^\infty u(x)\exp[-4^{-1}(1-\varepsilon)qRx]dx\Big)^{\frac{1}{q}} \times \Big(\int_{2^{-1}R}^R v^{1-p'}(y)dy\Big)^{\frac{1}{p'}} \leq A \qquad \text{for all } R > 0$$

and the condition (1.3) is satisfied.

A similar result was previously obtained by S. Bloom [Bm] with the condition (1.5) replaced by

$$(1.6)\quad \Big(\int_{R^{-1}}^\infty u(x)\exp[-Rx]dx\Big)^{\frac{1}{q}}\Big(\int_0^R v^{1-p'}(y)dy\Big)^{\frac{1}{p'}} \leq A \qquad \text{for all } R > 0.$$

Therefore our Theorem improves Bloom's one whenever $q > 4$.

In order to get a characterization result for certain weights, it would be useful to see connections between conditions (1.3) and (1.5).

Lemma 2. *Condition* (1.3) *implies* (1.5) *under one of the following assumptions*

i) $u(.)$ *is a decreasing function;*
ii) $v^{1-p'}(.) \in D$*;*
iii) $u(.) \in D$*;*
iv) $v(.)$ *is a increasing function.*

Here $w(.) \in D$ means that for some $C > 0$

$$\int_0^{2R} w(y)dy \leq C \int_0^R w(y)dy \qquad \text{for all } R > 0.$$

Theorem 1 and Lemma 2 can be combined to get

Corollary 3. *For* $p \leq q$*, the boundedness* $\mathcal{L} : L^p_v \to L^q_u$ *is equivalent to condition* (1.3) *whenever one of the following assumptions is satisfied:*

i) $u(.)$ *is a decreasing function;*
ii) $v^{1-p'}(.) \in D$*;*
iii) $v(.)$ *is an increasing function;*
iv) $u(.) \in D$*.*

Our second main result concerning the case $q < p$ is

Theorem 4. *Let* $q < p$*. Suppose that* $\mathcal{L} : L^p_v \to L^q_u$*. Then*

$$\int_0^\infty \left[\Big(\int_0^{x^{-1}} u(y)dy\Big)^{\frac{1}{q}} \Big(\int_0^x v^{1-p'}(z)dz\Big)^{\frac{1}{q'}}\right]^r v^{1-p'}(x)dx < \infty \tag{1.7}$$

and

$$\int_0^\infty \left[\Big(\int_{x^{-1}}^\infty u(y)\exp[-qxy]dy\Big)^{\frac{1}{q}} \Big(\int_0^x v^{1-p'}(z)dz\Big)^{\frac{1}{q'}}\right]^r v^{1-p'}(x)dx < \infty. \tag{1.8}$$

Conversely $\mathcal{L} : L^p_v \to L^q_u$ *whenever for some* $0 < \varepsilon \leq 1$

$$\int_0^\infty \left[\Big(\int_{x^{-1}}^\infty u(y)\exp[-4^{-1}(1-\varepsilon)qxy]dy\Big)^{\frac{1}{q}} \times \Big(\int_{2^{-1}x}^x v^{1-p'}(z)dz\Big)^{\frac{1}{q'}}\right]^r v^{1-p'}(x)dx < \infty. \tag{1.9}$$

and the condition (1.7) *is satisfied.*

Following S. Blom [Bm] for $q < p$, the boundedness $\mathcal{L} : L^p_v \to L^q_u$ holds whenever

$$(1.10)\quad \int_0^\infty \Big[\Big(\int_{x^{-1}}^\infty u(y)\exp[-xy]dy \Big)^{\frac{1}{q}} \Big(\int_0^x v^{1-p'}(z)dz \Big)^{\frac{1}{q'}} \Big]^r v^{1-p'}(x)dx < \infty.$$

and the condition (1.7) is satisfied. So this author's result is improved in Theorem 4 whenever $q > 4$.

As we have announced above, Theorems 1 and 4 are based on a sufficient condition for the boundedness $T : L^p_v \to L^q_u$, which is described by

Theorem 5. *For $p \le q$, the boundedness $T : L^p_v \to L^q_u$ holds if for some constants $A > 0$ and $0 < \varepsilon \le 1$*

$$(1.11)\quad \Big(\int_{2^{-1}R}^R u(x)dx \Big)^{\frac{1}{q}} \Big(\int_0^R v^{1-p'}(y)\exp[-4^{-1}(1-\varepsilon)p'Ry^{-1}]dy \Big)^{\frac{1}{p'}} \le A$$

for all $R > 0$.

