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Splendid Morita equivalences for the principal 2-blocks of 2-dimensional general linear groups in non-defining characteristic

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Abstract. We show that the principal 2-blocks of infinite series of 2-dimensional general linear groups $GL_2(q)$ with wreathed Sylow 2-subgroups are splendidly Morita equivalent. Consequently, Puig's conjecture holds in this case. To construct the splendid Morita equivalences, we use relative stable equivalences of Morita type introduced by Wang and Zhang.

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§1. Introduction

Let p be a prime, and $(K, \mathcal{O}, k)$ a p-modular system, that is, $\mathcal{O}$ is a complete discrete valuation ring with quotient field K of characteristic 0, and with residue field k of characteristic p. We assume here that k is algebraically closed. Let G and G' be finite groups, and let B and B' be blocks of $\mathcal{O}G$ and $\mathcal{O}G'$, respectively, with a common defect group P. We write $\Delta P = \{(u, u) \mid u \in P\}$ for the diagonal subgroup of $G \times G'$. A Morita equivalence between B and B' is said to be *splendid* if it is induced by a B-B'-bimodule M that is a ΔP-projective p-permutation module as an $\mathcal{O}[G \times G']$-module.

Puig's conjecture states that, for a given finite p-group P, there are only finitely many isomorphism classes of interior P-algebras arising as source algebras of p-blocks of finite groups with defect groups isomorphic to P (see [21, Conjecture 38.5]). This is equivalent to saying that there are only finitely many splendid Morita equivalence classes of p-blocks of finite groups with defect groups isomorphic to P.

Splendid Morita equivalence classes of principal 2-blocks of tame representation type have been classified (see [11], [12], and [13]). In the classifications, it was shown that there are only finitely many splendid Morita equivalence classes of the principal blocks arising from infinite series of finite groups. Thus the classifications imply that Puig's conjecture holds for the principal 2-blocks of tame representation type.

In this paper, we consider splendid Morita equivalences of the principal 2-blocks of wild representation type for 2-dimensional general linear groups $GL_2(q)$. If q_1 and q_2 are odd prime powers with $q_1 \equiv q_2 \equiv 1 \pmod 4$ and $(q_1 - 1)_2 = (q_2 - 1)_2$, where $(q_i - 1)_2$ means the 2-part of $q_i - 1$, $i = 1, 2$, then $GL_2(q_1)$ and $GL_2(q_2)$ have a common Sylow 2-subgroup isomorphic to the wreathed 2-group. Our main result is as follows.

Theorem 1.1. *Let k be an algebraically closed field of characteristic 2. Let $G_i = GL_2(q_i)$, $i = 1, 2$, where $q_1 \equiv q_2 \equiv 1 \pmod 4$, and $(q_1 - 1)_2 = (q_2 - 1)_2$. Then the principal blocks of kG_1 and kG_2 are splendidly Morita equivalent.*

We use Scott modules to construct the splendid Morita equivalences: in any decomposition into the direct sum of indecomposable modules, $k_H{\uparrow}^G$ has a unique indecomposable summand having k_G in its top. This indecomposable summand is called the *Scott module* with respect to H, and denoted by $S(G, H)$ (see [18, Chapter 4, Section 8]).

The splendid Morita equivalence in Theorem 1.1 is constructed by the Scott module $S(G_1 \times G_2, \Delta P)$, where P is a common Sylow 2-subgroup of G_1 and G_2. Since this module is a 2-permutation module and liftable to $\mathcal{O}$, we obtain the following.

Corollary 1.2. *Let $(K, \mathcal{O}, k)$ be a 2-modular system such that k is algebraically closed. Let $G_i = GL_2(q_i)$, $i = 1, 2$, where $q_1 \equiv q_2 \equiv 1 \pmod 4$, and $(q_1 - 1)_2 = (q_2 - 1)_2$. Then the principal blocks of $\mathcal{O}G_1$ and $\mathcal{O}G_2$ are splendidly Morita equivalent.*

Morita equivalences for the principal blocks of finite groups having a common Sylow p-subgroup P have been constructed by lifting stable equivalences of Morita type using Linckelmann's result [17] (see for example [11] and [13]). The stable equivalences of Morita type have been constructed using Broué's result [3], which constructs them by gluing Morita equivalences between the principal blocks of centralizers of the nontrivial subgroups of P. However Broué's method does not make sense in Theorem 1.1 since G_1 and G_2 have a common nontrivial central 2-subgroup, and its centralizers are G_1 and G_2 themselves.

Therefore we use the notion of relative stable equivalence of Morita type introduced by Wang and Zhang [22], which is a generalization of stable equivalence of Morita type. In [16], the current authors generalized the results due to Broué and Linckelmann to relative stable equivalences of Morita type. In the paper, we use the results to construct splendid Morita equivalences.

This paper is organized as follows. In Section 2, we establish some notation and facts used throughout the paper. In Section 3, we recall the definitions of relative projectivity and relative stable equivalences of Morita type, and recall some results that we use to prove Theorem 1.1. In Section 4, we recall the definition of relative projective covers, and collect some properties of homomorphisms factoring through relative projective modules. In Section 5, we collect some properties on Scott modules. In Section 6, we describe subgroups of $GL_2(q)$ with $q \equiv 1 \pmod 4$. In Section 7, we prove Theorem 1.1.

§2. Preliminaries

Throughout this paper, we assume that $(K, \mathcal{O}, k)$ is a p-modular system with k algebraically closed, G is a finite group, and modules are finitely generated right modules, unless otherwise stated.

We write $H \leq G$ if H is a subgroup of G. We write $\Delta G = \{(g, g) \mid g \in G\}$ for the diagonal subgroup of $G \times G$. We write $Z(G)$ for the center of G. For subgroups H and K of G, we write $[H\backslash G]$ for a set of representatives of the right cosets of H in G and write $[H\backslash G/K]$ for a set of representatives of the double cosets of H and K in G, and we set $H^g = g^{-1}Hg$ for $g \in G$.

