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Batalin-Vilkovisky algebra structures on the Hochschild cohomology of self-injective Nakayama algebras

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Abstract. In this paper, we determine the Batalin-Vilkovisky algebra structure on the Hochschild cohomology of self-injective Nakayama algebras with the diagonalizable Nakayama automorphism over an algebraically closed field K. Moreover, in the case that the characteristic of K divides the order of the Nakayama automorphism, we compute the Batalin-Vilkovisky algebra structure on cohomology of Hochschild complex related to the Nakayama automorphism.

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§1. Introduction

Hochschild cohomology is an invariant of derived equivalence and it has several algebraic structures; module structures, graded commutative ring structures and Gerstenhaber algebra structures, etc. These algebraic structures of Hochschild cohomology of algebras have been computed for many classes of algebras. For instance, for algebras which are stably equivalent to basic representation-finite self-injective algebras with associate tree $D_n (n \geq 4)$, Volkov computed their Hochschild cohomology groups and Hochschild cohomology rings ([17], [18], [19], [20] and [21]).

Tradler [16] discovered that Hochschild cohomology of arbitrary symmetric algebra has a Batalin-Vilkovisky (we say BV for short) algebra structure given by a symmetric bilinear form. Later, Lambre, Zhou and Zimmermann [9] discovered that Hochschild cohomology of Frobenius algebras with diagonalizable Nakayama automorphism has a BV algebra structure. However, it is not known if Hochschild cohomology of Frobenius algebras has a BV algebra structure in general.

Recently, for any Frobenius algebra A, Volkov [22] defined the cohomology $\mathrm{HH}^*(A)^{\nu\uparrow}$ of Hochschild complex related to Nakayama automorphism ν, which induces a Gerstenhaber algebra $(\mathrm{HH}^*(A)^{\nu\uparrow}, \smile, [\ ,\ ])$. Moreover, Volkov [22] also found a BV algebra structure on $(\mathrm{HH}^*(A)^{\nu\uparrow}, \smile, [\ ,\ ])$. In particular, if the Nakayama automorphism ν is diagonalizable, then $\mathrm{HH}^*(A)^{\nu\uparrow}$ is isomorphic to $\mathrm{HH}^*(A)$ and the BV differential on $(\mathrm{HH}^*(A)^{\nu\uparrow}, \smile, [\ ,\ ])$ induces the one on the Gerstenhaber algebra $(\mathrm{HH}^*(A), \smile, [\ ,\ ])$. In [22], the BV differentials on Hochschild cohomology of representation-finite self-injective algebras of tree type $D_n (n \geq 4)$ with diagonalizable Nakayama automorphism were calculated. However, there are few examples of complete calculation of BV differentials on Hochschild cohomology of Frobenius algebras which are not symmetric.

In this paper, we will compute BV differentials on Hochschild cohomology of self-injective Nakayama algebras. We will divide the computation into two cases: Case (a) the characteristic of the ground field does not divide the order of the Nakayama automorphism; Case (b) the characteristic of the ground field divides the order of the Nakayama automorphism. For a self-injective Nakayama algebra Λ in Case (b), we will compute $\mathrm{HH}^*(\Lambda)^{\nu\uparrow}$ and BV differentials on $(\mathrm{HH}^*(\Lambda)^{\nu\uparrow}, \smile, [\ ,\ ])$. This implies that $\mathrm{HH}^*(\Lambda)^{\nu\uparrow} \cong \mathrm{HH}^*(\Lambda)$ as algebras and $[\ ,\ ] = 0$. However, $[\ ,\ ] \neq 0$ on Hochschild cohomology of Λ in Case (a) in general. On the special case, when Λ is a truncated polynomial ring, BV algebra structures on the Hochschild cohomology of truncated polynomial rings were calculated in [15]. The Gerstenhaber brackets on Hochschild cohomology rings of truncated quiver algebras were calculated in [23].

This paper is organized as follows: In Section 2, we recall the definitions and the notation for Hochschild cohomology, Gerstenhaber brackets on Hochschild cohomology, BV algebras on Hochschild cohomology of Frobenius algebras. Moreover, we recall the bilinear form and the Nakayama automorphism for self-injective Nakayama algebras. In Section 3, we recall chain maps between Bardzell's projective resolution and a bar resolution for truncated quiver algebras. In Section 4, we compute BV differentials on Hochschild cohomology of self-injective Nakayama algebras under the assumption that the characteristic of base field dose not divide the order of the Nakayama automorphism. We will determine the image of the BV differentials for each basis element in n-th Hochschild cohomology group for each $n \geq 0$ (Theorem 4.4, Theorem 4.5 and Theorem 4.6). Theorem 4.4, Theorem 4.5 and Theorem 4.6 give precise formulas for the BV differentials $\Delta : \mathrm{HH}^n(\Lambda) \to \mathrm{HH}^{n-1}(\Lambda)$, for $n = 1$, $n = 2i$, $n = 2i + 1$, respectively. In Section 5, we determine the ring structure (Theorem 5.4 and Theorem 5.5), a BV algebra structure (Theorem 5.7) of the cohomology of Hochschild complex related to Nakayama automorphism of self-injective Nakayama algebras under the assumption that the characteristic of base field divides the order of the Nakayama automorphism. Theorem 5.7 gives precise formulas for the BV differential $\Delta : \mathrm{HH}^n(\Lambda)^{\nu\uparrow} \to \mathrm{HH}^{n-1}(\Lambda)^{\nu\uparrow}$ for $n \geq 1$.

Throughout this paper, we denote the tensor product $\otimes_K$ over K by $\otimes$ for simplicity, where K is an algebraically closed field.

§2. Preliminaries

In this section, we recall the definitions and the notation for Hochschild cohomology, Gerstenhaber algebras, Batalin-Vilkovisky (we say BV for short) algebras. Moreover, following [22], we recall the BV differential on Hochschild cohomology for Frobenius algebras. In order to compute BV differentials on Hochschild cohomology of self-injective algebras, we also describe the bilinear form of self-injective Nakayama algebras and the Nakayama automorphism.

2.1. Batalin-Vilkovisky differential on Hochschild cohomology of Frobenius algebras

Let K be an algebraically closed field and A a finite dimensional K-algebra.

Definition 2.1. The following complex $(C^*(A), \delta_*)$ is called the *Hochschild complex* of A:

$$0 \to C^0(A) \xrightarrow{\delta_0} C^1(A) \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{n-2}} C^{n-1}(A) \xrightarrow{\delta_{n-1}} C^n(A) \xrightarrow{\delta_n} C^{n+1}(A) \xrightarrow{\delta_{n+1}} \cdots$$

where $C^0(A) = \mathrm{Hom}_K(K, A) \cong A$, $C^n(A) = \mathrm{Hom}_K(A^{\otimes n}, A)$ and

$$\begin{aligned}&\delta_n(f)(a_1 \otimes \cdots \otimes a_{n+1}) = a_1 f(a_2 \otimes \cdots \otimes a_{n+1})\\ &\quad + \sum_{i=1}^{n} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) + (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) a_{n+1},\end{aligned}$$

for $f \in C^n(A)$ and $n \geq 1$. The n-th *Hochschild cohomology group* $\mathrm{HH}^n(A)$ of A is defined as the n-th cohomology of $(C^*(A), \delta_*)$.

