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REMARKS ON SCATTERING THEORY AND LARGE TIME ASYMPTOTICS OF SOLUTIONS TO HARTREE TYPE EQUATIONS WITH A LONG RANGE POTENTIAL

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Abstract. We study the scattering problem and asymptotics for large time of solutions to the Hartree type equations

$$\begin{cases} iu_t = -\frac{1}{2}\Delta u + f(|u|^2)u, & (t,x) \in \mathbf{R} \times \mathbf{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbf{R}^n, \quad n \geq 2, \end{cases}$$

where the nonlinear interaction term is $f(|u|^2) = V * |u|^2, V(x) = \lambda |x|^{-\delta}, \quad \lambda \in \mathbf{R}, 0 < \delta < 1$. We suppose that the initial data $u_0 \in H^{0,l}$ and the value $\epsilon = \|u_0\|_{H^{0,l}}$ is sufficiently small,where l is an integer satisfying $l \geq [\frac{n}{2}] + 3$, and $[s]$ denotes the largest integer less than s. Then we prove that there exists a unique final state $u_+ \in H^{0,l-2}$ such that for all $t > 1$

$$u(t,x) = \frac{1}{(it)^{\frac{n}{2}}}\hat{u}_+(\frac{x}{t})\exp(\frac{ix^2}{2t} - \frac{it^{1-\delta}}{1-\delta}f(|\hat{u}_+|^2)(\frac{x}{t}) + O(1+t^{1-2\delta})) + O(t^{-n/2-\delta})$$

uniformly with respect to $x \in \mathbf{R}^n$ with the following decay estimate $\|u(t)\|_{L^p} \leq C\epsilon t^{\frac{n}{p}-\frac{n}{2}}$, for all $t \geq 1$ and for every $2 \leq p \leq \infty$. Furthermore we show that for $\frac{1}{2} < \delta < 1$ there exists a unique final state $u_+ \in H^{0,l-2}$ such that for all $t \geq 1$

$$\|u(t) - \exp(-\frac{it^{1-\delta}}{1-\delta}f(|\hat{u}_+|^2)(\frac{x}{t}))U(t)u_+\|_{L^2} = O(t^{1-2\delta})$$

and uniformly with respect to $x \in \mathbf{R}^n$

$$u(t,x) = \frac{1}{(it)^{\frac{n}{2}}}\hat{u}_+(\frac{x}{t})\exp(\frac{ix^2}{2t} - \frac{it^{1-\delta}}{1-\delta}f(|\hat{u}_+|^2)(\frac{x}{t})) + O(t^{-n/2+1-2\delta}),$$

where $\hat{\phi}$ denotes the Fourier transform of the function ϕ, $H^{m,s} = \{\phi \in \mathcal{S}'; \|\phi\|_{m,s} = \|(1+|x|^2)^{s/2}(1-\Delta)^{m/2}\phi\|_{L^2} < \infty\}, m, s \in \mathbf{R}$. In [5] we assumed that $u_0 \in H^{m,0} \cap H^{0,m}$, $(m = n+2)$, and showed the same results as in this paper. Here we show that we do not need regularity conditions on the initial data by showing the local existence theorem in lower order Sobolev spaces.

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§1. Introduction

We study the asymptotic behavior for large time of solutions to the Cauchy problem for the Hartree type equation

$$\begin{cases} i\partial_t u = -\frac{1}{2}\Delta u + f(|u|^2)u, & (t,x) \in \mathbf{R} \times \mathbf{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbf{R}^n, \end{cases} \tag{1.1}$$

where

$$f(|u|^2) = V * |u|^2 = \int V(x-y)|u|^2(y)dy,$$

$$V(x) = \lambda |x|^{-\delta}, \quad \lambda \in \mathbf{R}, \quad 0 < \delta < 1 \quad \text{and} \quad n \geq 2.$$

From the point of view of the large time behavior of solutions we classify the equation (1.1) by the value δ into three cases. We call the equation (1.1) with $1 < \delta < n$ as the super-critical one. If $\delta = 1$ the equation (1.1) is known as the Hartree equation and is considered as the critical case in the scattering theory. We refer to the equation (1.1) with $0 < \delta < 1$ as the sub-critical case. It is known that the usual scattering states do not exist in the critical and sub-critical cases (see, e.g., [8]). Therefore the scattering problem in these cases is more difficult than that of the super-critical case. The critical case was considered in many papers, see, for example, [2, 4, 6, 9]. For the super-critical case, see, e.g., [3,7,8]. Recently in [5] we studied the sub-critical case $0 < \delta < 1$ and obtained the sharp time decay estimates of solutions. For $1/2 < \delta < 1$ we proved the existence of the modified scattering states under the conditions that the initial data $u_0 \in H^{m,0} \cap H^{0,m}, (m = n+2)$ and the norm $\|u_0\|_{m,0} + \|u_0\|_{0,m}$ is sufficiently small. Our purpose in this paper is to remove the regularity conditions on the initial data. More precisely, we will prove the results of [5] under the conditions that the initial data $u_0 \in H^{l,0}$ and the norm $\|u_0\|_{0,l}$ is sufficiently small, where l is an integer satisfying $l \geq [\frac{n}{2}]+3$ and $[s]$ denotes the largest integer less than s.