And for $q < p$, then $T : L^p_v \to L^q_u$ whenever

$$(1.12)\quad \int_0^\infty \Big[\Big(\int_{2^{-1}x}^x u(z)dz \Big)^{\frac{1}{p}} \times \Big(\int_0^{2x} v^{1-p'}(y)\exp[-4^{-1}(1-\varepsilon)p'xy^{-1}]dy \Big)^{\frac{1}{p'}} \Big]^r u(x)dx < \infty.$$

Actually the above boundedness holds under the condition

$$(1.13)\quad \sum_{k=-\infty}^{\infty} \Big[\Big(\int_{2^{-(k+1)}}^{2^{-k}} u(x)dx \Big)^{\frac{1}{q}} \times \Big(\int_0^{2^{-k}} v^{1-p'}(y)\exp[-4^{-1}(1-\varepsilon)p'2^{-k}y^{-1}]dy \Big)^{\frac{1}{p'}} \Big]^r < \infty$$

which is implied by (1.12).

A necessary condition for $T : L^p_v \to L^q_u$ to be true is that

$$(1.14)\quad \Big(\int_{2^{-1}R}^R u(x)dx \Big)^{\frac{1}{q}} \Big(\int_0^{2^{-1}R} v^{1-p'}(y)\exp[-p'Ry^{-1}]dy \Big)^{\frac{1}{p'}} \le A$$

for all $R > 0$.

Condition (1.11) seems new and quite different from those given in [An-Hg], [An] and [Hz1]. Moreover it appears not too far from the necessary condition (1.14).

While this paper was submitted, the author has been extended the present approach to treat boundednesses problems for the two-dimensional Laplace transform.

§2 Proofs of Results

First we prove Theorem 1, Lemma 2 and Theorem 4. And next we give the proof of Theorem 5 on which Theorems 1 and 4 are based.

Proof of Theorem 1

The Necessary Part.

Suppose that $\mathcal{L} : L^p_v \to L^q_u$. To get condition (1.4), let $R > 0$ and $f(.)$ a nonnegative function whose support is included in $]0, R[$. Since for $0 < x < \infty$

$$(\mathcal{L}f)(x) = \int_0^R f(y) \exp[-yx]dy \geq \Big(\exp[-Rx]\Big) \int_0^R f(y)dy$$

inequality (1.1) yields

$$\Big(\int_0^R f(y)dy\Big)^q \int_{R^{-1}}^\infty u(x) \exp[-qRx]dx \leq C^q \Big(\int_0^R f^p(x)v(x)dx\Big)^{\frac{q}{p}}.$$

Taking $f(.) = v^{1-p'}(.)$ on its support $]0, R[$ and using the identity $(1-p')p+1 = (1-p')$ then condition (1.4) appears immediately.

To get condition (1.3) the key is to observe that

$$(\mathcal{L}f)(x) \geq [\exp(-1)](Hf)(x^{-1}) = [\exp(-1)] \times \int_0^{x^{-1}} f(y)dy, \qquad 0 < x < \infty.$$

So inequality (1.1) yields $H : L^p_v \to L^q_w$ with

$$w(x) = u(x^{-1})x^{-2}.$$

As it is well-known in [Oc-Kr], this last boundedness implies that

$$\text{(2.1)} \qquad \Big(\int_R^\infty w(x)dx\Big)^{\frac{1}{q}} \Big(\int_0^R v^{1-p'}(y)dy\Big)^{\frac{1}{p'}} \leq A \qquad \text{for all } R > 0$$

and for some fixed constant $A > 0$. By the definition of $w(.)$, (2.1) is nothing else than condition (1.3).

The Sufficient Part.