We write k_G for the *trivial* kG*-module* and $B_0(kG)$ for the principal block of kG. For a kG-module M, we write $M{\downarrow}^G_H$ (or simply $M{\downarrow}_H$) for the *restriction* of M to H. For a kH-module N, we write $N{\uparrow}^G_H$ (or simply $N{\uparrow}^G$) for the *induced* kG*-module* of N. For modules V, we write $V^* = \mathrm{Hom}_k(V, k)$ for the k-dual of V. For modules U and V, we write $U \otimes V$ for $U \otimes_k V$. If U is a left module, and V is a right module, then we consider $U \otimes V$ as a bimodule, and V^* as a left module, unless otherwise stated.

We recall a fact on Scott modules. If H and H' are subgroups of G, and Q and Q' are Sylow p-subgroups of H and H', respectively, then $S(G, H)$ and $S(G, H')$ are isomorphic if and only if Q and Q' are conjugate in G (see [18, Chapter 4, Corollary 8.5]). In particular, it follows that $S(G, H) \cong S(G, Q)$, and $S(G, Q)$ has Q as a vertex. For further facts on Scott modules, we refer the reader to [18, Chapter 4, Section 8].

A kG-module is called a *p-permutation module* if it is a direct summand of $\bigoplus_{i=1}^{r} k_{H_i}{\uparrow}^G$ for some subgroups H_i of G. The Scott module $S(G, H)$ is an indecomposable p-permutation module.

For a kG-module M and a p-subgroup Q of G, the *Brauer construction* $M(Q)$ of M with respect to Q is the $kN_G(Q)$-module defined as follows:

$$M(Q) = M^Q / \sum_R \mathrm{tr}_R^Q(M^R),$$

where R runs over the proper subgroups of Q, M^Q is the set of fixed points of Q in M, and $\mathrm{tr}_R^Q : M^R \to M^Q$ is a linear map given by $\mathrm{tr}_R^Q(m) = \sum_{t\in[R\backslash Q]} mt$.

The *fusion system* of G over P is the category $\mathcal{F}_P(G)$ whose objects are the subgroups of P and whose morphisms are given by

$$\mathrm{Hom}_{\mathcal{F}_P(G)}(Q,R) = \{\varphi \in \mathrm{Hom}(Q,R) \mid \varphi = c_g \text{ for some } g \in G \text{ with } Q^g \leq R\},$$

where c_g is a conjugation map. For further notation and terminology on fusion system, we refer the reader to [1].

Remark 2.1. In this paper, it suffices to know the following facts for fusion systems.

(i) By the definition, we can take fully normalized subgroups as representatives of $\mathcal{F}_P(G)$-conjugacy classes of subgroups of P.

(ii) If $\mathcal{F}_P(G)$ is saturated, then any fully normalized subgroup is fully centralized.

(iii) If P is a Sylow p-subgroup of G, then $\mathcal{F}_P(G)$ is saturated.

(iv) If P is a Sylow p-subgroup of G, then $C_P(Q)$ is a Sylow p-subgroup of $C_G(Q)$ for any fully centralized subgroup Q of P.

§3. Relative stable equivalences of Morita type

In this section, we recall the definitions of relative projectivity and relative stable equivalences of Morita type, and also recall results from [16], which we use to prove Theorem 1.1.

Let W be a kG-module. In [19], Okuyama introduced the notion of projectivity relative to a kG-module (see also [4, Section 8]). We say that a kG-module U is *relatively W-projective* if U is a direct summand of $W \otimes V$ for some kG-module V, where $W \otimes V$ is considered as a kG-module via the diagonal action. We say that a short exact sequence of kG-modules

$$E:\ 0 \longrightarrow U_1 \xrightarrow{\ f\ } U_2 \xrightarrow{\ g\ } U_3 \longrightarrow 0$$

is *W-split* if $E \otimes W$ is split. Then f is called a *W-split monomorphism*, and g is called a *W-split epimorphism.* These properties are described in Section 4.

We define the *W-stable category* $\underline{\mathrm{mod}}^W(kG)$ of $\mathrm{mod}(kG)$ whose objects are the same as those of $\mathrm{mod}(kG)$, and whose morphisms are given by

$$\underline{\mathrm{Hom}}_{kG}^W(U,V) = \mathrm{Hom}_{kG}(U,V)/\mathrm{Hom}_{kG}^W(U,V),$$

where $\mathrm{Hom}^W_{kG}(U,V)$ is the subspace of $\mathrm{Hom}_{kG}(U,V)$ consisting of all homomorphisms factoring through a W-projective kG-module. For a block B of kG, we write $\underline{\mathrm{mod}}^W(B)$ for the full subcategory of $\underline{\mathrm{mod}}^W(kG)$ whose objects are all finitely generated B-modules. In [5], it was shown that $\underline{\mathrm{mod}}^W(kG)$ is a triangulated category. In [22, Proposition 3.1], it was also shown that, for a block B of kG, the subcategory $\underline{\mathrm{mod}}^W(B)$ is triangulated.

Wang and Zhang [22] introduced the notion of relative stable equivalence of Morita type:

Definition 3.1. (see [22, Definition 5.1]) Let G and G' be finite groups and B and B' blocks of kG and kG', respectively. Let W be a kG-module and W' a kG'-module. For a B-B'-bimodule M, and a B'-B-bimodule N, we say that the pair (M,N) induces a relative (W,W')-stable equivalence of Morita type between B and B' if M and N are finitely generated and projective as left modules and right modules with the property that there are isomorphisms of bimodules

$$M \otimes_{B'} N \cong B \oplus X \quad \text{and} \quad N \otimes_B M \cong B' \oplus Y,$$

where X is $W^* \otimes W$-projective as a $k[G\times G]$-module and Y is $W'^* \otimes W'$-projective as a $k[G'\times G']$-module.

In this paper, we mainly consider subgroup versions of the notions above. Let H be a subgroup of G. Then it follows from Frobenius reciprocity that a kG-module U is H-projective if and only if U is $k_H{\uparrow}^G$-projective. Therefore, using $W = k_H{\uparrow}^G$, projectivity relative to modules is a generalization of projectivity relative to subgroups. We say that a short exact sequence of kG-modules is *H-split* if its restriction to H is split. It follows that a short exact sequence of kG-modules is $k_H{\uparrow}^G$-split if and only if it is H-split.