The bar resolution $(\mathrm{Bar}_*(A), b_*)$ of A is the following:

$$\cdots \xrightarrow{b_{n+1}} \mathrm{Bar}_n(A) \xrightarrow{b_n} \mathrm{Bar}_{n-1}(A) \xrightarrow{b_{n-1}} \cdots \xrightarrow{b_2} \mathrm{Bar}_1(A) \xrightarrow{b_1} \mathrm{Bar}_0(A) \xrightarrow{b_0} A \to 0$$

where, for $n \geq 0$, $\mathrm{Bar}_n(A) = A^{\otimes n+2}$ and

$$b_n(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}.$$

For $n \geq 0$, the isomorphism $\mathrm{Hom}_K(A^{\otimes n}, A) \cong \mathrm{Hom}_{A^e}(A^{\otimes n+2}, A)$ induces an isomorphism $(\mathrm{Hom}_{A^e}(\mathrm{Bar}_*(A), A), \mathrm{Hom}_{A^e}(b_*, A)) \cong (C^*(A), \delta_*)$, where A^e is the enveloping algebra of A. Therefore, $\mathrm{HH}^n(A) \cong \mathrm{Ext}^n_{A^e}(A, A)$.

The cup product on the Hochschild complex $(C^*(A), \delta_*)$ is given as follows: for $f \in C^n(A)$ and $g \in C^m(A)$, $f \smile g \in C^{m+n}(A)$ is given by

$$(f \smile g)(a_1 \otimes \cdots \otimes a_{n+m}) = f(a_1 \otimes \cdots \otimes a_n)g(a_{n+1} \otimes \cdots \otimes a_{n+m}).$$

The cup product induces the one on Hochschild cohomology $\smile: \mathrm{HH}^n(A) \times \mathrm{HH}^m(A) \to \mathrm{HH}^{n+m}(A)$. Then, $\mathrm{HH}^*(A) := \oplus_{n\geq 0}\mathrm{HH}^n(A)$ is a commutative graded algebra. We remark that the Yoneda product on $\mathrm{Ext}^*_{A^e}(A, A) = \oplus_{n\geq 0}\mathrm{Ext}^n_{A^e}(A, A)$ coincides with the cup product on the Hochschild cohomology $\mathrm{HH}^*(A)$.

Following [6], we recall the Lie bracket $[\ ,\ ]$ on the Hochschild cohomology ring $\mathrm{HH}^*(A)$. First, we recall the definition of Gerstenhaber algebras.

Definition 2.2. A *Gerstenhaber algebra* over an algebraically closed field K is $(V^*, \cup, [\ ,\ ])$, where $V^* = \oplus_{k\geq 0}V^k$ is a graded K-vector space, $\cup : V^n \times V^m \to V^{n+m}$ $(n, m \geq 0)$ is a cup product of degree zero and $[\ ,\ ] : V^n \times V^m \to V^{n+m-1}$ $(n, m \geq 0$ and $V^{-1} = 0)$ is a Lie bracket of degree -1 such that the following conditions hold:

(i) $(V^*, \cup)$ is a graded commutative associative algebra with unit $1 \in V^0$.

(ii) $(V^*, \cup, [\ ,\ ])$ is a graded Lie algebra.

(iii) For arbitrary homogeneous elements a, b and c in V^*,

$$[a, b \cup c] = [a, b] \cup c + (-1)^{(|a|-1)|b|} b \cup [a, c],$$

where the notation $|a|$ means the degree of the homogeneous element a.

For $f \in C^n(A)$ and $g \in C^m(A)$ $(n + m \geq 1)$, we define $[f, g] \in C^{n+m-1}(A)$ as follows: If $n, m \geq 1$, then, for $1 \leq i \leq n$, $f \circ_i g \in C^{n+m-1}(A)$ is given by

$$\begin{aligned}&(f \circ_i g)(a_1 \otimes \cdots \otimes a_{n+m-1})\\ &\quad = f(a_1 \otimes \cdots \otimes a_{i-1} \otimes g(a_i \otimes \cdots \otimes a_{i+m-1}) \otimes a_{i+m} \otimes \cdots \otimes a_{n+m-1}).\end{aligned}$$

If $n \geq 1$ and $m = 0$, then, for $1 \leq i \leq n$, $f \circ_i g \in C^{n-1}(A)$ is given by

$$(f \circ_i g)(a_1 \otimes \cdots \otimes a_{n+m-1}) = f(a_1 \otimes \cdots \otimes a_{i-1} \otimes g \otimes a_i \otimes \cdots \otimes a_{n-1}),$$

where g is regarded as an element of A.

We set

$$f \circ g := \sum_{i=1}^{n} (-1)^{(m-1)(i-1)} f \circ_i g$$

and

$$[f,g] := f \circ g - (-1)^{(n-1)(m-1)} g \circ f \in C^{n+m-1}(A).$$

Then $[\ ,\ ]$ induces $[\ ,\ ] : \mathrm{HH}^n(A) \times \mathrm{HH}^m(A) \to \mathrm{HH}^{n+m-1}(A)$, and then $(\mathrm{HH}^*(A), \smile, [\ ,\ ])$ is a Gerstenhaber algebra.

Definition 2.3. A *Batalin-Vilkovisky algebra* is a Gerstenhaber algebra $(V^*, \cup, [\ ,\ ])$ with an operator $\Delta : V^* \to V^{*-1}$ of degree -1 such that $\Delta \circ \Delta = 0$ and

$$[a,b] = -(-1)^{(|a|-1)|b|}(\Delta(a \cup b) - \Delta(a) \cup b - (-1)^{|a|} a \cup \Delta(b)),$$

for homogeneous elements $a, b \in V^*$.

Let A be a Frobenius algebra with the bilinear form $\langle\ ,\ \rangle : A \times A \to K$ and the Nakayama automorphism ν. Following [22], we recall a BV differential on $(\mathrm{HH}^*(A)^{\nu\uparrow}, \smile, [\ ,\ ])$ focusing on Frobenius algebras. For $n \geq 0$, the map $\phi_\nu : C^n(A) \to C^n(A)$ can be defined by

$$(\phi_\nu(f))(a_1 \otimes \cdots \otimes a_n) = \nu^{-1}(f(\nu(a_1) \otimes \cdots \otimes \nu(a_n))),$$

for $f \in C^n(A)$ and $a_i \in A$. Then $\phi_\nu(\delta_n f) = \delta_n(\phi_\nu(f))$, so ϕ_ν induces an automorphism of the Hochschild cohomology. Let $C^n(A)^\nu = \{f \in C^n(A) \mid \phi_\nu(f) = f\}$. Then, δ_n restricts to a differential $\delta_n^\nu : C^n(A)^\nu \to C^{n-1}(A)^\nu$ and let $\mathrm{HH}^n(A)^{\nu\uparrow}$ be the n-th cohomology of the complex $(C^*(A)^\nu, \delta_*^\nu)$. Then $\mathrm{HH}^n(A)^{\nu\uparrow} \cong \mathrm{HH}^n(A)$ if ν is diagonalizable by [22, Corollary 2]. The cup product on the Hochschild complex can restrict to $(C^*(A)^\nu, \delta_*^\nu)$ and $\mathrm{HH}^*(A)^{\nu\uparrow}$ has a ring structure. The Gerstenhaber algebra structure on $\mathrm{HH}^*(A)$ induces the Gerstenhaber algebra structure on $\mathrm{HH}^*(A)^{\nu\uparrow}$.