As used in [An-Hg] and [Hg], the main point is to observe that

$$(\mathcal{L}f)(x) \leq \Big((Hf)(x^{-1}) + (T^*f)(x^{-1})\Big) \qquad 0 < x < \infty$$

where the dual T^* of the operator T is defined by

$$(T^*f)(y) = \int_y^\infty f(x) \exp[-xy^{-1}]dx, \qquad 0 < y < \infty$$

and $\int_0^\infty (T^* f)(y)g(y)dy = \int_0^\infty f(x)(Tg)(x)dx$. With the above observation to derive (1.1) it remains to get

$$H : L_v^p \to L_w^q \qquad \text{and} \qquad T^* : L_v^p \to L_w^q \tag{2.2.}$$

As it is known in [Oc-Kr], for $p \leq q$, the boundedness $H : L_v^p \to L_w^q$ holds whenever the test condition (2.1) (which is the same as (1.3)) holds.

By duality arguments, $T^* : L_v^p \to L_w^q$ is equivalent to $T : L_{w^{1-q'}}^{q'} \to L_{v^{1-p'}}^{p'}$. And by Theorem 5, this last boundedness holds whenever for some $A > 0$ and $0 < \varepsilon \leq 1$

$$\Big(\int_0^R w(y)\exp[-4^{-1}(1-\varepsilon)qRy^{-1}]dy\Big)^{\frac{1}{q}}\Big(\int_{2^{-1}R}^R v^{1-p'}(x)dx\Big)^{\frac{1}{p'}} \leq A$$

for all $R > 0$. Using the definition of $w(.)$, clearly the validity of this last inequality is ensured by condition (1.5).

Proof of Lemma 2

Condition (1.3) implies (1.5) whenever $u(.)$ is a decreasing function since for $R > 0$

$$\begin{aligned}\int_{R^{-1}}^\infty u(x)\exp[-4^{-1}(1-\varepsilon)qRx]dx &\leq u(R^{-1})\int_{R^{-1}}^\infty \exp[-4^{-1}(1-\varepsilon)qRx]dx\\ &= u(R^{-1})R^{-1}\int_1^\infty \exp[-4^{-1}(1-\varepsilon)qz]dz\\ &= c_1 u(R^{-1})R^{-1} \leq c_1\int_0^{R^{-1}} u(y)dy.\end{aligned}$$

The implication (1.3) $\Longrightarrow$ (1.5) for $v^{1-p'}(.) \in D$ can be seen as follows

$$\begin{aligned}&\Big(\int_{R^{-1}}^\infty u(x)\exp[-4^{-1}(1-\varepsilon)qRx]dx\Big)\Big(\int_{2^{-1}R}^R v^{1-p'}(y)dy\Big)^{\frac{q}{p'}}\\ &= \sum_{k=1}^\infty \Big(\int_{2^{k-1}R^{-1}}^{2^kR^{-1}} u(x)\exp[-4^{-1}(1-\varepsilon)qRx]dx\Big)\Big(\int_{2^{-1}R}^R v^{1-p'}(y)dy\Big)^{\frac{q}{p'}}\\ &\leq \sum_{k=1}^\infty \Big[\exp[-8^{-1}(1-\varepsilon)q2^k]\Big]\Big(\int_{2^{k-1}R^{-1}}^{2^kR^{-1}} u(x)dx\Big)\Big(\int_0^R v^{1-p'}(y)dy\Big)^{\frac{q}{p'}}\\ &\leq \sum_{k=1}^\infty 2^{k\tau}\Big[\exp[-8^{-1}(1-\varepsilon)q2^k]\Big]\Big(\int_0^{2^kR^{-1}} u(x)dx\Big)\Big(\int_0^{2^{-k}R} v^{1-p'}(y)dy\Big)^{\frac{q}{p'}}\\ &\leq A^q\sum_{k=1}^\infty 2^{k\tau}\Big[\exp[-8^{-1}(1-\varepsilon)q2^k]\Big] = c_2A^q.\end{aligned}$$

Here we have used the fact that $w(.) \in D$ implies that for some $\tau > 0$

$$\int_0^{2^m R} w(y)dy \leq 2^{m\tau} \int_0^R w(y)dy \qquad \text{for all } R > 0 \text{ and all integers } m \geq 1.$$

The implication (1.3) $\Longrightarrow$ (1.5), for $u(.) \in D$, can be easily seen as above. This implication is also true provided $v(.)$ is an increasing function, since in this case $v^{1-p'}(.)$ is a decreasing function and consequently $v^{1-p'}(.) \in D$.