In what follows we consider the positive time t only since for the negative one the results are analogous. We use the following notation and function spaces. We let $\partial_j = \partial/\partial x_j, \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in (\mathbf{N} \cup \{0\})^n$, $|\alpha| = \sum_{j=1}^n \alpha_j$. And let $\mathcal{F}\phi$ or $\hat{\phi}$ be the Fourier transform of ϕ defined by $\mathcal{F}\phi(\xi) = \frac{1}{(2\pi)^{n/2}} \int e^{-ix\cdot\xi}\phi(x)dx$ and $\mathcal{F}^{-1}\phi(x)$ be the inverse Fourier transform of ϕ, i.e.
$\mathcal{F}^{-1}\phi(x) = \frac{1}{(2\pi)^{n/2}} \int e^{ix\cdot\xi}\phi(\xi)d\xi$.

We introduce some function spaces. $L^p = \{\phi \in \mathcal{S}'; \|\phi\|_p < \infty\}$, where $\|\phi\|_p = (\int |\phi(x)|^p dx)^{1/p}$ if $1 \leq p < \infty$ and $\|\phi\|_\infty = \text{ess.sup}\{|\phi(x)|; x \in \mathbf{R}^n\}$

if $p=\infty$. For simplicity we let $\|\phi\|=\|\phi\|_2$. Weighted Sobolev space $H^{m,s}=\{\phi\in\mathcal{S}';\|\phi\|_{m,s}=\|(1+|x|^2)^{s/2}(1-\Delta)^{m/2}\phi\|<\infty\}$, $m,s\in\mathbf{R}$ and the homogeneous Sobolev space $\dot{H}^{m,s}=\{\phi\in\mathcal{S}';\||x|^s(-\Delta)^{m/2}\phi\|<\infty\}$ with seminorm $\|\phi\|_{\dot{H}^{m,s}}=\||x|^s(-\Delta)^{m/2}\phi\|$. We let $(\psi,\varphi)=\int\psi(x)\cdot\overline{\varphi}(x)dx$. By $C(I;E)$ we denote the space of continuous functions from an interval I to a Banach space E.

The free Schrödinger evolution group $U(t)=e^{it\Delta/2}$ gives us the solution of the linear Cauchy problem (1.1) (with $f=0$). It can be represented explicitly in the following manner

$$U(t)\phi=\frac{1}{(2\pi it)^{n/2}}\int e^{i(x-y)^2/2t}\phi(y)dy=\mathcal{F}^{-1}e^{-it\xi^2/2}\mathcal{F}\phi.$$

Note that $U(t)=M(t)D(t)\mathcal{F}M(t)$, where $M=M(t)=\exp(\frac{ix^2}{2t})$ and $D(t)$ is the dilation operator defined by $(D(t)\psi)(x)=\frac{1}{(it)^{n/2}}\psi(\frac{x}{t})$. Then since $D(t)^{-1}=i^nD(\frac{1}{t})$ we have $U(-t)=\overline{M}\mathcal{F}^{-1}D(t)^{-1}\overline{M}=\overline{M}i^n\mathcal{F}^{-1}D(\frac{1}{t})\overline{M}$, where $\overline{M}=M(-t)=\exp(-\frac{ix^2}{2t})$.

Different positive constants might be denoted by the same letter C.

We now state our results in this paper.

Theorem 1.1. *Let $0<\delta<1$. Suppose that the initial data $u_0\in H^{0,l}$, and the value $\epsilon=\|u_0\|_{H^{0,l}}$ is sufficiently small, where l is an integer satisfying $l\ge[\frac{n}{2}]+3$. Then there exists a unique global solution of the Hartree type equation (1.1) such that $U(-t)u(t)\in C([0,\infty);H^{0,l})$ and $\|U(-t)u(t)\|_{0,l}\le C\epsilon(1+t)^{(1-\delta)l}$. Moreover the following decay estimate*

$$\|u(t)\|_p\le C\epsilon t^{\frac{n}{p}-\frac{n}{2}}$$

is valid for all $t\ge1$, where $2\le p\le\infty$.

Remark 1.1. The decay rate in Theorem 1.1 is the same as that of the solutions to the linear Schrödinger equation.

Theorem 1.2. *Let u be the solution of (1.1) obtained in Theorem 1.1. Then for any u_0 satisfying the condition of Theorem 1.1, there exists a unique final state $\hat{u}_+\in H^{l-\gamma,0}$, $0<\gamma\le1$ such that the following asymptotics*

$$u(t,x)=\frac{1}{(it)^{\frac{n}{2}}}\hat{u}_+(\frac{x}{t})\exp\left(\frac{ix^2}{2t}-\frac{it^{1-\delta}}{1-\delta}f(|\hat{u}_+|^2)(\frac{x}{t})+O(1+t^{1-2\delta})\right)+O(t^{-\frac{n}{2}-\delta})$$

is valid as $t\to\infty$ uniformly with respect to $x\in\mathbf{R}^n$.