We write

$$\underline{\mathrm{mod}}^H(kG) = \underline{\mathrm{mod}}^{k_H\uparrow^G}(kG), \quad \text{and} \quad \underline{\mathrm{mod}}^H(B) = \underline{\mathrm{mod}}^{k_H\uparrow^G}(B),$$

where B is a block of kG. In Definition 3.1, assume further that B and B' have a common defect group P. Then for a subgroup Q of P, we say that (M,N) induces a *relative Q-stable equivalence of Morita type* between B and B' if (M,N) induces a relative (W,W')-stable equivalence of Morita type for $W = k_Q{\uparrow}^G$ and $W' = k_Q{\uparrow}^{G'}$. Note that, with this definition, X and Y in Definition 3.1 are $Q\times Q$-projective since it follows that

$$(k_Q{\uparrow}^G)^* \otimes k_Q{\uparrow}^G \cong kG \otimes_{kQ} k_Q \otimes k_Q \otimes_{kQ} kG \cong k_{Q\times Q}{\uparrow}^{G\times G}.$$

Note that a relative (W,W')-stable equivalence of Morita type does not, in general, induce an equivalence between $\underline{\mathrm{mod}}^W(B)$ and $\underline{\mathrm{mod}}^W(B')$ (see [16, Section 4]). However, a relative Q-stable equivalence of Morita type between

B and B' induces an equivalence between $\underline{\mathrm{mod}}^Q(B)$ and $\underline{\mathrm{mod}}^Q(B')$ as triangulated categories under certain conditions (see [16, Corollary 4.6]).

In Section 7, we use following results to prove Theorem 1.1.

Theorem 3.2. (see [16, Theorem 1.1]) *Let G and G' be finite groups with a common Sylow p-subgroup P such that $\mathcal{F}_P(G) = \mathcal{F}_P(G')$. Assume that Z is a subgroup of P and central in G and G'. Let $M = S(G \times G', \Delta P)$. Then the following are equivalent.*

(i) *The pair $(M(\Delta Q), M(\Delta Q)^*)$ induces a Morita equivalence between the principal blocks of $kC_G(Q)$ and $kC_{G'}(Q)$ for any subgroup Q of P properly containing Z.*

(ii) *The pair (M, M^*) induces a relative Z-stable equivalence of Morita type between the principal blocks of kG and kG'.*

Remark 3.3. By the proof of [16, Theorem 1.1], it suffices to check the condition (i) of Theorem 3.2 only for representatives of the conjugacy classes of subgroups of P properly containing Z.

Theorem 3.4. (see [16, Theorem 1.2]) *Let G and G' be finite groups, and B and B' blocks of kG and kG', respectively, with a common nontrivial defect group P such that $\mathcal{F}_P(G) = \mathcal{F}_P(G')$. Let M be a B-B'-bimodule that is a ΔP-projective p-permutation $k[G \times G']$-module. Assume that Z is a proper subgroup of P that is central in G and G', and (M, M^*) induces a relative Z-stable equivalence of Morita type between B and B'. Then the following hold.*

(i) *If M is an indecomposable p-permutation module with vertex ΔP, then for any simple B-module S, the B'-module $S \otimes_B M$ is indecomposable, and non Z-projective, considered as a kG'-module.*

(ii) *The pair (M, M^*) induces a Morita equivalence between B and B' if and only if for any simple B-module S, the B'-module $S \otimes_B M$ is simple.*

In Theorem 1.1, P has a nontrivial subgroup Z contained in both $Z(G_1)$ and $Z(G_2)$. Our approach to the proof of Theorem 1.1 is to construct a relative Z-stable equivalence of Morita type between $B_0(kG_1)$ and $B_0(kG_2)$ using Theorem 3.2 and lift it to a Morita equivalence using Theorem 3.4.

The following lemma is useful to construct a relative Z-stable equivalence of Morita type using Theorem 3.2.

Lemma 3.5. (see [15, Lemma 2.5]) *Let G be a finite group with a p-subgroup P such that $\mathcal{F}_P(G)$ is saturated. If Q is a fully normalized subgroup of P, then $S(C_G(Q), C_P(Q))$ is a direct summand of $S(G, P)(Q)\downarrow_{C_G(Q)}$.*

Next we recall the definition of relative Brauer indecomposability from [16]. Let M be a kG-module and R a p-subgroup of G. We say that M is relatively R-Brauer indecomposable if for any p-subgroup Q of G containing R, the Brauer construction $M(Q)$ is indecomposable (or zero) as a $kQC_G(Q)$-module.

Note that relative 1-Brauer indecomposability is just Brauer indecomposability introduced in [8]. Also, if M is Brauer indecomposable, then M is relatively R-Brauer indecomposable for any p-subgroup R of G.

In the condition (i) of Theorem 3.2, $M(\Delta Q)$ must be indecomposable as a $B_0(kC_G(Q))$-$B_0(kC_{G'}(Q))$-bimodule for any nontrivial subgroup Q of P properly containing Z. Thus the relative ΔZ-Brauer indecomposability of M is useful for constructing a relative Z-stable equivalence of Morita type using Theorem 3.2.

§4. Relative projective covers and relative injective hulls

Throughout this section, let W be a kG-module. In this section, we recall the definitions of relative W-projective covers and relative W-injective hulls. We also collect some properties of homomorphisms factoring through W-projective modules.

Proposition 4.1. (see [19, Lemma 9.5] and [5, Proposition 2.4]) *Let X be a kG-module. Then the following are equivalent.*

(i) *The module X is W-projective.*

(ii) *For any kG-homomorphisms $\beta : U \to V$ and $\alpha : X \to V$, if β is a W-split epimorphism, then there is a kG-homomorphism $\gamma : X \to U$ such that $\beta\gamma = \alpha$.*

(iii) *Any W-split epimorphism $U \to X$ is split.*

(iv) *For any kG-homomorphisms $\beta : U \to V$ and $\alpha : U \to X$, if β is a W-split monomorphism, then there is a kG-homomorphism $\gamma : V \to X$ such that $\gamma\beta = \alpha$.*

(v) *Any W-split monomorphism $X \to U$ is split.*

In [19], Okuyama introduced the notion of relative projective cover with respect to a kG-module as follows.