Let $f \in C^n(A)$, $n \geq 1$. Then, for i such that $1 \leq i \leq n$ we define $\Delta_i f \in C^{n-1}(A)$ by

$$\langle \Delta_i f(a_1 \otimes \cdots \otimes a_{n-1}), a_n \rangle = \langle f(a_i \otimes \cdots \otimes a_n \otimes \nu a_1 \otimes \cdots \otimes \nu a_{i-1}), 1 \rangle$$

and we set $\Delta := \sum_{i=1}^n (-1)^{i(n-1)} \Delta_i : C^n(A) \to C^{n-1}(A)$.

Theorem 2.4 ([22, Theorem 2.2]). *Let A be a Frobenius algebra with the bilinear form $\langle\ ,\ \rangle$ and the Nakayama automorphism ν. Then, Δ induces a BV algebra structure on the Gerstenhaber algebra* $(\mathrm{HH}^*(A)^{\nu\uparrow}, \smile, [\ ,\ ])$.

Note that this generalizes Tradler's result. If ν is diagonalizable, then Δ induces a BV differential on the Gerstenhaber algebra $(\mathrm{HH}^*(A), \smile, [\ ,\ ])$. In [22], BV algebra structures on $(\mathrm{HH}^*(A), \smile, [\ ,\ ])$ for self-injective algebras of tree type D_n is determined. However, there are few examples known of BV algebra structure on Hochschild cohomology for Frobenius algebras.

2.2. Self-injective Nakayama algebras

We are focusing on representation-finite self-injective basic algebras over an algebraically closed field. They are classified by associate trees of their stable AR-quivers, and the trees are Dynkin diagrams $A_m (m \geq 1), D_n (n \geq 4), E_i (6 \leq i \leq 8)$ in [12]. If the tree is A_m, then the algebra is stably equivalent either to a serial self-injective algebra (self-injective Nakayama algebra) or to Möbius algebra [13]. Moreover, their Hochschild cohomology groups and rings have been computed (cf. [3], [5], [8] and [10]). For algebras which are stably equivalent to representation-finite self-injective basic algebras with associate tree $D_n (n \geq 4)$, Volkov computed their Hochschild cohomology groups and Hochschild cohomology rings ([17], [18], [19], [20] and [21]).

In this subsection, according to [3], we recall the notation of self-injective Nakayama algebras. Let $e (\geq 2)$ and $N (\geq 2)$ be integers, and Z_e a cyclic quiver with e vertices:

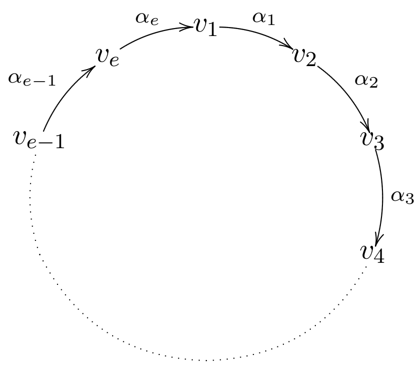

Figure 1: Z_e

Moreover, let $\Lambda = KZ_e/J^N$, where J is the arrow ideal of KZ_e. For an integer l, we denote a path of length $n (\geq 0)$ with the start point v_l by γ_l^n, that is, $\gamma_l^n = \alpha_l \alpha_{l+1} \cdots \alpha_{l+n-1}$, where we regard subscripts l of v_l, α_l and γ_l^n modulo e. Since Λ is basic, Λ is a Frobenius algebra. The bilinear form $\langle\ ,\ \rangle : \Lambda \times \Lambda \to K$ is given by

$$\langle \gamma_l^j, \gamma_{l+j}^{N-1-j} \rangle = 1,$$

for $1 \leq l \leq e$ and $0 \leq j \leq N-1$, and thus a Nakayama automorphism ν of Λ is given by

$$\nu(v_l) = v_{l+N-1},\ \nu(\gamma_l^j) = \gamma_{l+N-1}^j.$$

Then, it is easy to show that $\mathrm{ord}(\nu) \leq e$. It is well-known that Λ is symmetric if and only if $N \equiv 1 \pmod e$.

Let $\mathcal{B} = \{\gamma_l^j \mid 1 \leq l \leq e \text{ and } 0 \leq j \leq N-1\}$ be a basis of Λ. We use the notation γ_l^0 as v_l if there is no confusion. For $b \in \mathcal{B}$, we denote by b^* the element in $\mathcal{B}$ such that $\langle b^*, b\rangle = 1$. Then, for $\gamma_l^j \in \mathcal{B}$, $(\gamma_l^j)^* = \gamma_{l+j-(N-1)}^{N-j-1}$. Thus, for $f \in \mathrm{Hom}_{\Lambda^e}(\mathrm{Bar}_n(\Lambda), \Lambda)$, $\Delta_i f$ can be computed by

$$\Delta_i f(\alpha \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes \beta) = \sum_{b \in \mathcal{B}} \langle \Delta_i f(1 \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes 1), b\rangle \alpha b^* \beta.$$

Remark 2.5. In [7], a BV algebra structure on Hochschild cohomology of the group algebra of the quaternion group of order eight in characteristic two was computed by means of the dual basis, a symmetric bilinear form and chain maps between the bar resolution and the projective resolution which gave a description of its Hochschild cohomology ring.

§3. Chain maps between Bardzell's projective resolution and the bar resolution for truncated quiver algebras

Note that self-injective Nakayama algebras are monomial algebras. In [2], Bardzell gave projective resolutions of monomial algebras as bimodules, and Hochschild cohomology of truncated quiver algebras were calculated by means of this resolution. Self-injective Nakayama algebras are truncated quiver algebras. For truncated quiver algebras, Sköldberg [14] also gave the projective resolution and Ames, Cagliero and Tirao gave comparison morphisms between Sköldberg's projective resolution and E-normalized projective resolution by Cibils [4], where E is the algebra generated by all vertices. For monomial algebras, comparison morphisms between Bardzell's projective resolution and the bar resolution were given by Redondo and Roman [11].

In this section, we focus on truncated quiver algebras and we describe chain maps between the bar resolution and the Bardzell's projective resolution through Cibils' projective resolution in order to compute BV differentials on Hochschild cohomology of self-injective Nakayama algebras. This chain map coincides with the comparison morphism given in [11]. In order to compute BV differentials easily, we divide comparison morphisms in [11] for truncated quiver algebras into several chain maps.