For the values $\delta\in(\frac{1}{2},1)$ we obtain the existence of the modified scattering states.

Theorem 1.3. *Let u be the solution of (1.1) obtained in Theorem 1.1 and $\frac{1}{2} < \delta < 1$. Then there exists a unique final state $\hat{u}_+ \in H^{l-\gamma,0}$, $0 < \gamma \leq 1$ such that the following asymptotics for $t \to \infty$ is valid uniformly with respect to $x \in \mathbf{R}^n$*

$$u(t,x) = \frac{1}{(it)^{\frac{n}{2}}}\hat{u}_+(\frac{x}{t}) \exp\left(\frac{ix^2}{2t} - \frac{it^{1-\delta}}{1-\delta} f(|\hat{u}_+|^2)(\frac{x}{t})\right) + O(t^{-\frac{n}{2}+1-2\delta})$$

and the estimate

$$\|u(t) - \exp(-\frac{it^{1-\delta}}{1-\delta} f(|\hat{u}_+|^2)(\frac{x}{t}))U(t)u_+\| \leq Ct^{1-2\delta}$$

is true for all $t \geq 1$.

In Section 2 we prepare some preliminary estimates. Lemma 2.1 is the usual Sobolev inequality. We show the local in time existence of solutions to (1.1) in Theorem 2.2. Lemma 2.3 is necessary to treat the nonlinear term. Section 3 is devoted to the proof of Theorems 1.1-1.3. First we prove Theorem 3.1 and Theorem 3.2 where we estimate the solutions of auxiliary system (3.1). And then we prove Theorems 1.1-1.3.

§2. Preliminaries

We first state the well-known Sobolev embedding inequality (for the proof, see, e. g., [1]).

Lemma 2.1. *Let q, r be any numbers satisfying $1 \leq q, r \leq \infty$, and let j, m be any real numbers satisfying $0 \leq j < m$. Then the following inequality is valid*

$$\|(-\Delta)^{j/2}u\|_p \leq C\|(-\Delta)^{m/2}u\|_r^a \|u\|_q^{1-a}$$

if the right-hand side is bounded, where C is a constant depending only on m, n, j, q, r, a, here $\frac{1}{p} = \frac{j}{n} + a(\frac{1}{r} - \frac{m}{n}) + (1-a)\frac{1}{q}$ and a is any real number from the interval $\frac{j}{m} \leq a \leq 1$, with the following exception: if $m - j - \frac{n}{r}$ is nonnegative and integer, then $a = \frac{j}{m}$.

Theorem 2.2. *Suppose that the initial data u_0 satisfy the condition of Theorem 1.1. Then there exists a time $T > 1$ and a unique solution u of the Cauchy problem (1.1) such that $U(-t)u(t) \in C([0,T]; H^{0,l})$ and $\|U(-t)u(t)\|_{0,l} \leq 2\epsilon$ for $t \in [0,T]$.*

Proof. We introduce the function space

$$X_T = \{\varphi \in C([0,T]; L^2); \|\varphi\|_{X_T} \equiv \sup_{0 \leq t \leq T} \|U(-t)\varphi(t)\|_{0,l} < \infty\}.$$

We denote by $X_{T,\rho}$ the closed ball of X_T with a center at the origin and a radius ρ. We consider the linearized version of the equation (1.1):

$$\begin{cases} i\partial_t u = -\frac{1}{2}\Delta u + f(|v|^2)v, & (t,x) \in \mathbf{R} \times \mathbf{R}^n \\ u(0,x) = u_0(x), & x \in \mathbf{R}^n, \end{cases}$$

where $v \in X_{T,\rho}$. This Cauchy problem defines the mapping $\mathcal{A} : u = \mathcal{A}v$ acting in X_T. Using Lemma 2.1, Hölder's inequality and the fact that the operator $J = U(t)xU(-t)$ commutes with the linear Schrödinger operator $i\partial_t + \frac{1}{2}\Delta$ we obtain

$$\begin{aligned}
&\frac{d}{dt}\|J^l u(t)\|^2 \le 2|\mathrm{Im}(\overline{J^l u}, J^l f(|v|^2)v)| \\
&\le C \sum_{1\le k\le l} \left|\mathrm{Im}\left(\overline{J^l u}, \left((it\nabla)^k f\left(|\bar{M}v|^2\right)\right) J^{l-k} v\right)\right| \\
&\le C \sum_{1\le k\le l} \|J^l u\| \|(it\nabla)^k f\left(|\bar{M}v|^2\right)\|_{n/\delta} \|J^{l-k}v\|_{2n/(n-2\delta)} \\
&\le C \sum_{1\le k\le l} \|J^l u\| \|J^k v\|^2 \|\nabla M(-t) J^{l-k} v\|^\delta \|M(-t)J^{l-k}v\|^{1-\delta} \\
&\le C t^{-\delta} \|J^l u\| (1 + \|J^l v\|)^3.
\end{aligned}$$

Whence we can easily see that the mapping $\mathcal{A}$ is a contraction mapping from $X_{T,\rho}$ into itself if we take ρ sufficiently small. This implies Theorem 2.2. □

The following lemma is used for obtaining estimates of the nonlinear term.