Definition 4.2. A short exact sequence of kG-modules

$$0 \longrightarrow V \longrightarrow X \longrightarrow U \longrightarrow 0$$

is called a *relative W-projective cover* of U if it satisfies the following conditions.

(i) X is W-projective.

(ii) the sequence is W-split.

(iii) V has no W-projective summand.

Dually, the exact sequence above is called a *relative W-injective hull* of V if it satisfies the condition (i), (ii) and the following.

(iii′) U has no W-projective summand.

Note that, for a family $\mathcal{H}$ of subgroups of G, a W-projective cover for $W = \bigoplus_{H\in\mathcal{H}} k_H{\uparrow}^G$ coincides with an $\mathcal{H}$-projective cover introduced by Knörr [9]. Therefore we refer a W-projective cover (respectively a W-injective hull) for $W = \bigoplus_{H\in\mathcal{H}} k_H{\uparrow}^G$ as a *$\mathcal{H}$-projective cover* (respectively an *$\mathcal{H}$-injective hull*).

Let A be an algebra over a field. An A-homomorphism $f : L \to M$ is said to be *left minimal* if every homomorphism $h \in \mathrm{End}_A(M)$ such that $hf = f$ is an automorphism. An A-homomorphism $g : M \to N$ is said to be *right minimal* if every homomorphism $h \in \mathrm{End}_A(M)$ such that $gh = g$ is an automorphism.

We say that a W-split monomorphism $f : L \to M$ is *W-essential* if a kG-homomorphism $h : M \to U$ is a W-split monomorphism whenever hf is a W-split monomorphism. We say that a W-split epimorphism $g : M \to N$ of kG-module is *W-essential* if a kG-homomorphism $h : U \to M$ is a W-split epimorphism whenever gh is a W-split epimorphism. Note that these definitions are slightly different from those of essential homomorphisms in [20].

Thévenaz [20] defined relative $\mathcal{H}$-projective covers for a family $\mathcal{H}$ of subgroups of G using right minimal epimorphisms and also $\mathcal{H}$-essential epimorphisms. We can also define relative W-projective covers using right minimal epimorphisms and W-essential epimorphisms. For this purpose, we use the following lemma.

Lemma 4.3. (see [20, Lemma 1.4]) *Let*

$$0 \longrightarrow V \xrightarrow{\ \alpha\ } X \xrightarrow{\ \beta\ } U \longrightarrow 0$$

be a short exact sequence of kG-modules. Let $X = \bigoplus_i X_i$ be a direct sum of indecomposable kG-modules. Then the following hold.

(i) *Let $U = \bigoplus_j U_j$ be a direct sum of indecomposable kG-modules, and (β_{ij}) the matrix form of β with $\beta_{ij} : X_j \to U_i$. Then α is left minimal if and only if none of β_{ij} is an isomorphism.*

(ii) *Let* $V = \bigoplus_j V_j$ *be a direct sum of indecomposable* kG*-modules, and* (α_{ij}) *the matrix form of* α *with* $\alpha_{ij} : V_j \to X_i$*. Then* β *is right minimal if and only if none of* α_{ij} *is an isomorphism.*

We collect equivalent properties of relative W-injective hulls and relative W-projective covers in the following two propositions.

Proposition 4.4. *Let*

$$E:\ 0 \longrightarrow V \xrightarrow{\ \iota\ } X \xrightarrow{\ \pi\ } U \longrightarrow 0$$

be a W*-split short exact sequence of* kG*-modules, where* X *is* W*-projective. Then the following are equivalent:*

(i) E *is a* W*-injective hull of* V*.*

(ii) ι *is left minimal.*

(iii) ι *is a* W*-essential monomorphism.*

Proof. (ii)⇒(iii): Let $\alpha : X \to V'$ be a kG-homomorphism such that $\alpha\iota$ is a W-split monomorphism. By Proposition 4.1 (iv), there is a kG-homomorphism $\beta : V' \to X$ such that $\beta\alpha\iota = \iota$. By (ii), $\beta\alpha$ is an isomorphism, and hence α is a split monomorphism.

(iii)⇒(ii): Let $\gamma : X \to X$ be a kG-homomorphism with $\gamma\iota = \iota$. By (iii), γ is a W-split monomorphism, which implies that γ is an isomorphism.

(i)⇒(ii): Let $X = \bigoplus_i X_i$ and $U = \bigoplus_j U_j$ be the direct sums of indecomposable kG-modules, and let $\pi = (\pi_{ij})$ be the matrix form with $\pi_{ij} : X_j \to U_i$. Suppose that ι is not left minimal. Then, by Lemma 4.3, π_{ij} is an isomorphism for some i and j. Hence U has the W-projective module X_j as a direct summand, a contradiction.

(ii)⇒(i): Suppose that U has a W-projective summand U'. We may assume that U' is indecomposable. Let $U = \bigoplus_j U'_j$ be a direct sum of indecomposable kG-modules, where $U'_1 = U'$, and let $\varepsilon_1 : U \to U'_1$ be the projection. Since $\varepsilon_1\pi$ is a W-split epimorphism, $\varepsilon_1\pi$ is split by Proposition 4.1 (iii). This means that X has a decomposition into a direct sum of indecomposable modules such that π has an isomorphism in the matrix form. This is a contradiction. □

Proposition 4.5. *Let*

$$E:\ 0 \longrightarrow V \xrightarrow{\ \iota\ } X \xrightarrow{\ \pi\ } U \longrightarrow 0$$

be a W*-split short exact sequence of* kG*-modules, where* X *is* W*-projective. Then the following are equivalent:*

(i) E *is a* W*-projective cover of* U*.*

(ii) π *is right minimal.*

(iii) π *is a* W*-essential epimorphism.*

Lemma 4.6. *Let* W *be a* kG*-module. Let*

$$0 \longrightarrow V \longrightarrow X \xrightarrow{\pi} U \longrightarrow 0$$

and

$$0 \longrightarrow V' \longrightarrow X' \xrightarrow{\pi'} U \longrightarrow 0$$

be W*-split short exact sequences of* kG*-modules, where* X *and* X' *are* W*-projective. If the first sequence is a* W*-projective cover of* U*, then there are split epimorphisms* $\alpha : X' \to X$ *and* $\beta : V' \to V$ *such that the following diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V' & \longrightarrow & X' & \xrightarrow{\pi'} & U & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} & & \| & & \\ 0 & \longrightarrow & V & \longrightarrow & X & \xrightarrow{\pi} & U & \longrightarrow & 0 \end{array}$$

is commutative.