Let $A = KQ/J^N$ $(N \geq 2)$ be a truncated quiver algebra, where J is the arrow ideal of KQ. We denote by Q_i the set of all path of length $i \geq 0$. For a path p in Q, $s(p)$ and $t(p)$ denote source of p and target of p, respectively. We focus on the truncated quiver algebra A, we recall Bardzell's projective resolution $\boldsymbol{P}$:

$$\boldsymbol{P} : \cdots \xrightarrow{d_{i+1}} P_i \xrightarrow{d_i} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A \to 0$$

where $P_i = \coprod_{R_i \in AP(i)} As(R_i)\otimes t(R_i)A$. Here, for $c \geq 1$, $AP(0) = Q_0$, $AP(1) = Q_1$, $AP(2c) = Q_{cN}$, $AP(2c+1) = Q_{cN+1}$, d_0 is the multiplication map and the differentials d_i are given by

$$d_{2c}(s(R_{2c}) \otimes t(R_{2c})) = \sum_{j=0}^{m-1} s(R_{2c})\alpha_1 \cdots \alpha_j \otimes \alpha_{(c-1)m+2+j} \cdots \alpha_{cN} t(R_{2c}),$$

$$\begin{aligned} &d_{2c+1}(s(R_{2c+1}) \otimes t(R_{2c+1})) \\ &\quad = s(R_{2c+1})\alpha'_1 \otimes t(R_{2c+1}) - s(R_{2c+1}) \otimes \alpha'_{cN+1} t(R_{2c+1}), \end{aligned}$$

for $R_{2c} = \alpha_1 \cdots \alpha_{cN} \in AP(2c)$ and $R_{2c+1} = \alpha'_1 \cdots \alpha'_{cN+1} \in AP(2c+1)$, where $\alpha_k, \alpha'_l \in Q_1$.

On the other hand, in [14], Sköldberg gave a similar projective resolution to compute the Hochschild homology of truncated quiver algebras.

Theorem 3.1 ([14, Theorem 1])**.** *Let $\mathbb{N}$ be the set of all non negative integers and $A = KQ/J^N$ a truncated quiver algebra, where J is the arrow ideal of KQ. The following is a projective $\mathbb{N}$-graded resolution of A as a left A^{e}-module*:

$$\boldsymbol{P'} : \cdots \xrightarrow{d'_{i+1}} P'_i \xrightarrow{d'_i} \cdots \xrightarrow{d'_2} P'_1 \xrightarrow{d'_1} P'_0 \xrightarrow{\varepsilon} A \longrightarrow 0$$

Here the modules are defined by

$$P'_i = A \otimes_{KQ_0} K\Gamma^{(i)} \otimes_{KQ_0} A,$$

where $K\Gamma^{(i)}$ is the vector space generated by $\Gamma^{(i)}$ and the set $\Gamma^{(i)}$ is given by

$$\Gamma^{(i)} = \begin{cases} Q_{cN} & \text{if } i = 2c \ (c \geq 0), \\ Q_{cN+1} & \text{if } i = 2c+1 \ (c \geq 0). \end{cases}$$

The differentials are defined by

$$\begin{aligned} &d'_{2(c+1)}(\alpha \otimes \alpha_1 \cdots \alpha_{(c+1)N} \otimes \beta) \\ &\qquad = \sum_{j=0}^{N-1} \alpha\alpha_1 \cdots \alpha_j \otimes \alpha_{1+j} \cdots \alpha_{cN+1+j} \otimes \alpha_{cN+2+j} \cdots \alpha_{(c+1)N}\beta, \\ &d'_{2c+1}(\alpha \otimes \alpha_1 \cdots \alpha_{cN+1} \otimes \beta) \\ &\qquad = \alpha\alpha_1 \otimes \alpha_2 \cdots \alpha_{cN+1} \otimes \beta - \alpha \otimes \alpha_1 \cdots \alpha_{cN} \otimes \alpha_{cN+1}\beta, \end{aligned}$$

for $c \geq 0$. The augmentation $\varepsilon\colon A \otimes_{KQ_0} KQ_0 \otimes_{KQ_0} A \cong A \otimes_{KQ_0} A \to A$ is defined by $\varepsilon(\alpha \otimes \beta) = \alpha\beta$.

These projective resolutions are isomorphic. In fact, for $n \geq 0$, $AP(n) = \Gamma^{(n)}$ and the isomorphism $\boldsymbol{P} \to \boldsymbol{P'}$ is given by

$$P_n \to P'_n,\ s(R_n) \otimes t(R_n) \mapsto s(R_n) \otimes_{KQ_0} R_n \otimes_{KQ_0} t(R_n),$$

for $R_n \in \Gamma^{(n)}$.

The following is the comparison morphism given in [11] from Bardzell's projective resolution to bar resolution for truncated quiver algebras.

Proposition 3.2 (cf. [11]). *Define the map* $\Phi : \boldsymbol{P} \to (\mathrm{Bar}_*(A), b_*)$ *as follows*:

$$\begin{aligned}
&\Phi_0(e_i \otimes e_i) = e_i \otimes e_i,\\
&\Phi_1(s(\alpha_1) \otimes t(\alpha_1)) = s(\alpha_1) \otimes \alpha_1 \otimes t(\alpha_1),\\
&\Phi_{2c}(s(R_{2c}) \otimes t(R_{2c}))\\
&\ = \sum_{0 \le j_1, \dots, j_c \le N-2} s(R_{2c}) \otimes \alpha_1 \cdots \alpha_{1+j_1} \otimes \alpha_{2+j_1} \otimes \alpha_{3+j_1} \cdots \alpha_{3+j_1+j_2} \otimes \alpha_{4+j_1+j_2}\\
&\quad \otimes \cdots \otimes \alpha_{2c-1+j_1+\cdots+j_{c-1}} \cdots \alpha_{2c-1+j_1+\cdots+j_c}\\
&\qquad\qquad \otimes \alpha_{2c+j_1+\cdots+j_c} \alpha_{2c+1+j_1+\cdots+j_c} \cdots \alpha_{cN} t(R_{2c}),\\
&\Phi_{2c+1}(s(R_{2c+1}) \otimes \alpha_1 \cdots \alpha_{cN+1} \otimes t(R_{2c+1}))\\
&\ = \sum_{0 \le j_1, \dots, j_c \le N-2} s(R_{2c+1}) \otimes \alpha_1 \otimes \alpha_2 \cdots \alpha_{2+j_1} \otimes \alpha_{3+j_1} \otimes \alpha_{4+j_1} \cdots \alpha_{4+j_1+j_2}\\
&\quad \otimes \alpha_{5+j_1+j_2} \otimes \cdots \otimes \alpha_{2c+j_1+\cdots+j_{c-1}} \cdots \alpha_{2c+j_1+\cdots+j_c} \otimes \alpha_{2c+1+j_1+\cdots+j_c}\\
&\qquad \otimes \alpha_{2c+2+j_1+\cdots+j_c} \cdots \alpha_{cN+1} t(R_{2c+1}),
\end{aligned}$$

where $e_i \in Q_0$ *and* $\alpha_1, \alpha_2, \ldots \in Q_1$. *Then,* Φ *is a chain map.*

Next, in order to describe a chain map $(\mathrm{Bar}_*(A), b_*) \to \boldsymbol{P}$, we recall Cibils' projective resolution.