Lemma 2.3. *We have the following estimates*

$$\|\phi\psi\|_{l,0} \le C\|\phi\|_{l,0}(\|\psi\|_\infty + \|\psi\|_{\dot{H}^{l,0}}),$$

$$\sum_{j=1}^n |Re(\partial_j^l \phi, \partial_j^l(\nabla\psi \cdot \nabla\phi))| \le C\|\phi\|_{l,0}^2(\|\psi\|_\infty + \|\psi\|_{\dot{H}^{k,0}})$$

and

$$|(\partial_j^k \psi, \partial_j^k (\nabla\psi)^2)| \le C(\|\psi\|_\infty + \|\psi\|_{\dot{H}^{k,0}})^2 \|\partial_j^k \psi\|,$$

if the right-hand sides are bounded, where ψ is a real valued function, ϕ is a complex valued function, $l \ge [\frac{n}{2}] + 3, k = l+2, n \ge 2$.

For the proof, see [5, Lemma 2.2]. □

§3. Proof of Theorems

In the same way as in [5,(3.2)] we have

$$\begin{cases} w_t = \frac{1}{t^2}\nabla w \nabla g + \frac{i}{2t^2}\Delta w + \frac{1+i}{2t^2} w\Delta g, \\ g_t = t^{-\delta} f(|w|^2) + \frac{1}{2t^2}(\nabla g)^2 + \frac{1}{2t^2}\Delta g, \\ g(1) = 0, \quad w(1) = v(1) = \mathcal{F}M(-1)U(-1)u(1) \end{cases} \tag{3.1}$$

by putting $w = e^{ig}\mathcal{F}MU(-t)u(t)$. In order to obtain the desired result we prove the global existence of solutions to (3.1) under the condition that $\|v(1)\|_{l,0}$ is sufficiently small. The later is true by virtue of Theorem 2.2 since $\|v(1)\|_{l,0} = \|U(-1)u(1)\|_{0,l}$.

We first prove the local existence theorem for the system of equations (3.1).

Theorem 3.1. *Suppose that the initial data $v(1)$ satisfies $\|v(1)\|_{l,0} \le \rho$, where ρ is sufficiently small. Then there exists a time $T > 2$ and a unique solution to the Cauchy problem for the system of equations (3.1) such that $w \in C([1,T]; H^{l,0})$, $g \in C([1,T], \dot{H}^{k,0} \cap L^\infty)$, and the following estimates are valid*

$$\|w\|_{l,0} + t^{\delta-1}(\|g\|_\infty + \|g\|_{\dot{H}^{l,0}}) + t^{\frac{\delta}{2}-1}\|g\|_{\dot{H}^{k,0}} \le 2\rho,$$

for any $t \in [1,T]$, where $l \ge [\frac{n}{2}] + 3$, $k = l+2$.

Proof. We consider the linearized version of the system of equations (3.1):

$$\begin{cases} w_t = \frac{1}{t^2}\nabla w \nabla \tilde{g} + \frac{i}{2t^2}\Delta w + \frac{1+i}{2t^2} w\Delta \tilde{g}, \\ g_t = t^{-\delta} f(|\tilde{w}|^2) + \frac{1}{2t^2}(\nabla \tilde{g})^2 + \frac{1}{2t^2}\Delta g, \\ g(1) = 0, w(1) = v(1) = \mathcal{F}M(-1)U(-1)u(1). \end{cases} \tag{3.2}$$

The Cauchy problem (3.2) defines a mapping $\mathcal{A}$

$$\begin{pmatrix} w \\ g \end{pmatrix} = \mathcal{A}\begin{pmatrix} \tilde{w} \\ \tilde{g} \end{pmatrix}.$$

We introduce the function space

$$\mathcal{X}_T = \left\{ \begin{pmatrix} w \\ g \end{pmatrix}; w \in C([1,T]; H^{l,0}), g \in C([1,T); L^\infty \cap \dot{H}^{l+2,0}); \quad \left\| \begin{pmatrix} w \\ g \end{pmatrix} \right\|_{\mathcal{X}_T} < \infty \right\},$$

where

$$\left\| \begin{pmatrix} w \\ g \end{pmatrix} \right\|_{\mathcal{X}_T} = \sup_{1\le t\le T}\left(\|w(t)\|_{l,0} + t^{\delta-1}(\|g(t)\|_\infty + \|g(t)\|_{\dot{H}^{l,0}}) + t^{\delta/2-1}\|g(t)\|_{\dot{H}^{l+2,0}} \right).$$

We denote by $\mathcal{X}_{T,2\rho}$ the closed ball in $\mathcal{X}_T$ with a center at the origin and a radius 2ρ. We now let

$$\begin{pmatrix} \tilde{w} \\ \tilde{g} \end{pmatrix} \in \mathcal{X}_{T,2\rho}.$$

For the first equation in the system (3.2) the estimates in $H^{l,0}$ are easily obtained by the usual energy method. The second equation of the system (3.2) is parabolic and therefore possesses a regularizing effect so we do not encounter a derivative loss. Then the standard contraction mapping yields the result. $\square$

We next prove the following theorem.