Proof. By Proposition 4.1 (ii), there is a kG-homomorphism $\alpha : X' \to X$ such that $\pi\alpha = \pi'$. Since π is a W-essential epimorphism by Proposition 4.5, α is a W-split epimorphism. Hence α is split by Proposition 4.1 (iii). It is clear that there is a split epimorphism $\beta : V' \to V$ which makes the diagram commutative. □

Theorem 4.7. (see [19, Theorem 9.6] and [5, Proposition 2.6]) *Any* kG*-module has a* W*-projective cover and a* W*-injective hull, both of which are unique up to isomorphism.*

If the short exact sequence

$$0 \longrightarrow V \longrightarrow X \longrightarrow U \longrightarrow 0$$

of kG-modules is a W-projective cover of U, then we write $P_W(U)$ for X and $\Omega_W(U)$ for V. If it is a W-injective hull of V, then we write $I_W(V)$ for X and $\Omega_W^{-1}(V)$ for U.

Proposition 4.8. (see [4, Lemma 8.10]) *There are the following isomorphisms for* kG*-modules* U *and* V*.*

$$P_W(U \oplus V) \cong P_W(U) \oplus P_W(V) \quad \textit{and} \quad \Omega_W(U \oplus V) \cong \Omega_W(U) \oplus \Omega_W(V).$$

Next we give some properties of homomorphisms factoring through W-projective modules, which are analogous to those of homomorphisms factoring through projective modules.

Lemma 4.9. *Let $\varphi : U \to V$ be a homomorphism of kG-modules. Then the following are equivalent.*

(i) *φ factors through a W-projective kG-module.*

(ii) *φ factors through any W-split monomorphism $U \to M$.*

(iii) *φ factors through any W-split epimorphism $N \to V$.*

Proof. (i) $\Rightarrow$ (ii): Assume that there are kG-homomorphisms $\alpha : U \to X$ and $\beta : X \to V$ such that $\beta\alpha = \varphi$, where X is a W-projective kG-module. Let $\gamma : U \to M$ be a W-split monomorphism. By Proposition 4.1 (iv), there is a kG-homomorphism $\delta : M \to X$ such that $\delta\gamma = \alpha$. Hence we have $\varphi = \beta\alpha = \beta\delta\gamma$, and (ii) follows.

(ii) $\Rightarrow$ (i): By (ii), φ factors through a W-injective hull $U \to I_W(U)$, and hence (i) follows.

(i) $\Leftrightarrow$ (iii) is proved dually. □

Lemma 4.10. *Let $\varphi : U \to V$ be a nonzero kG-homomorphism. Then the following hold.*

(i) *Assume that φ is a W-split monomorphism. If φ factors through a W-projective kG-module, then V has a nonzero W-projective summand.*

(ii) *Assume that φ is a W-split epimorphism. If φ factors through a W-projective kG-module, then U has a nonzero W-projective summand.*

Proof. (i) By Lemma 4.9, φ factors through a W-injective hull $\iota : U \to I_W(U)$ of U, that is, there is a kG-homomorphism $\alpha : I_W(U) \to V$ such that $\alpha\iota = \varphi$. Since ι is an W-essential monomorphism by Proposition 4.4, α is a W-split monomorphism, and hence, Y is a direct summand of V by Proposition 4.1 (v).

(ii) This is proved dually. □

§5. Properties of Scott modules

In this section, we collect some properties of Scott modules.

Lemma 5.1. (see [20, Proposition 3.8]) *If H is a subgroup of G, then $S(G, H)$ is an H-projective cover of k_G.*

Lemma 5.2. *Let Q be a p-subgroup of G and Z a subgroup of $Q \cap Z(G)$. If $\varphi : k_G \to \Omega_Q(k_G)$ is a nonzero kG-homomorphism, then φ is a Z-split monomorphism.*

Proof. We have that

$$S(G,Q){\downarrow_Z} \mid k_Q{\uparrow^G}{\downarrow_Z} \cong \bigoplus_{t\in[Q\backslash G/Z]} k_{Q^t\cap Z}{\uparrow^Z} = k_Z^{|Q\backslash G/Z|}.$$

Hence $S(G,Q){\downarrow_Z}$ is semisimple, and so is $\Omega_Q(k_G){\downarrow_Z}$. This implies the result. □

Lemma 5.3. (see [11, Lemma 3.3] and the proof of [11, Lemma 3.4 (a)]) *Let G and G' be finite groups with a common Sylow p-subgroup P such that $\mathcal{F}_P(G) = \mathcal{F}_P(G')$. Let M be a ΔP-projective p-permutation $k[G\times G']$-module. If Q is a subgroup of P, then the following are equivalent.*

(i) *The module $S(G',Q)$ is a direct summand of $k_G \otimes_{kG} M$.*

(ii) *The module $S(G\times G',\Delta Q)$ is a direct summand of M.*

In particular, $k_{G'}$ is a direct summand of $k_G \otimes_{kG} S(G\times G',\Delta P)$.

Lemma 5.4. *Let G and G' be finite groups with a common Sylow p-subgroup P such that $\mathcal{F}_P(G) = \mathcal{F}_P(G')$. Let $M = S(G\times G',\Delta P)$. If Q is a subgroup of P, then the following hold.*

(i) *The module $S(G',Q)$ is a direct summand of $S(G,Q)\otimes_{B_0(kG)} M$.*

(ii) *The module $\Omega_Q(k_{G'})$ is a direct summand of $\Omega_Q(k_G)\otimes_{B_0(kG)} M$.*

Proof. Let

$$0 \to \Omega_Q(k_G) \to P_Q(k_G) \to k_G \to 0$$

be a relative Q-projective cover of k_G. Then the exact sequence

$$0 \to \Omega_Q(k_G)\otimes_{kG} M \to P_Q(k_G)\otimes_{kG} M \to k_G\otimes_{kG} M \to 0$$

is Q-split, and $P_Q(k_G)\otimes_{kG} M$ is Q-projective by [16, Lemma 4.5]. Hence, by Lemma 4.6, $P_Q(k_G\otimes_{kG} M)$ is a direct summand of $P_Q(k_G)\otimes_{kG} M$. By Lemma 5.3, $k_{G'}$ is a direct summand of $k_G\otimes_{kG} M$, and hence, by Proposition 4.8, $P_Q(k_{G'})$ is a direct summand of $P_Q(k_G\otimes_{kG} M)$. Thus Lemma 5.1 implies (i). This also shows (ii). □

§6. Subgroups of $GL_2(q)$

Throughout this section, let $G = GL_2(q)$, where q is an odd prime power with $q \equiv 1 \pmod 4$. In this section, we describe subgroups of G and the structure of the Scott kG-module with respect to the standard Borel subgroup of G.