Proof of Theorem 4

The Necessary part

As in the proof of Theorem 1, to get the condition (1.7), the key is to observe that the boundedness $\mathcal{L} : L_v^p \to L_u^q$ implies the weighted Hardy inequality $H : L_v^p \to L_w^q$ with $w(.)$ defined as in (2.2). As it is well-known in [Oc-Kr], for $q < p$, this last boundedness implies that

$$\int_0^\infty \Big[\Big(\int_x^\infty w(y)dy \Big)^{\frac{1}{q}} \Big(\int_0^x v^{1-p'}(z)dz \Big)^{\frac{1}{q'}} \Big]^r v^{1-p'}(x)dx < \infty. \tag{2.3}$$

Using the definition of $w(.)$, then (2.3) is nothing else than condition (1.7).

Condition (1.8) can be derived immediatly from $(\mathcal{L}u)(z) \geq (T^*u)(z^{-1})$ and the inequality

$$\int_0^\infty \Big[\Big((\mathcal{L}u)(qx) \Big)^{\frac{1}{q}} \Big(\int_0^x v^{1-p'}(z)dz \Big)^{\frac{1}{q'}} \Big]^r v^{1-p'}(x)dx < \infty.$$

The fact that this last is a necessary condition for $\mathcal{L} : L_v^p \to L_u^q$ was proved in [Bm].

The Sufficient part.

As we have explained in the proof of Theorem 1, it remains to get the boundednesses of H and T^* as in (2.2).

For $q < p$, again by well-known results as written in [Oc-Kr], then $H : L_v^p \to L_w^q$ whenever the test condition (2.3) (which is the same as (1.7)) holds.

Theorem 5 applied to $q < p$ and $T : L^{q'}_{w^{1-q'}} \to L^{p'}_{v^{1-p'}}$, lead to state that the boundedness $T^* : L_v^p \to L_w^q$ holds whenever

$$\int_0^\infty \Big[\Big(\int_{2^{-1}x}^x v^{1-p'}(y)dy \Big)^{\frac{1}{q'}} \times \\ \Big(\int_0^{2x} w(z) \exp[-4^{-1}(1-\varepsilon)qxz^{-1}]dz \Big)^{\frac{1}{q}} \Big]^r v^{1-p'}(x)dx < \infty.$$

Clearly, after using the definition of $w(.)$, the validity of this inequality is equivalent to

$$\int_0^\infty \Big[\Big(\int_{(2x)^{-1}}^\infty u(y) \exp[-4^{-1}(1-\varepsilon)qxy]dy \Big)^{\frac{1}{q}} \times \\ \Big(\int_{2^{-1}x}^x v^{1-p'}(z)dz \Big)^{\frac{1}{q'}} \Big]^r v^{1-p'}(x)dx < \infty.$$

This inequality will be a consequence of condition (1.9) and

$$\int_0^\infty \Big[\Big(\int_{(2x)^{-1}}^{x^{-1}} u(y)\exp[-4^{-1}(1-\varepsilon)qxy]dy\Big)^{\frac{1}{q}} \times \Big(\int_{2^{-1}x}^{x} v^{1-p'}(z)dz\Big)^{\frac{1}{q'}}\Big]^r v^{1-p'}(x)dx < \infty.$$

This last inequality is true because of condition (1.7) and
$\exp[-4^{-1}(1-\varepsilon)qxy] < \exp[-8^{-1}(1-\varepsilon)q]$ whenever $(2x)^{-1} < y < x^{-1}$.

Proof of Theorem 5

The Sufficient part

The main point to get $T : L^p_v \to L^q_u$ is to cut the operator as

$$\int_0^\infty (Tf)^q(x)u(x)dx \leq \sum_{k=-\infty}^{\infty}\Big[\sum_{j=0}^{\infty}\lambda_j\Big(\int_{2^{-(2+j+k)}}^{2^{-(j+k)}} f^p(x)v(x)dx\Big)^{\frac{1}{p}}\Theta_k\Big]^q \tag{2.4}$$

for all functions $f(.) \geq 0$. Here

$$\lambda_j = \exp[-\varepsilon 2^j] \qquad \text{with} \qquad 0 < \varepsilon \leq 1$$

and

$$\begin{aligned}\Theta_k =&\Theta_k(p,q,v,u)\\ =&\Big(\int_{2^{-(1+k)}}^{2^{-k}} u(y)dy\Big)^{\frac{1}{q}}\Big(\int_0^{2^{-k}} v^{1-p'}(x)\exp[-4^{-1}(1-\varepsilon)p'2^{-k}x^{-1}]dx\Big)^{\frac{1}{p'}}.\end{aligned}$$

It can be noted that $\sum_{l=0}^{\infty}\lambda_l = c_0 < \infty$. We will postpone below the proof of (2.4).