Theorem 3.2. *Suppose that the initial data $v(1)$ are such that the value $\|v(1)\|_{l,0} \leq 2\epsilon$, where ϵ is sufficiently small. Then there exists a unique solution to the Cauchy problem for the system of equations (3.1) such that $w \in C([1,\infty); H^{l,0})$, $g \in C([1,\infty), \dot{H}^{k,0} \cap L^\infty)$, and the following estimates are valid*

$$\|w\|_{l,0} + t^{\delta-1}(\|g\|_\infty + \|g\|_{\dot{H}^{l,0}}) + t^{\frac{\delta}{2}-1}\|g\|_{\dot{H}^{k,0}} \leq 3\epsilon,$$

where $l = [\frac{n}{2}] + 3$, $k = l + 2$.

Proof. We estimate the following norms $J(t) = \|w(t)\|_{l,0}$ and $I(t) = t^{\frac{\delta}{2}-1}(\|g\|_\infty + \sum_{|\alpha|=k} \|\partial^\alpha g\|)$ of the functions w and g on the time interval $[1,T]$. Differentiating (3.1) with respect to x_j and using the usual energy method we get

$$\frac{d}{dt}\|\partial_j^l w\|^2 = \mathrm{Re}\frac{2}{t^2}(\partial_j^l w, \partial_j^l(\nabla g \cdot \nabla w)) + \mathrm{Re}\frac{1+i}{t^2}(\partial_j^l w, \partial_j^l(w\Delta g)),$$

whence by the first two estimates of Lemma 2.3 and Theorem 3.1 we obtain

$$\frac{d}{dt}J(t) \leq Ct^{-1-\delta/2}I(t)J(t) \leq C\rho^2 t^{-1-\delta/2}$$

and integration with respect to t gives

$$J(t) \leq 2\epsilon + C\rho^2. \tag{3.3}$$

Analogously by virtue of the third estimate of Lemma 2.3 we find

$$\begin{aligned}\frac{d}{dt}\|\partial_j^k g\|^2 &\leq 2t^{-\delta}|(\partial_j^k g, \partial_j^k f(|w|^2))| + \frac{1}{t^2}|(\partial_j^k g, \partial_j^k(\nabla g)^2)| - \frac{1}{t^2}\|\nabla\partial_j^k g\|^2 \\ &\leq Ct^{-\delta}\|(-\Delta)^{\delta/2}r_j\|\|\partial_j^k(-\Delta)^{-\delta/2}f(|w|^2)\| + Ct^{-\delta}\|r_j\|I^2 - \frac{1}{t^2}\|\nabla r_j\|^2,\end{aligned}$$

where $r_j = \partial_j^k g$ and $k = l + 2$. From Lemma 2.1 we have the estimate $\|(-\Delta)^{\delta/2}r_j\| \leq C\|r_j\|^{1-\delta}\|\nabla r_j\|^\delta$ since $\delta \in (0,1)$. Then using the Young's inequality $ab \le \frac{a^p}{p} + \frac{b^q}{q}$, where we take $a = C\|r_j\|^{1-\delta}\|\partial_j^k(-\Delta)^{-\delta/2}f\|$ and $b = t^{-\delta}\|\nabla r_j\|^{\delta}$, $p = \frac{2}{2-\delta}$, $q = \frac{2}{\delta}$, so that $\frac{1}{p} + \frac{1}{q} = 1$, we get

$$\begin{aligned}\frac{d}{dt}\|r_j\|^2 &\le C(\|r_j\|^{1-\delta}\|\partial_j^k(-\Delta)^{-\delta/2}f\|)^{\frac{2}{2-\delta}} + Ct^{-\delta}\|r_j\|I^2\\ &\le CJ^{\frac{4}{2-\delta}}\|r_j\|^{\frac{2-2\delta}{2-\delta}} + Ct^{-\delta}\|r_j\|I^2 \le C\rho^2 t^{1-\delta}\end{aligned}$$

since $f(|w|^2) = (-\Delta)^{-\frac{n-\delta}{2}}|w|^2$ (see [10]) we have by Lemma 2.1

$$\begin{aligned}&\|\partial_j^k(-\Delta)^{-\delta/2}f(|w|^2)\| \le C\|\partial_j^k(-\Delta)^{-n/2}|w|^2\|\\ &\le C\|(-\Delta)^{\frac{1}{2}(l+2)-\frac{n}{2}}|w|^2\| \le C\|(-\Delta)^{\frac{1}{2}(l+2)-\frac{n}{4}}|w|^2\|_1\\ &\le C\|(1-\Delta)^{l/2}|w|^2\|_1 \le C\|w\|_{l,0}^2 \quad \text{for} \quad n \ge 4\end{aligned}$$

and for $n = 2, 3$

$$\begin{aligned}&\|\partial_j^k(-\Delta)^{-\delta/2}f(|w|^2)\| \le C\|\partial_j^k(-\Delta)^{-n/2}|w|^2\|\\ &\le C\|(-\Delta)^{\frac{1}{2}(l+2)-\frac{n}{2}}|w|^2\| \le C\||w|^2\|_{l,0} \le C\|w\|_{l,0}^2.\end{aligned}$$