Lemma 3.3 ([4, Lemma 1.1]). *Let* Q *be a finite quiver and* KQ_0 *the subalgebra of the truncated quiver algebra* $A = KQ/J^N$ *generated by* Q_0 *and* $r = J/J^N$ *the Jacobson radical of* A. *Then the following is a projective resolution of* A *as a left* A^e*-module*:

$$\begin{aligned}
\boldsymbol{P''} : \cdots \longrightarrow A \otimes_{KQ_0} r^{\otimes^i_{KQ_0}} \otimes_{KQ_0} A \xrightarrow{d''_i} A \otimes_{KQ_0} r^{\otimes^{i-1}_{KQ_0}} \otimes_{KQ_0} A \longrightarrow \cdots\\
\longrightarrow A \otimes_{KQ_0} r \otimes_{KQ_0} A \xrightarrow{d''_1} A \otimes_{KQ_0} A \xrightarrow{d''_0} A \longrightarrow 0
\end{aligned}$$

where $d''_0(\lambda[\quad]\mu) = \lambda\mu$, *and*

$$\begin{aligned}
d''_i(\lambda[x_1|\cdots|x_i]\mu) = \lambda x_1[x_2|\cdots|x_i]\mu + \sum_{j=1}^{i-1}(-1)^i \lambda[x_1|\cdots|x_j x_{j+1}|\cdots|x_i]\mu\\
+ (-1)^i \lambda[x_1|\cdots|x_{i-1}]x_i\mu,
\end{aligned}$$

for $i \ge 1$. *Here, we use the bar notation* $\lambda[x_1|\cdots|x_i]\mu$ *for* $\lambda \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_i \otimes \mu$.

There is a chain map θ from the bar resolution to Cibils' projective resolution given by $\theta(a_1 \otimes \cdots \otimes a_{n+2}) = a_1[a_2|\cdots|a_{n+1}]a_{n+2}$. Moreover, Ames, Cagliero and Tirao [1] gave the following chain map $\boldsymbol{P''} \to \boldsymbol{P'}$ to compute the ring structure of Hochschild cohomology of truncated quiver algebras.

Proposition 3.4 ([1])**.** *Let $A = KQ/J^N$ be a truncated quiver algebra, $x_1, x_2, \ldots$ paths in Q and $m_1, m_2, \ldots$ the lengths of $x_1, x_2, \ldots$, respectively. We set $x_1 = \alpha_1\alpha_2\cdots\alpha_{m_1}$, $x_2 = \alpha_{m_1+1}\alpha_{m_1+2}\cdots\alpha_{m_1+m_2}, \ldots$, where $\alpha_1, \alpha_2, \ldots \in Q_1$. Then there exists a chain map $\pi : \boldsymbol{P''} \to \boldsymbol{P'}$ defined by the following equations*:

$$\pi_0(\alpha[\quad]\beta) = \alpha \otimes \beta,$$

$$\pi_1(\alpha[x_1]\beta) = \sum_{j=1}^{m_1} \alpha\alpha_1\cdots\alpha_{j-1} \otimes \alpha_j \otimes \alpha_{j+1}\cdots\alpha_{m_1}\beta,$$

$$\pi_{2c}(\alpha[x_1|x_2|\cdots|x_{2c}]\beta) = \begin{cases} \alpha \otimes \alpha_1\cdots\alpha_{cN} \otimes \alpha_{cN+1}\cdots\alpha_{m_1+\cdots+m_{2c}}\beta \\ \qquad\qquad \textit{if} \quad m_{2i-1} + m_{2i} \geq N \ (1 \leq i \leq c), \\ 0 \qquad\qquad \textit{otherwise}, \end{cases}$$

$$\pi_{2c+1}(\alpha[x_1|x_2|\cdots|x_{2c+1}]\beta) = \begin{cases} \sum_{j=1}^{m_1} \alpha\alpha_1\cdots\alpha_{j-1} \otimes \alpha_j\cdots\alpha_{j+cN}\otimes \\ \qquad\qquad \alpha_{j+cN+1}\cdots\alpha_{m_1+\cdots+m_{2c+1}}\beta \\ \qquad\qquad \textit{if} \quad m_{2i} + m_{2i+1} \geq N \ (1 \leq i \leq c), \\ 0 \qquad\qquad \textit{otherwise}, \end{cases}$$

for $\alpha, \beta \in A$.

We denote by Ψ the composition map of the following chain maps:

$$\mathrm{Bar}_*(A) \xrightarrow{\theta} \boldsymbol{P''} \xrightarrow{\pi} \boldsymbol{P'} \xrightarrow{\sim} \boldsymbol{P}$$

Then, $\Psi\Phi = \mathrm{id}_{\boldsymbol{P}}$. In particular, the chain map Ψ coincides with the chain map G given in [11] for truncated quiver algebras.

We will compute BV differentials on Hochschild cohomology of self-injective Nakayama algebras by means of Φ and Ψ.

§4. Hochschild cohomology of self-injective Nakayama algebras 1: Case (a)

Frobenius algebras over K with a Nakayama automorphism ν of finite order are divided into the following two cases:

Case (a): $\operatorname{char} K$ does not divide $\operatorname{ord} \nu$.

Case (b): $\operatorname{char} K$ divides $\operatorname{ord} \nu$.

In Case (a), ν is always diagonalizable.

In this section, we assume Case (a) and we recall the basis of the n-th Hochschild cohomology groups of self-injective Nakayama algebras given in [3]. Moreover, by means of comparison morphisms in Section 3, we compute BV differentials. Focusing on the self-injective Nakayama algebra $\Lambda = KZ_e/J^N$ in Section 2, we recall the notation for complexes derivered from projective resolutions as bimodules for truncated quiver algebras in [3].

For paths $p \in \Lambda$ and $q \in AP(n)$, (p, q) denotes the pair of paths p, q such that $s(p) = s(q)$ and $t(p) = t(q)$. In [3], the Hochschild cohomology group $\mathrm{HH}^n(\Lambda)$ of Λ is computed by means of the following diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\Lambda^e}(P_n, \Lambda) \arrow[rr, "\mathrm{Hom}_{\Lambda^e}(d_{n+1}, \Lambda)"] & & \mathrm{Hom}_{\Lambda^e}(P_{n+1}, \Lambda) \arrow[d, "\cong"] \\
\coprod_{p \in AP(n)} \mathrm{Hom}_{\Lambda^e}(\Lambda s(p) \otimes t(p)\Lambda, \Lambda) \arrow[u, "\cong"] & & \coprod_{q \in AP(n+1)} \mathrm{Hom}_{\Lambda^e}(\Lambda s(q) \otimes t(q)\Lambda, \Lambda) \arrow[d, "\mu^{-1}"] \\
\coprod_{p \in AP(n)} (s(p)\Lambda t(p), p) \arrow[u, "\mu"] \arrow[rr, "\phi^*_{n+1}"] & & \coprod_{q \in AP(n+1)} (s(q)\Lambda t(q), q)
\end{tikzcd}$$

where $\mu : \coprod_{p \in AP(n)} (s(p)\Lambda t(p), p) \to \coprod_{p \in AP(n)} \mathrm{Hom}_{\Lambda^e}(\Lambda s(p) \otimes t(p)\Lambda, \Lambda)$ is defined by

$$[\mu(m_p, p)](s(q) \otimes t(q)) = \begin{cases} m_p & \text{if } q = p, \\ 0 & \text{if } q \neq p, \end{cases}$$

for $m_p \in s(p)\Lambda t(p)$ and $p, q \in AP(n)$. Moreover, ϕ^*_n is given by

$$\phi^*_{2i+1}((m_p, p)_{p \in AP(2i)}) = ({}^1q m_{q_2} - m_{q_1} q^1, q)_{q \in AP(2i+1)},$$
$$\phi^*_{2i}((m_p, p)_{p \in AP(2i-1)}) = (\sum_{j=1}^{N} {}^{j-1}q m_{q_j} q^{N-j}, q)_{q \in AP(2i)},$$

where, for $q \in AP(n)$, ${}^kq$ denotes a path of length k such that $q = {}^kq q'$ for some path q', q^l denotes a path of length l such that $q = q'' q^l$ for some path q'', ${}^1q q_2 = q_1 q^1 = q \in AP(2i+1)$, and ${}^{j-1}q q_j q^{N-j} = q \in AP(2i)$.