Now consider the case $p \leq q$. By condition (1.11) (with $R = 2^{-k}$) then

$$\Theta_k \leq A.$$

Using this last fact and the cutting out (2.4), the boundedness $T : L^p_v \to L^q_u$ appears as follows

$$\begin{aligned}\int_0^\infty (Tf)^q(x)u(x)dx \leq& \sum_{k=-\infty}^{\infty}\Big[\sum_{j=0}^{\infty}\lambda_j\Big(\int_{2^{-(2+j+k)}}^{2^{-(j+k)}} f^p(x)v(x)dx\Big)^{\frac{1}{p}}\Theta_k\Big]^q\\ \leq& A^q\sum_{k=-\infty}^{\infty}\Big[\sum_{j=0}^{\infty}\lambda_j\Big(\int_{2^{-(2+j+k)}}^{2^{-(j+k)}} f^p(x)v(x)dx\Big)^{\frac{1}{p}}\Big]^q\end{aligned}$$

$$\begin{aligned}
&\leq A^q \sum_{k=-\infty}^{\infty} \Big[\sum_{j=0}^{\infty} \lambda_j \int_{2^{-(2+j+k)}}^{2^{-(j+k)}} f^p(x)v(x)dx\Big]^{\frac{q}{p}} \Big[\sum_{l=0}^{\infty} \lambda_l\Big]^{\frac{q}{p'}} \\
&\leq c_1 A^q \Big[\sum_{j=0}^{\infty} \lambda_j \sum_{k=-\infty}^{\infty} \int_{2^{-(2+j+k)}}^{2^{-(j+k)}} f^p(x)v(x)dx\Big]^{\frac{q}{p}} \qquad \text{since } \frac{q}{p} \geq 1 \\
&\leq c_2 A^q \Big[\sum_{j=0}^{\infty} \lambda_j\Big]^{\frac{q}{p}} \Big(\int_0^{\infty} f^p(x)v(x)dx\Big)^{\frac{q}{p}} \\
&= c_3 A^q \Big(\int_0^{\infty} f^p(x)v(x)dx\Big)^{\frac{q}{p}}.
\end{aligned}$$

For the case $q < p$, observe that by condition (1.13)

$$\sum_{k=-\infty}^{\infty} \Theta_k^r < A^r.$$

We will differ below the proof of the fact that (1.13) is implied by condition (1.12).

Using this observation, the boundedness $T : L_v^p \to L_u^q$ appears as follows

$$\begin{aligned}
\int_0^{\infty} (Tf)^q(x)u(x)dx &\leq \sum_{k=-\infty}^{\infty} \Big[\sum_{j=0}^{\infty} \lambda_j \Big(\int_{2^{-(2+j+k)}}^{2^{-(j+k)}} f^p(x)v(x)dx\Big)^{\frac{1}{p}} \Theta_k\Big]^q \\
&\leq \sum_{k=-\infty}^{\infty} \Big[\sum_{j=0}^{\infty} \lambda_j \int_{2^{-(2+j+k)}}^{2^{-(j+k)}} f^p(x)v(x)dx\Big]^{\frac{q}{p}} \Theta_k^q \Big[\sum_{l=0}^{\infty} \lambda_l\Big]^{\frac{q}{p'}} \\
&\leq c_1 \sum_{k=-\infty}^{\infty} \Big[\sum_{j=0}^{\infty} \lambda_j \int_{2^{-(2+j+k)}}^{2^{-(j+k)}} f^p(x)v(x)dx\Big]^{\frac{q}{p}} \Theta_k^q \\
&\leq c_1 \Big[\sum_{k=-\infty}^{\infty} \sum_{j=0}^{\infty} \lambda_j \int_{2^{-(2+j+k)}}^{2^{-(j+k)}} f^p(x)v(x)dx\Big]^{\frac{q}{p}} \Big[\sum_{m=-\infty}^{\infty} \Theta_m^r\Big]^{1-\frac{q}{p}} \\
&\leq c_1 A^q \Big[\sum_{j=0}^{\infty} \lambda_j \sum_{k=-\infty}^{\infty} \int_{2^{-(2+j+k)}}^{2^{-(j+k)}} f^p(x)v(x)dx\Big]^{\frac{q}{p}} \\
&\leq c_2 A^q \Big[\sum_{j=0}^{\infty} \lambda_j\Big]^{\frac{q}{p}} \Big(\int_0^{\infty} f^p(x)v(x)dx\Big)^{\frac{q}{p}} \\
&\leq c_3 A^q \Big(\int_0^{\infty} f^p(x)v(x)dx\Big)^{\frac{q}{p}}.
\end{aligned}$$