Integration with respect to t yields

$$\|r_j\|^2 \le C\rho^2 t^{2-\delta}. \tag{3.4}$$

For the L^∞ norm by (3.1) and Lemma 2.1 we see that there exists a positive constant $\tilde{\epsilon} < 1/2$ such that

$$\begin{aligned}\|g\|_\infty = \|\int_1^t g_t dt\|_\infty &\le \int_1^t t^{-\delta}\|f(|w|^2)\|_\infty dt + \int_1^t (\|(\nabla g)^2\|_\infty + \|\Delta g\|_\infty)\frac{dt}{t^2}\\ &\le \int_1^t t^{-\delta}\|f(|w|^2)\|_\infty dt + \int_1^t (\|(\nabla g)^2\|_\infty + \tilde{\epsilon}\|g\|_\infty + C\|g\|_{\dot{H}^{k,0}})\frac{dt}{t^2}\\ &\le C\rho^2 t^{1-\delta} + \tilde{\epsilon} t^{1-\delta}\sup_{1\le t\le T}\|g\|_\infty\end{aligned}$$

since $\|\Delta g\|_\infty \le C\|g\|_\infty^{1-a}\|g\|_{\dot{H}^{k,0}}^a \le \tilde{\epsilon}\|g\|_\infty + C\|g\|_{\dot{H}^{k,0}}$, where $a = 4/(2k-n)$.

Therefore we have

$$\|g\|_\infty \le C\rho^2 t^{1-\delta}. \tag{3.5}$$

In the same way we estimate the norm in $\dot{H}^{l,0}$ to get

$$\begin{aligned}\|g\|_{\dot{H}^{l,0}} = \|\int_1^t g_t dt\|_{\dot{H}^{l,0}} &\le \int_1^t t^{-\delta}\|f(|w|^2)\|_{\dot{H}^{l,0}} dt + \int_1^t (\|(\nabla g)^2\|_{\dot{H}^{l,0}}\\ &+ \|\Delta g\|_{\dot{H}^{l,0}})\frac{dt}{t^2} < C\rho^2 t^{1-\delta}.\end{aligned} \tag{3.6}$$

From (3.3)-(3.6) we see that

$$\left\| \begin{pmatrix} w \\ g \end{pmatrix} \right\|_{X_T} \leq 2\epsilon + C\rho^2 \leq 3\epsilon,$$

if we take ρ satisfying $C\rho^2 \leq \epsilon$. Thus Theorem 3.1 and the standard continuation argument yield the result. $\square$

We are now in a position to prove Theorems 1.1 - 1.3.

Proof of Theorem 1.1. From the identity

$$\mathcal{F}MU(-t)u(t) = w(t)\exp(-ig(t))$$

we have

$$\|U(-t)u(t)\|_{0,l} = \|\mathcal{F}MU(-t)u(t)\|_{l,0} = \|w(t)\exp(-ig(t))\|_{l,0}.$$

Whence applying Lemma 2.1 we obtain

$$\begin{aligned} \|w(t)\exp(-ig(t))\|_{l,0} &\leq C\|w\|_{l,0}(1 + \|g\|_\infty + \|g\|_{\dot{H}^{k,0}})^l \\ &\leq C\epsilon(1+t)^{(1-\delta)l}. \end{aligned}$$

Hence by Theorem 2.2 and Theorem 3.2 we see that there exists a unique solution u of (1.1) such that $U(-t)u(t) \in C([0,\infty); H^{0,l})$ and $\|U(-t)u(t)\|_{0,l} \leq C\epsilon(1+t)^{(1-\delta)l}$. By virtue of the identity

$$u(t) = M(t)D(t)w(t)\exp(-ig) = \frac{1}{(it)^{n/2}}M(t)w(t, \frac{x}{t})\exp(-ig(t, \frac{x}{t}))$$

we easily get

$$\begin{aligned} \|u(t)\|_p &\leq Ct^{-n/2}\|w(t, \frac{\cdot}{t})\|_p \leq Ct^{-n/2}(\int |w(t, \frac{x}{t})|^p dx)^{1/p} \\ &= Ct^{n/p-n/2}(\int |w(t,y)|^p dy)^{1/p} = Ct^{n/p-n/2}\|w\|_p \\ &\leq Ct^{n/p-n/2}\|w\|_{n/2-n/p,0} \leq C\epsilon t^{n/p-n/2} \end{aligned}$$