A basis of $\coprod_{p \in AP(n)} (s(p)\Lambda t(p), p)$ is given by

$$\begin{cases} \{(\gamma_l^j, \gamma_l^{Ni}) \mid 1 \le l \le e,\ 0 \le j \le N-1\} & \text{if } n = 2i, \\ \{(\gamma_l^j, \gamma_l^{Ni+1}) \mid 1 \le l \le e,\ 0 \le j \le N-1\} & \text{if } n = 2i+1. \end{cases}$$

Moreover, we regard the notation μ again for the composition

$$\coprod_{p \in AP(n)} (s(p)\Lambda t(p), p) \xrightarrow{\mu} \coprod_{p \in AP(n)} \mathrm{Hom}_{\Lambda^e}(\Lambda s(p) \otimes t(p)\Lambda, \Lambda) \xrightarrow{\sim} \mathrm{Hom}_{\Lambda^e}(P_n, \Lambda)$$

if there is no confusion.

We will compute the BV differential by dividing the Hochschild cohomology groups $\mathrm{HH}^n(\Lambda)$ into the following cases: $\mathrm{HH}^1(\Lambda)$; $\mathrm{HH}^{2i}(\Lambda)$; $\mathrm{HH}^{2i+1}(\Lambda)$ for $i \geq 1$.

Let $N = me+t (m \geq 0, 0 \leq t \leq e-1)$, $g_0 = \gcd(N-1, e)$ and $e_0 = \frac{e}{g_0}$. Then $\mathrm{ord}(\nu) = e_0$ and $\mathrm{char}\, K \nmid e_0$. We recall the basis of Hochschild cohomology groups $\mathrm{HH}^n(\Lambda)$ in [3].

Proposition 4.1 ([3, Proposition 5.1]). *If $N > 2$, then a K-basis of $\mathrm{HH}^0(\Lambda)$ is given by*

$$\begin{cases} B = \left\{ \sum_{l=1}^{e} (\gamma_l^{ae}, v_l) \mid 0 \leq a \leq \left[\dfrac{N-2}{e}\right] \right\} & \text{if } N \not\equiv 1 \pmod{e}, \\ B \cup \{ (\gamma_l^{N-1}, v_l) \mid 0 \leq l \leq e \} & \text{if } N \equiv 1 \pmod{e}. \end{cases}$$

Proposition 4.2 ([3, Proposition 5.2]). *For $i \geq 1$, a K-basis of $\mathrm{HH}^{2i}(\Lambda)$ is given by*

$$\begin{cases} B = \left\{ \sum_{l=1}^{e} (\gamma_l^{j}, \gamma_l^{Ni}) \mid 0 \leq j \leq N-2 \text{ and } j \equiv Ni \pmod{e} \right\} \\ \qquad\qquad \text{if } \mathrm{char}\, K \nmid N \text{ or } Ni \not\equiv N-1 \pmod{e}, \\ B \cup \{ \sum_{l=1}^{e} (\gamma_l^{N-1}, \gamma_l^{Ni}) \} \quad \text{if } \mathrm{char}\, K \mid N \text{ and } Ni \equiv N-1 \pmod{e}. \end{cases}$$

Proposition 4.3 ([3, Proposition 5.3]). *For $i \geq 1$, a K-basis of $\mathrm{HH}^{2i-1}(\Lambda)$ is given by*

$$\begin{cases} B = \\ \left\{ \sum_{k=0}^{e_0-1} (\gamma_{1+kg_0}^{j+1}, \gamma_{1+kg_0}^{N(i-1)+1}) \mid 0 \leq j \leq N-2 \text{ and } j \equiv N(i-1) \pmod{e} \right\} \\ \qquad\qquad \text{if } \mathrm{char}\, K \nmid N \text{ or } Ni \not\equiv N-1 \pmod{e}, \\ B \cup \{ \sum_{l=1}^{e} (v_l, \gamma_l^{N(i-1)+1}) \} \quad \text{if } \mathrm{char}\, K \mid N \text{ and } Ni \equiv N-1 \pmod{e}. \end{cases}$$

For the images of μ with the basis elements above, we will compute BV differentials. We fix the K-basis $\mathcal{B} = \{\gamma_l^j \mid 1 \leq l \leq e \text{ and } 0 \leq j \leq N-1\}$ of Λ.

Theorem 4.4. *The equation $\Delta(\sum_{k=0}^{e_0-1} (\gamma_{1+kg_0}^{j+1}, \gamma_{1+kg_0}^{1})) = \frac{N-j-1}{g_0} \sum_{l=1}^{e} (\gamma_l^j, v_l)$ holds for $\sum_{k=0}^{e_0-1} (\gamma_{1+kg_0}^{j+1}, \gamma_{1+kg_0}^{1}) \in \mathrm{HH}^1(\Lambda)$, $0 \leq j \leq N-2$ and $j \equiv 0 \pmod{e}$.*

Proof. We set $\omega_j = \mu(\sum_{k=0}^{e_0-1} (\gamma_{1+kg_0}^{j+1}, \gamma_{1+kg_0}^{N(i-1)+1}))$, where $0 \leq j \leq N-2$ and $j \equiv 0 \pmod{e}$. Then,

$$\Delta\omega_j(v_l \otimes v_l) = \Delta_1 \omega_j(v_l \otimes v_l)$$

$$
\begin{aligned}
&= \sum_{b\in\mathcal{B}} \langle \omega_j \Psi_1(1\otimes b\otimes 1), 1\rangle v_l b^* v_l \\
&= \sum_{k=0}^{[\frac{N-j-1}{e}]} \langle \omega_j \Psi_1(1\otimes \gamma_{l+j}^{N-j-1-ek}\otimes 1), 1\rangle v_l (\gamma_{l+j}^{N-j-1-ek})^* v_l \\
&= \langle \omega_j \Psi_1(1\otimes \gamma_{l+j}^{N-j-1}\otimes 1), 1\rangle v_l (\gamma_{l+j}^{N-j-1})^* v_l \\
&= \frac{N-j-1}{g_0}\gamma_l^j.
\end{aligned}
$$

Thus we have $\Delta\omega_j = \dfrac{N-j-1}{g_0}\mu(\sum_{l=1}^{e}(\gamma_l^j, v_l))$. $\square$

Theorem 4.5. *Let i be a positive integer. For any $w \in \mathrm{HH}^{2i}(\Lambda)$, $\Delta(w) = 0$ holds.*