Let $(q-1)_2 = 2^\ell$, $\ell \geq 2$. Then the multiplicative group $\mathbb{F}_q^\times$ has an element s of order 2^ℓ. Let

$$a = \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}, c = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, t = ab = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}.$$

Let $P = \langle a, b, c \rangle = (\langle a \rangle \times \langle b \rangle) \rtimes \langle c \rangle$. Then we have that

$$a^{2^\ell} = b^{2^\ell} = c^2 = 1, \quad c^{-1}ac = b, \quad ab = ba.$$

Hence P is the wreathed 2-group of order $2^{2\ell+1}$ and a Sylow 2-subgroup of G. Let $Z = \langle t \rangle$. Then we have $Z = Z(G) \cap P$. Let $I = \{1, 2, 2^2, \ldots, 2^{\ell-1}\}$.

Lemma 6.1. *The proper subgroups of P properly containing Z up to G-conjugate are as follows.*

(i) $\langle t, a^{2i}, ac \rangle$, $i \in I$,

(ii) $\langle t, a^{2i}, c \rangle$, $i \in I \backslash \{2^{\ell-1}\}$,

(iii) $\langle t, a^i \rangle$, $i \in I$.

Proof. Let $\bar{P} = P/Z$. The group $\bar{P} = \langle aZ \rangle \rtimes \langle cZ \rangle$ is isomorphic to the dihedral group of order $2^{\ell+1}$, and hence $\langle t, c \rangle$ and the subgroups in the assertion are the preimages under the canonical epimorphism $P \to \bar{P}$ of representatives of the $\bar{P}$-conjugacy classes of nontrivial proper subgroups of $\bar{P}$. In order to determine whether they are conjugate in G or not, it suffices to consider the three subgroups $\langle t, a^i \rangle$, $\langle t, a^{2i}, c \rangle$, and $\langle t, a^{2i}, ac \rangle$ for each $i \in I$, which have the same order $2^{2\ell}/i$.

In case $i = 2^{\ell-1}$, we see that $\langle t, c \rangle$ is conjugate in G with $\langle t, a^{2^{\ell-1}} \rangle$. On the other hand, $\langle t, c \rangle$ is not conjugate to $\langle t, ac \rangle$ since $\langle t, c \rangle$ has no element of the same order as ac, which is $2^{\ell+1}$. Next assume that $i \neq 2^{\ell-1}$. Then $\langle t, a^{2i}, c \rangle$ and $\langle t, a^{2i}, ac \rangle$ are not abelian, and hence are not conjugate to $\langle t, a^i \rangle$. Since $\langle t, a^{2i}, c \rangle$ also has no element of the same order as ac, it is not conjugate to $\langle t, a^{2i}, ac \rangle$. □

We calculate the centralizers of the subgroups listed in Lemma 6.1 to see that they are 2-nilpotent.

Lemma 6.2. *The centralizers of P and the subgroups in Lemma 6.1 are as follows.*

(i) $C_G(P) = Z(G)$.

(ii) $C_G(\langle t, a^{2i}, ac \rangle) = Z(G)$, $i \in I \backslash \{2^{\ell-1}\}$.

(iii) $C_G(\langle t, ac\rangle) = \left\{ \begin{pmatrix} \alpha & s\beta \\ \beta & \alpha \end{pmatrix} \;\middle|\; \alpha, \beta \in \mathbb{F}_q,\ (\alpha, \beta) \neq (0,0) \right\} \cong C_{q^2-1}$.

(iv) $C_G(\langle t, a^{2i}, c\rangle) = Z(G)$, $i \in I \backslash \{2^{\ell-1}\}$.

(v) $C_G(\langle t, a^i\rangle) = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \;\middle|\; \alpha, \beta \in \mathbb{F}_q^\times \right\} \cong C_{q-1} \times C_{q-1}$, $i \in I$.

In particular, all the centralizers above are 2*-nilpotent.*

Proof. Straightforward verifications show the result. □

Assume that k has characteristic 2. By [7, Theorem 7.11], $B_0(kG)$ has two simple modules k_G and S, which correspond to the partitions (2) and (1^2), respectively. Let $\mathbf{B}$ be the standard Borel subgroup of G, the subgroup consisting of the upper triangular matrices in G, and it has order $q(q-1)^2$.

Lemma 6.3. *The Scott module* $S(G, \mathbf{B})$ *has the following Loewy and socle series*

$$S(G, \mathbf{B}) = \begin{pmatrix} k_G \\ S \\ k_G \end{pmatrix}.$$

Proof. Since $|\mathbf{B}\backslash G/\mathbf{B}| = 2$ and $|G : \mathbf{B}| = q+1$, we have that $1_\mathbf{B}{\uparrow}^G = 1_G + \chi$, where $1_\mathbf{B}$ is the trivial character of $\mathbf{B}$, and χ is an irreducible character of degree q of G. Suppose that $k_\mathbf{B}{\uparrow}^G$ is not indecomposable. Then the uniquely lifted permutation $\mathcal{O}G$-module $\mathcal{O}_\mathbf{B}{\uparrow}^G$ is not indecomposable, and it is decomposed as $\mathcal{O}_\mathbf{B}{\uparrow}^G = \mathcal{O}_G \oplus U$, where U is an $\mathcal{O}G$-module corresponding to χ. This implies $2 \nmid |G : \mathbf{B}|$, a contradiction.