for all $p \geq 2$. This completes the proof of Theorem 1.1. $\square$

Proof of Theorem 1.2. We have via Lemma 2.3 and Theorem 3.2

$$\begin{aligned} \|w(t) - w(s)\|_{l-2,0} &\leq \int_s^t \|w_\tau(\tau)\|_{l-2,0} d\tau \leq C\int_s^t (\|\nabla g \nabla w\|_{l-2,0} \\ &+ \|\Delta w\|_{l-2,0} + \|w\Delta g\|_{l-2,0})\frac{d\tau}{\tau^2} \leq C\epsilon \int_s^t \frac{d\tau}{\tau^{1+\delta}} \leq C\epsilon s^{-\delta} \end{aligned} \tag{3.7}$$

for all $1 < s < t$. Therefore there exists a unique limit $W_+ \in H^{l-2,0}$ such that $\lim_{t\to\infty} w(t) = W_+$ in $H^{l-2,0}$ and thus we get

$$u(t,x) = \frac{1}{(it)^{\frac{n}{2}}} M(t) w(t, \frac{x}{t}) e^{-ig(t,\frac{x}{t})} = \frac{1}{(it)^{\frac{n}{2}}} M(t) W_+(\frac{x}{t}) e^{-ig(t,\frac{x}{t})} + O(\epsilon t^{-\frac{n}{2}-\delta})$$

uniformly with respect to $x \in \mathbf{R}^n$ since for all $2 \le p \le \infty$ we have the estimate

$$\begin{aligned}\|u(t) - \frac{1}{(it)^{n/2}} M(t) W_+(\frac{\cdot}{t}) e^{-ig(t,\frac{\cdot}{t})}\|_p &\le Ct^{-n/2} \|w(t,\frac{\cdot}{t}) - W_+(\frac{\cdot}{t})\|_p \\ &\le Ct^{n/p-n/2}\|w(t) - W_+\|_p \le Ct^{n/p-n/2}\|w(t) - W_+\|_{n/2-n/p,0} \\ &\le C\epsilon t^{n/p-n/2-\delta}.\end{aligned}$$

By Lemma 2.1, Theorem 3.2 and (3.7) we have for $0 < \gamma \le 2$

$$\|w(t) - w(s)\|_{l-\gamma,0} \le C\|w(t) - w(s)\|_{l,0}^{1-\gamma/2} \|w(t) - w(s)\|_{l-2,0}^{\gamma/2} \le C\epsilon s^{-\delta\gamma/2}.$$

Therefore $W_+ \in H^{l-\gamma,0}$. For the phase g we write the identity

$$g(t) = \int_1^t f(|w|^2)\frac{d\tau}{\tau^\delta} + \int_1^t ((\nabla g)^2 + \Delta g)\frac{d\tau}{2\tau^2} = f(|W_+|^2)\frac{t^{1-\delta}}{1-\delta} + \Phi(t),$$

where

$$\begin{aligned}\Phi(t) = &-\frac{1}{1-\delta} f(|W_+|^2) + \Psi(t) + (f(|w(t)|^2) - f(|W_+|^2))\frac{(t^{1-\delta}-1)}{1-\delta} \\ &+ \int_1^t ((\nabla g)^2 + \Delta g)\frac{d\tau}{2\tau^2},\end{aligned}$$

$$\Psi(t) = \int_1^t (f(|w(\tau)|^2) - f(|w(t)|^2))\frac{d\tau}{\tau^\delta}.$$

By Lemma 2.1

$$\begin{aligned}\|f(|w(t)|^2) - f(|w(\tau)|^2)\|_\infty &\le C\|\nabla(|w(t)|^2 - |w(\tau)|^2)\|^a \||w(t)|^2 - |w(\tau)|^2\|_1^{1-a} \\ &\le C\epsilon\|w(t) - w(\tau)\|_{1,0} \le C\epsilon^2\tau^{-\delta}.\end{aligned}$$

Hence we get $g = \frac{t^{1-\delta}}{1-\delta} f(|W_+|^2) + O(1 + t^{1-2\delta})$ uniformly in $x \in \mathbf{R}^n$. From these estimates the result of Theorem 1.2 follows with $\hat{u}_+ = W_+$. □

Proof of Theorem 1.3. We have

$$\begin{aligned}\Phi(t) - \Phi(s) = &\int_s^t (f(|w(\tau)|^2) - f(|w(t)|^2))\frac{d\tau}{\tau^\delta} - (f(|w(t)|^2) - f(|w(s)|^2))\frac{s^{1-\delta}-1}{1-\delta} \\ &+ (f(|w(t)|^2) - f(|W_+|^2))\frac{t^{1-\delta}-1}{1-\delta} - (f(|w(s)|^2) - f(|W_+|^2))\frac{s^{1-\delta}-1}{1-\delta} \\ &+ \int_s^t ((\nabla g(\tau))^2 + \Delta g(\tau))\frac{d\tau}{2\tau^2},\end{aligned} \tag{3.8}$$

where $1 < s < t$. We apply Theorem 3.2 and (3.7) to (3.8) to get $\|\Phi(t) - \Phi(s)\|_{\dot{H}^{l,0}} + \|\Phi(t) - \Phi(s)\|_\infty \leq C\epsilon s^{1-2\delta}$ for $1 < s < t$. This implies that there exists a unique limit $\Phi_+ = \lim_{t\to\infty} \Phi(t) \in \dot{H}^{l,0} \cap L^\infty$ such that

$$\|\Phi(t) - \Phi_+\|_{\dot{H}^{l,0}} + \|\Phi(t) - \Phi_+\|_\infty \leq C\epsilon t^{1-2\delta} \tag{3.9}$$

since we now consider the case $\frac{1}{2} < \delta < 1$.