It is now time to prove (2.4). This inequality is true since

$$\int_0^{\infty} (Tf)^q(x)u(x)dx = \int_0^{\infty} \Big[\sum_{j=0}^{\infty} \int_{2^{-(j+1)}x}^{2^{-j}x} f(y)\exp[-xy^{-1}]dy\Big]^q u(x)dx$$

$$= \sum_{k=-\infty}^{\infty} \int_{2^{-(k+1)}}^{2^{-k}} \Big[\sum_{j=0}^{\infty} \int_{2^{-(j+1)}x}^{2^{-j}x} f(y)\exp[-xy^{-1}]dy\Big]^q u(x)dx$$

$$\leq \sum_{k=-\infty}^{\infty} \Big[\sum_{j=0}^{\infty} \Big(\int_{2^{-(j+2+k)}}^{2^{-(j+k)}} f(y)dy\Big)\exp[-2^j]\Big]^q \Big(\int_{2^{-(k+1)}}^{2^{-k}} u(x)dx\Big)$$

$$\leq \sum_{k=-\infty}^{\infty} \Big[\sum_{j=0}^{\infty} \Big(\int_{2^{-(j+2+k)}}^{2^{-(j+k)}} f^p(y)v(y)dy\Big)^{\frac{1}{p}} \times$$

$$\Big(\int_{2^{-(j+2+k)}}^{2^{-(j+k)}} v^{1-p'}(z)dz\Big)^{\frac{1}{p'}} \Big(\int_{2^{-(k+1)}}^{2^{-k}} u(x)dx\Big)^{\frac{1}{q}} \exp[-2^j]\Big]^q$$

$$\leq \sum_{k=-\infty}^{\infty} \Big[\sum_{j=0}^{\infty} \exp[-\varepsilon 2^j]\Big(\int_{2^{-(j+2+k)}}^{2^{-(j+k)}} f^p(y)v(y)dy\Big)^{\frac{1}{p}} \times$$

$$\Big(\int_{2^{-(j+2+k)}}^{2^{-(j+k)}} v^{1-p'}(z)\exp[-4^{-1}(1-\varepsilon)p'2^{-k}z^{-1}]dz\Big)^{\frac{1}{p'}} \times$$

$$\Big(\int_{2^{-(k+1)}}^{2^{-k}} u(x)dx\Big)^{\frac{1}{q}}\Big]^q$$

$$\leq \sum_{k=-\infty}^{\infty} \Big[\sum_{j=0}^{\infty} \exp[-\varepsilon 2^j]\Big(\int_{2^{-(j+2+k)}}^{2^{-(j+k)}} f^p(y)v(y)dy\Big)^{\frac{1}{p}} \times$$

$$\Big(\int_0^{2^{-k}} v^{1-p'}(z)\exp[-4^{-1}(1-\varepsilon)p'2^{-k}z^{-1}]dz\Big)^{\frac{1}{p'}} \Big(\int_{2^{-(k+1)}}^{2^{-k}} u(x)dx\Big)^{\frac{1}{q}}\Big]^q$$

$$= \sum_{k=-\infty}^{\infty} \Big[\sum_{j=0}^{\infty} \lambda_j \Big(\int_{2^{-(j+2+k)}}^{2^{-(j+k)}} f^p(y)v(y)dy\Big)^{\frac{1}{p}} \Theta_k\Big]^q.$$

Now we can show how does the condition (1.12) imply (1.13). Here the main point is the elementary equality

$$\Big(\int_a^b u(z)dz\Big)^{\frac{r}{q}} = \frac{r}{q}\int_a^b \Big(\int_a^x u(z)dz\Big)^{\frac{r}{p}} u(x)dx, \qquad 0 < a < b < \infty.$$