We have that $1_\mathbf{B}{\uparrow}^G{\downarrow}_H = 1_{\mathbf{B}\cap H}{\uparrow}^H$, where $H = SL_2(q)$, and hence by [2, Section 3.2.3 and Table 9.1], we have that

$$k_\mathbf{B}{\uparrow}^G{\downarrow}_H = 2k_H + \bar{\mathrm{St}}_+^k + \bar{\mathrm{St}}_-^k \quad \text{(as composition factors)}.$$

This implies that

$$k_\mathbf{B}{\uparrow}^G = 2k_G + S \quad \text{(as composition factors)}.$$

Since $k_\mathbf{B}{\uparrow}^G$ is indecomposable, we have $k_\mathbf{B}{\uparrow}^G = S(G, \mathbf{B})$, and the result follows. □

§7. Proof of Theorem 1.1

In this section, we prove Theorem 1.1.

We use the following lemmas to lift relative stable equivalences of Morita type to Morita equivalences using Theorem 3.4. These are analogous to [10, Lemma 1.11] and [14, Lemma A.1].

Lemma 7.1. (see [10, Lemma 1.11]) *Let W be a kG-module and X a W-projective kG-module. Then the following hold:*

(i) *Let $\varphi : V \to U \oplus X$ be a kG-homomorphism. If $V \xrightarrow{\varphi} U \oplus X \xrightarrow{\pi_U} U$ is a W-split monomorphism, where π_U is the projection, then there is a submodule U' of $U \oplus X$ such that $U \oplus X = U' \oplus X$ and $\mathrm{Im}(\varphi) \subseteq U'$.*

(ii) *Let $\psi : U \oplus X \to V$ be a kG-homomorphism. If $U \xrightarrow{\iota_U} U \oplus X \xrightarrow{\psi} V$ is a W-split epimorphism, where ι_U is the injection, then there is a submodule X' of $U \oplus X$ such that $U \oplus X = U \oplus X'$ and $X' \subseteq \mathrm{Ker}(\psi)$.*

Proof. (i) By Proposition 4.1, there is a kG-homomorphism $\alpha : U \to X$ such that the following diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & V & \xrightarrow{\pi_U\varphi} & U \\ & & {\scriptstyle \pi_X\varphi}\downarrow & \swarrow{\scriptstyle \alpha} & \\ & & X & & \end{array}$$

is commutative. Let $U' = \{(u, \alpha(u)) \mid u \in U\}$. Then it follows that $U' \cap X = 0$ and $U \cong U'$. Hence we have $U \oplus X = U' \oplus X$. We also have that

$$\mathrm{Im}(\varphi) = \{(\pi_U\varphi(v), \pi_X\varphi(v)) \mid v \in V\} = \{(\pi_U\varphi(v), \alpha(\pi_U\varphi(v))) \mid v \in V\},$$

and hence $\mathrm{Im}(\varphi)$ is contained in U'.

(ii) This is proved dually. □

Lemma 7.2. (see [14, Lemma A.1]) *Let B and B' be blocks of finite groups kG and kG', respectively, having a common defect group P with $\mathcal{F}_P(G) = \mathcal{F}_P(G')$. Let M be a B-B'-bimodule that is a ΔP-projective p-permutation $k[G \times G']$-module. Assume that, for a subgroup Q of P, (M, M^*) induces a relative Q-stable equivalence of Morita type between B and B'. Let X be a B-module having no nonzero Q-projective summand and U a submodule of X. Then $X \otimes_B M = Y \oplus L$ holds, where Y is a B'-module having no nonzero Q-projective summand, and L is a Q-projective B'-module. Moreover, the following hold.*

(i) *If $U \otimes_B M \to X \otimes_B M \xrightarrow{\pi_Y} Y$ is a Q-split monomorphism, where π_Y is the projection, then we can take Y so that Y contains $U \otimes_B M$, and hence we have that*

$$(X/U) \otimes_B M \cong Y/(U \otimes_B M) \oplus (Q\text{-projective}).$$

(ii) *If $Y \xrightarrow{\iota_Y} X \otimes_B M \to (X/U) \otimes_B M$ is a Q-split epimorphism, where ι_Y is the injection, then we can take L so that L is contained in $\mathrm{Ker}(\pi \otimes \mathrm{id}_M)$, and hence we have that*

$$U \otimes_B M \cong \mathrm{Ker}(Y \xrightarrow{\iota_Y} X \otimes_B M \to (X/U) \otimes_B M) \oplus (Q\text{-projective})$$

Proof. Note that we can write $X \otimes_B M = Y \oplus L$, since $- \otimes_B M$ and $- \otimes_{B'} M^*$ induce an equivalence between $\underline{\mathrm{mod}}^Q(B)$ and $\underline{\mathrm{mod}}^Q(B')$ (see [16, Lemma 4.5 and Corollary 4.6]).

(i) By Lemma 7.1 (i), there is a submodule Y' of $X \otimes_B M$ such that $Y \oplus L = Y' \oplus L$, and Y' contains $U \otimes_B M$. Hence we have that

$$(X/U) \otimes_B M \cong (X \otimes_B M)/(U \otimes_B M) \cong Y'/(U \otimes_B M) \oplus L.$$

If we take Y' as Y, then the result follows.

(ii) We write $\pi \otimes \mathrm{id}_M$ for $X \otimes_B M \to (X/U) \otimes_B M$. By Lemma 7.1 (ii), there is a submodule L' of $X \otimes_B M$ such that $Y \oplus L = Y \oplus L'$, and L' is contained in $\mathrm{Ker}(\pi \otimes \mathrm{id}_M)$. Hence $\mathrm{Ker}((\pi \otimes \mathrm{id}_M)\iota_Y) \oplus L'$ is contained in $\mathrm{Ker}(\pi \otimes \mathrm{id}_M)$. Since $(\pi \otimes \mathrm{id}_M)\iota_Y$ is an epimorphism, the result follows. □

In the rest of the section, we assume that k has characteristic 2. We also assume that q_1 and q_2 are odd prime powers such that $q_1 \equiv q_2 \equiv 1 \pmod 4$ and $(q_1 - 1)_2 = (q_2 - 1)_2 =: 2^\ell$, $\ell \geq 2$. Let $G_i = GL_2(q_i)$ and $B_i = B_0(kG_i)$, $i = 1, 2$.