By virtue of (3.9) we have

$$\|g(t) - \frac{t^{1-\delta}}{1-\delta} f(|W_+|^2) - \Phi_+\|_\infty \leq C\epsilon t^{1-2\delta}. \tag{3.10}$$

We now put $\hat{u}_+ = W_+ \exp(-i\Phi_+)$. Then we obtain the asymptotics for $t \to \infty$ uniformly with respect to $x \in \mathbf{R}^n$

$$u(t,x) = \frac{1}{(it)^{\frac{n}{2}}} \hat{u}_+(\frac{x}{t}) \exp(\frac{ix^2}{2t} - \frac{it^{1-\delta}}{1-\delta} f(|\hat{u}_+|^2)(\frac{x}{t})) + O(t^{-n/2+1-2\delta}).$$

Via (3.10) and (3.7) we have

$$\begin{aligned}
&\|\mathcal{F}MU(-t)u(t) - \hat{u}_+ \exp(-i\frac{t^{1-\delta}}{1-\delta} f(|\hat{u}_+|^2))\| \\
&\quad = \|w(t)\exp(-ig(t)) - W_+ \exp(-i\frac{t^{1-\delta}}{1-\delta} f(|\hat{W}_+|^2) - i\Phi_+)\| \\
&\quad \leq \|w(t) - W_+\| + \|W_+\| \|g(t) - f(|W_+|^2)\frac{t^{1-\delta}}{1-\delta} - \Phi_+\|_\infty \leq C\epsilon t^{1-2\delta},
\end{aligned}$$

whence we get

$$\begin{aligned}
&\|u(t) - \exp(-i\frac{t^{1-\delta}}{1-\delta} f(|\hat{u}_+|^2)(\frac{x}{t}))U(t)u_+\| \\
&\quad = \|u(t) - M(t)D(t)\exp(-i\frac{t^{1-\delta}}{1-\delta} f(|\hat{u}_+|^2))\mathcal{F}M(t)u_+\| \\
&\quad \leq \|M(t)D(t)(\mathcal{F}M(t)U(-t)u(t) - \hat{u}_+ \exp(-i\frac{t^{1-\delta}}{1-\delta} f(|\hat{u}_+|^2)))\| \\
&\quad + \|M(t)D(t)\exp(-i\frac{t^{1-\delta}}{1-\delta} f(|\hat{u}_+|^2))\mathcal{F}(M(t)-1)u_+\| \\
&\quad \leq Ct^{1-2\delta} + \|\mathcal{F}(M(t)-1)u_+\| \leq Ct^{1-2\delta} + Ct^{-1}\|x^2 u_+\| \leq Ct^{1-2\delta}
\end{aligned}$$

since $\|x^2 u_+\| = \|\Delta \hat{u}_+\| = \|\Delta(W_+ e^{i\Phi_+})\| \leq C\epsilon$. This completes the proof of Theorem 1.3. $\square$

References

1. A.Friedman, *Partial Differential Equations*, New York, Holt-Rinehart and Winston, 1969.
2. J.Ginibre and T.Ozawa, *Long range scattering for nonlinear Schrödinger and Hartree equations in space dimension* $n \geq 2$, Commun. Math. Phys. **151** (1993), 619-645.
3. J.Ginibre and G.Velo, *On a class of nonlinear Schrödinger equations with non-local interactions*, Math. Z. **170** (1980), 109-136.
4. N.Hayashi and P.I.Naumkin, *Asymptotic behavior in time of solutions to nonlinear Schrödinger and the Hartree equations*, Amer. J. Math. **120** (1998), 369-389.
5. N.Hayashi and P.I.Naumkin, *Scattering theory and asymptotics for large time of solutions to the Hartree type equations with long range potential*, submitted for pubication in J. Funct.Anal. (1997).
6. N. Hayashi, P. I. Naumkin and T. Ozawa, *Scattering theory for the Hartree equation*, SIAM, J. Math. Anal.(in press).
7. N.Hayashi and T.Ozawa, *Scattering theory in the weighted* $L^2(\mathbf{R}^n)$ *spaces for some Schrödinger equations*, Ann. IHP (Phys. Théor.) **48** (1988), 17-37.
8. N.Hayashi and Y.Tsutsumi, *Scattering theory for Hartree type equations*, Ann. IHP (Phys. Théor) **46** (1987), 187-213.
9. T.Ozawa, *Long range scattering for nonlinear Schrödinger equations in one space dimension*, Commun. Math. Phys. **139** (1991), 479-493.
10. E.M.Stein, *Singular Integral and Differentiability Properties of Functions*, vol. 30, Princeton Univ. Press, Princeton Math. Series, 1970.

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