Proof. We set $\omega_j = \mu(\sum_{l=1}^{e}(\gamma_l^j, \gamma_l^{Ni})) \in \mathrm{HH}^{2i}(\Lambda)$, where $0 \le j \le N-2$ and $j \equiv Ni \pmod e$. We have

$$
\begin{aligned}
&\Phi_{2i-1}(s(\gamma_l^{N(i-1)+1})\otimes t(\gamma_l^{N(i-1)+1})) \\
&\quad = \sum_{0\le j_1,\dots,j_{i-1}\le N-2} s(\gamma_l^{N(i-1)+1})\otimes \gamma_l^1 \otimes \gamma_{l+1}^{j_1+1}\otimes \gamma_{l+2+j_1}^1 \otimes \gamma_{l+3+j_1}^{j_2+1} \\
&\quad\quad \otimes \gamma_{l+4+j_1+j_2}^1 \otimes\cdots\otimes \gamma_{l-1+2(i-1)+j_1+\cdots+j_{i-2}}^{j_{i-1}+1} \otimes \gamma_{l+2(i-1)+j_1+\cdots+j_{i-1}}^1 \\
&\quad\quad\quad \otimes \gamma_{l+1+2(i-1)+j_1+\cdots+j_{i-1}}^{N(i-1)-(2(i-1)+j_1+\cdots+j_{i-1})}.
\end{aligned}
$$

By the map π_{2c}, we only need to consider the summand

$$
\begin{aligned}
s(\gamma_l^{N(i-1)+1}) &\otimes \gamma_l^1 \otimes \gamma_{l+1}^{N-1}\otimes \gamma_{l+N}^1 \otimes\cdots \\
&\otimes \gamma_{l+N(i-2)+1}^{N-1}\otimes \gamma_{l+N(i-1)}^1 \otimes t(\gamma_l^{N(i-1)+1}).
\end{aligned}
$$

For $b \in \mathcal{B}$, we set

$$
x_b := 1\otimes \gamma_l^1 \otimes \gamma_{l+1}^{N-1}\otimes \gamma_{l+N}^1 \otimes\cdots\otimes \gamma_{l+N(i-2)+1}^{N-1}\otimes \gamma_{l+N(i-1)}^1 \otimes b\otimes 1.
$$

Then, it suffices to assume that $N-1 \equiv Ni \pmod e$. By $j \le N-2$,

$$
\begin{aligned}
&\sum_{b\in\mathcal{B}} \langle \omega_j(\Psi(x_b)), 1\rangle s(\gamma_l^{N(i-1)+1}) b^* t(\gamma_l^{N(i-1)+1}) \\
&= \langle \omega_j(\Psi(x_{\gamma_{l+N(i-1)+1}^{N-1}})), 1\rangle s(\gamma_l^{N(i-1)+1})(\gamma_{l+N(i-1)+1}^{N-1})^* t(\gamma_l^{N(i-1)+1}) \\
&= \langle \gamma_l^j, 1\rangle s(\gamma_l^{N(i-1)+1})(\gamma_{l+N(i-1)+1}^{N-1})^* t(\gamma_l^{N(i-1)+1}) \\
&= 0.
\end{aligned}
$$

Thus we obtain $\Delta_1\omega_j = 0$. Similarly, we have $\Delta_k\omega_j = 0$ for $1 \leq k \leq 2i$. Therefore, $\Delta\omega_j = 0$.

We suppose that $\operatorname{char} K \mid N$ and $Ni \equiv N-1 \pmod{e}$ and we set $\omega = \mu(\sum_{l=1}^{e}(\gamma_l^{N-1}, \gamma_l^{Ni})) \in \mathrm{HH}^{2i}(\Lambda)$. We will compute $\Delta_1\omega$. By the above,

$$\begin{aligned}
&\sum_{b\in\mathcal{B}} \langle \omega(\Psi(x_b)), 1\rangle s(\gamma_l^{N(i-1)+1}) b^* t(\gamma_l^{N(i-1)+1})\\
&= \langle \omega(\Psi(x_{\gamma_{l+N(i-1)+1}^{N-1}})), 1\rangle s(\gamma_l^{N(i-1)+1})(\gamma_{l+N(i-1)+1}^{N-1})^* t(\gamma_l^{N(i-1)+1})\\
&= \langle \gamma_l^{N-1}, 1\rangle s(\gamma_l^{N(i-1)+1}) v_{l+N(i-1)+1} t(\gamma_l^{N(i-1)+1})\\
&= v_l.
\end{aligned}$$

Thus, $\Delta_1\omega(s(\gamma_l^{N(i-1)+1}) \otimes t(\gamma_l^{N(i-1)+1})) = v_l$. Similarly, for $2 \leq k \leq 2i$, we have $\Delta_k\omega(s(\gamma_l^{N(i-1)+1}) \otimes t(\gamma_l^{N(i-1)+1})) = v_l$, so $\Delta_k\omega = \mu(\sum_{l=1}^{e}(v_l, \gamma_l^{N(i-1)+1}))$. Therefore,

$$\Delta\omega = \sum_{k=1}^{2i}(-1)^{k(2i-1)}\mu(\sum_{l=1}^{e}(v_l, \gamma_l^{N(i-1)+1})) = 0.$$

$\square$

Theorem 4.6. *Let i be a positive integer. For $0 \leq j \leq N-2$ and $j \equiv Ni \pmod{e}$, we have $\Delta(\sum_{k=0}^{e_0-1}(\gamma_{1+kg_0}^{j+1}, \gamma_{1+kg_0}^{Ni+1})) = (Ni+N-j-1)(\sum_{l=1}^{e}(\gamma_l^{j}, \gamma_l^{Ni}))$. If $\operatorname{char} K \mid N$ and $Ni \equiv N-1 \pmod{e}$, then $\Delta(\sum_{l=1}^{e}(v_l, \gamma_l^{Ni+1})) = 0$ holds.*

Proof. For $s(\gamma_l^{Ni}) \otimes t(\gamma_l^{Ni}) \in \Lambda s(\gamma_l^{Ni}) \otimes t(\gamma_l^{Ni})\Lambda$, we have

$$\begin{aligned}
&\Phi_{2i}(s(\gamma_l^{Ni}) \otimes t(\gamma_l^{Ni}))\\
&= \sum_{0\leq j_1,\dots,j_i\leq N-2} s(\gamma_l^{Ni}) \otimes \gamma_l^{j_1+1} \otimes \gamma_{l+1+j_1}^{1} \otimes \gamma_{l+2+j_1}^{j_2+1} \otimes \gamma_{l+3+j_1+j_2}^{1} \otimes \cdots\\
&\qquad \otimes \gamma_{l+2i-2+j_1+\cdots+j_{i-1}}^{j_i+1} \otimes \gamma_{l+2i-1+j_1+\cdots+j_i}^{1} \otimes \gamma_{l+2i+j_1+\cdots+j_i}^{Ni-(2i+j_1+\cdots+j_i)} t(\gamma_l^{Ni}).
\end{aligned}$$