Let s be an element of order 2^ℓ in $\mathbb{F}_{q_1}^\times$. We identify s with an element of order 2^ℓ in $\mathbb{F}_{q_2}^\times$. Let P and Z be subgroups as in Section 6. Then P is a common Sylow 2-subgroup of G_1 and G_2 isomorphic to the wreathed 2-group of order $2^{2\ell+1}$. Note that we have $\mathcal{F}_P(G_1) = \mathcal{F}_P(G_2)$ (see [6, Theorem 5.3]). We also have that $Z = Z(G_1) \cap P = Z(G_2) \cap P$.

Let $M = S(G_1 \times G_2, \Delta P)$. As mentioned in Section 3, the relative ΔZ-Brauer indecomposability of M is useful for constructing a relative Z-stable equivalence of Morita type between B_1 and B_2. However, the Brauer indecomposability of M, which is stronger than the relative ΔZ-Brauer indecomposability of M, has been shown:

Lemma 7.3. (see [15, Theorem 1.1]) *The Scott module M is Brauer indecomposable.*

Lemma 7.4. *The pair (M, M^*) induces a relative Z-stable equivalence of Morita type between B_1 and B_2.*

Proof. Note that, by the assumption that $\mathcal{F}_P(G_1) = \mathcal{F}_P(G_2)$, the fusion system $\mathcal{F}_{\Delta P}(G_1 \times G_2)$ is equivalent to $\mathcal{F}_P(G_1) = \mathcal{F}_P(G_2)$, and hence is saturated.

Let Q be any fully normalized subgroup of P properly containing Z. Then ΔQ is a fully normalized subgroup in $\mathcal{F}_{\Delta P}(G_1 \times G_2)$. Hence, Lemma 3.5 and Lemma 7.3 give that

$$M(\Delta Q){\downarrow}_{C_{G_1}(Q) \times C_{G_2}(Q)} \cong S(C_{G_1}(Q) \times C_{G_2}(Q), C_{\Delta P}(\Delta Q)). \tag{7.1}$$

Since P is a Sylow p-subgroup of G_1 and G_2, the subgroup Q is also fully centralized in $\mathcal{F}_P(G_1)$ and $\mathcal{F}_P(G_2)$, and hence $C_P(Q)$ is a common Sylow p-subgroup of $C_{G_1}(Q)$ and $C_{G_2}(Q)$. By Lemma 6.2, $C_{G_1}(Q)$ and $C_{G_2}(Q)$ are 2-nilpotent, and hence, [11, Lemma 3.1] and (7.1) imply that $M(\Delta Q)$ and its dual induce a Morita equivalence between $B_0(kC_{G_1}(Q))$ and $B_0(kC_{G_2}(Q))$. This implies the result by Theorem 3.2 and Remark 3.3. □

Let $\mathbf{B}_i$ be the standard Borel subgroup of G_i, $i = 1, 2$. Then we have that $\mathbf{B}_1 \cap P = \mathbf{B}_2 \cap P =: L$, and L is a common Sylow 2-subgroup of $\mathbf{B}_1$ and $\mathbf{B}_2$, which has order $2^{2\ell}$. As mentioned in Section 6, B_i has two simple modules k_{G_i} and S_i, $i = 1, 2$.

We now prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 7.4, and Theorem 3.4 (ii), it suffices to show that $k_{G_1} \otimes_{B_1} M$ and $S_1 \otimes_{B_1} M$ are simple. Note that, by Lemma 7.4 and Theorem 3.4 (i), $k_{G_1} \otimes_{B_1} M$ and $S_1 \otimes_{B_1} M$ are indecomposable and non Z-projective. Hence, by Lemma 5.3, $k_{G_1} \otimes_{B_1} M \cong k_{G_2}$. We next show that $S_1 \otimes_{B_1} M \cong S_2$.

By Lemma 5.1, $P_L(k_{G_1}) \cong S(G_1, L)$, and hence we have a monomorphism $\iota : k_{G_1} \to \Omega_L(k_{G_1})$. By Lemma 7.4 and Lemma 5.4, we may write

$$\Omega_L(k_{G_1}) \otimes_{B_1} M = \Omega_L(k_{G_2}) \oplus (Z\text{-projective}).$$

By Lemma 5.2, ι is a Z-split monomorphism, and $\Omega_L(k_{G_1})$ has no nonzero L-projective summand, and in particular, no nonzero Z-projective summand as $Z \leq L$. Hence, by Lemma 4.10 (i), ι does not factor through a Z-projective kG-module. Since $- \otimes_{B_1} M$ induces an equivalence between $\underline{\text{mod}}^Z(B_1)$ and $\underline{\text{mod}}^Z(B_2)$ (see [16, Corollary 4.6]), $\iota \otimes \text{id}_M$ does not factor through a Z-projective kG'-module. This implies that

$$k_{G_1} \otimes_{B_1} M \xrightarrow{\iota \otimes \text{id}_M} \Omega_L(k_{G_1}) \otimes_{B_1} M \xrightarrow{\text{projection}} \Omega_L(k_{G_2}) \tag{7.2}$$

is nonzero. Indeed, suppose that it is zero, then $\iota \otimes \text{id}_M$ factors through the Z-projective part of $\Omega_L(k_{G_1}) \otimes_{B_1} M$, a contradiction.

Since $k_{G_1} \otimes_{B_1} M \cong k_{G_2}$, again by Lemma 5.2, (7.2) is a Z-split monomorphism. By Lemma 7.2 (i), we have that

$$(\Omega_L(k_{G_1})/k_{G_1}) \otimes_{B_1} M \cong \Omega_L(k_{G_2})/(k_{G_1} \otimes_{B_1} M) \oplus (Z\text{-projective}). \tag{7.3}$$

By Lemma 6.3, we have

$$\Omega_L(k_{G_i}) = \begin{pmatrix} S_i \\ k_{G_i} \end{pmatrix}.$$

Hence, (7.3) implies

$$S_1 \otimes_{B_1} M \cong S_2 \oplus (Z\text{-projective}).$$

This forces $S_1 \otimes_{B_1} M \cong S_2$ since $S_1 \otimes_{B_1} M$ is an indecomposable non Z-projective module. □

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