For shortness the term $4^{-1}(1-\varepsilon)p'$ is merely denoted by C. The implication (1.12) $\Longrightarrow$ (1.13) can be seen as follows

$$\sum_{k=-\infty}^{\infty} \Big[\Big(\int_0^{2^{-k}} v^{1-p'}(y)\exp[-C2^{-k}y^{-1}]dy\Big)^{\frac{1}{p'}} \Big(\int_{2^{-(k+1)}}^{2^{-k}} u(x)dx\Big)^{\frac{1}{q}}\Big]^r$$

$$\approx \sum_{k=-\infty}^{\infty} \Big(\int_0^{2^{-k}} v^{1-p'}(y)\exp[-C2^{-k}y^{-1}]dy\Big)^{\frac{r}{p'}} \times$$

$$\int_{2^{-(k+1)}}^{2^{-k}} \Big(\int_{2^{-(k+1)}}^{x} u(z)dz\Big)^{\frac{r}{p}} u(x)dx$$

$$
\begin{aligned}
&= \sum_{k=-\infty}^{\infty} \int_{2^{-(k+1)}}^{2^{-k}} \Big[\Big(\int_{2^{-(k+1)}}^{x} u(z)dz \Big)^{\frac{1}{p}} \times \\
&\qquad\qquad \Big(\int_0^{2^{-k}} v^{1-p'}(y) \exp[-C2^{-k}y^{-1}]dy \Big)^{\frac{1}{p'}} \Big]^r u(x)dx \\
&\leq \sum_{k=-\infty}^{\infty} \int_{2^{-(k+1)}}^{2^{-k}} \Big[\Big(\int_{2^{-1}x}^{x} u(z)dz \Big)^{\frac{1}{p}} \times \\
&\qquad\qquad \Big(\int_0^{2x} v^{1-p'}(y) \exp[-Cxy^{-1}]dy \Big)^{\frac{1}{p'}} \Big]^r u(x)dx \\
&= \int_0^{\infty} \Big[\Big(\int_{2^{-1}x}^{x} u(z)dz \Big)^{\frac{1}{p}} \Big(\int_0^{2x} v^{1-p'}(y) \exp[-Cxy^{-1}]dy \Big)^{\frac{1}{p'}} \Big]^r u(x)dx \\
&\leq A^r \qquad \text{by using (1.12).}
\end{aligned}
$$

The Necessary part

To get the condition (1.14) from the boundedness $T : L^p_v \to L^q_u$, by a duality argument, it can be assumed that $T^* : L^{q'}_{u^{1-q'}} \to L^{p'}_{v^{1-p'}}$ or equivalently for some constant $C > 0$

$$
\begin{aligned}
&\Big(\int_0^{\infty} \Big[\int_y^{\infty} f(x)u(x) \exp[-xy^{-1}]dx \Big]^{p'} v^{1-p'}(y)dy \Big)^{\frac{1}{p'}} \\
&\qquad\qquad \leq C \Big(\int_0^{\infty} f^{q'}(z)u(z)dz \Big)^{\frac{1}{q'}} \qquad \text{for all } f(.) \geq 0.
\end{aligned}
$$

Take $R > 0$ and $f(.)$ a nonnegative function whose support is included $]2^{-1}R, R[$. Since for $0 < y < 2^{-1}R$

$$
\begin{aligned}
\int_y^{\infty} f(x)u(x) \exp[-xy^{-1}]dx &= \int_{2^{-1}R}^{R} f(x)u(x) \exp[-xy^{-1}]dx \\
&\geq \Big(\int_{2^{-1}R}^{R} f(x)u(x)dx \Big) \exp[-Ry^{-1}]
\end{aligned}
$$

the above inequality yields

$$
\begin{aligned}
&\Big(\int_{2^{-1}R}^{R} f(x)u(x)dx \Big)^{p'} \int_0^{2^{-1}R} v^{1-p'}(y) \exp[-p'Ry^{-1}]dy \\
&\qquad\qquad \leq C^{p'} \Big(\int_{2^{-1}R}^{R} f^{q'}(z)u(z)dz \Big)^{\frac{p'}{q'}}.
\end{aligned}
$$

Taking $f(.) = 1$ on its support $]2^{-1}R, R[$ then condition (1.14) appears immediately.

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