Let $\omega_j = \mu(\sum_{k=0}^{e_0-1}(\gamma_{1+kg_0}^{j+1}, \gamma_{1+kg_0}^{Ni+1}))$, where $0 \leq j \leq N-2$ and $j \equiv Ni \pmod{e}$. First, we will compute Δ_{2c-1} for $1 \leq c \leq i$. We compute the following formula:

$$\begin{aligned}
&(\Delta_{2c-1}\omega_j)(s(\gamma_l^{Ni}) \otimes t(\gamma_l^{Ni}))\\
&= \sum_{b\in\mathcal{B}} \Bigg\langle \omega_j \Psi_{2i+1}\Bigg(\sum_{0\leq j_1,\dots,j_i\leq N-2} 1 \otimes \gamma_{l+2(c-1)+j_1+\cdots+j_{c-1}}^{j_c+1} \otimes \gamma_{l+2c-1+j_1+\cdots+j_c}^{1}\\
&\qquad \otimes \cdots \otimes \gamma_{l+2i-2+j_1+\cdots+j_{i-1}}^{j_i+1} \otimes \gamma_{l+2i-1+j_1+\cdots+j_i}^{1} \otimes b \otimes \nu(\gamma_l^{j_1+1}) \otimes \nu(\gamma_{l+1+j_1}^{1})\\
&\qquad \otimes \cdots \otimes \nu(\gamma_{l+2(c-2)+j_1+\cdots+j_{c-2}}^{j_{c-1}+1}) \otimes \nu(\gamma_{l+2c-3+j_1+\cdots+j_{c-1}}^{1}) \otimes 1\Bigg), 1\Bigg\rangle
\end{aligned}$$

$$s(\gamma_l^{Ni})b^*\gamma_{l+2i+j_1+\cdots+j_i}^{Ni-(2i+j_1+\cdots+j_i)}.$$

By the map π_{2c+1}, we can suppose that $j_1 = \cdots = j_{c-1} = j_{c+1} = \cdots = j_i = N-2$. Then we have

$$\begin{aligned}
&(\Delta_{2c-1}\omega_j)(s(\gamma_l^{Ni}) \otimes t(\gamma_l^{Ni})) \\
&= \sum_{b\in\mathcal{B}} \Big\langle \omega_j \Psi_{2i+1} \Big(\sum_{0\le j_c \le N-2} 1 \otimes \gamma_{l+N(c-1)}^{j_c+1} \otimes \gamma_{l+N(c-1)+1+j_c}^{1} \otimes \gamma_{l+N(c-1)+2+j_c}^{N-1} \\
&\qquad \otimes \cdots \otimes \gamma_{l+N(i-2)+1+j_c}^{1} \otimes \gamma_{l+N(i-2)+2+j_c}^{N-1} \otimes \gamma_{l+N(i-1)+1+j_c}^{1} \otimes b \\
&\qquad \otimes \nu(\gamma_l^{N-1}) \otimes \nu(\gamma_{l+N-1}^{1}) \otimes \cdots \otimes \nu(\gamma_{l+N(c-2)}^{N-1}) \otimes \nu(\gamma_{l+N(c-1)-1}^{1}) \otimes 1 \Big), 1 \Big\rangle \\
&\qquad\qquad s(\gamma_l^{Ni}) b^* \gamma_{l+N(i-1)+2+j_c}^{N-(2+j_c)} t(\gamma_l^{Ni}) \\
&= \Big\langle \omega_j \Psi_{2i+1} \Big(\sum_{0\le j_c \le N-2} 1 \otimes \gamma_{l+N(c-1)}^{j_c+1} \otimes \gamma_{l+N(c-1)+1+j_c}^{1} \otimes \gamma_{l+N(c-1)+2+j_c}^{N-1} \\
&\qquad \otimes \cdots \otimes \gamma_{l+N(i-2)+1+j_c}^{1} \otimes \gamma_{l+N(i-2)+2+j_c}^{N-1} \otimes \gamma_{l+N(i-1)+1+j_c}^{1} \otimes \gamma_l^{N-1} \\
&\qquad \otimes \nu(\gamma_l^{N-1}) \otimes \nu(\gamma_{l+N-1}^{1}) \otimes \cdots \otimes \nu(\gamma_{l+N(c-2)}^{N-1}) \otimes \nu(\gamma_{l+N(c-1)-1}^{1}) \otimes 1 \Big), 1 \Big\rangle \\
&\qquad\qquad s(\gamma_l^{Ni}) (\gamma_l^{N-1})^* \gamma_{l+N(i-1)+2+j_c}^{N-(2+j_c)} t(\gamma_l^{Ni}).
\end{aligned}$$

By the map w_j and the bilinear form $\langle\ ,\ \rangle$,

$$\begin{aligned}
&\Delta_{2c-1}\omega_j(s(\gamma_l^{Ni}) \otimes t(\gamma_l^{Ni})) \\
&= \Big\langle \omega_j \Psi_{2i+1} \Big(1 \otimes \gamma_{l+N(c-1)}^{N-j-1} \otimes \gamma_{l+Nc-j-1}^{1} \otimes \gamma_{l+Nc-j}^{N-1} \otimes \cdots \otimes \gamma_{l+N(i-1)-j-1}^{1} \\
&\qquad \otimes \gamma_{l+Ni-j}^{N-1} \otimes \gamma_{l+Ni-j-1}^{1} \otimes \gamma_l^{N-1} \otimes \nu(\gamma_l^{N-1}) \otimes \nu(\gamma_{l+N-1}^{1}) \otimes \cdots \\
&\qquad \otimes \nu(\gamma_{l+N(c-2)}^{N-1}) \otimes \nu(\gamma_{l+N(c-1)-1}^{1}) \otimes 1 \Big), 1 \Big\rangle \\
&\qquad\qquad s(\gamma_l^{Ni}) (\gamma_l^{N-1})^* \gamma_{l+Ni-j}^{j} t(\gamma_l^{Ni}) \\
&= \Big\langle \omega_j \Big(\sum_{p=1}^{N-j-1} \gamma_{l+N(c-1)}^{p-1} \otimes \gamma_{l+N(c-1)+Ni+p}^{N-j-p-1} v_{l+Nc-1} \Big), 1 \Big\rangle s(\gamma_l^{Ni}) \gamma_l^{j} t(\gamma_l^{Ni}).
\end{aligned}$$

Next, we will compute Δ_{2c} for $1 \le c \le i$. Then,

$$\begin{aligned}
&\Delta_{2c}\omega_j\ (s(\gamma_l^{Ni}) \otimes t(\gamma_l^{Ni})) \\
&= \sum_{b\in\mathcal{B}} \Big\langle \omega_j \Psi_{2i+1} \Big(\sum_{0\le j_1,\ldots,j_i \le N-2} 1 \otimes \gamma_{l+2c-1+j_1+\cdots+j_c}^{1} \otimes \gamma_{l+2c+j_1+\cdots+j_c}^{j_{c+1}+1} \otimes \cdots \\
&\otimes \gamma_{l+2i-3+j_1+\cdots+j_{i-1}}^{1} \otimes \gamma_{l+2i-2+j_1+\cdots+j_{i-1}}^{j_i+1} \otimes \gamma_{l+2i-1+j_1+\cdots+j_i}^{1} \otimes b \otimes \nu(\gamma_l^{j_1+1})
\end{aligned}$$

$$\otimes\,\nu(\gamma^{1}_{l+1+j_1})\otimes\cdots\otimes\nu(\gamma^{j_{c-1}+1}_{l+2(c-2)+j_1+\cdots+j_{c-2}})\otimes\nu(\gamma^{1}_{l+2c-3+j_1+\cdots+j_{c-1}})$$
$$\otimes\,\nu(\gamma^{j_c+1}_{l+2(c-1)+j_1+\cdots+j_{c-1}})\otimes 1\Big),1\Big\rangle s(\gamma^{Ni}_l)b^*\gamma^{Ni-(2i+j_1+\cdots+j_i)}_{l+2i+j_1+\cdots+j_i}t(\gamma^{Ni}_l).$$

By the map π_{2c+1}, it is sufficient to consider only the case $j_2=\cdots=j_i=N-2$. Moreover, by the bilinear form $\langle\ ,\ \rangle$, we have

$$\Delta_{2c}\omega_j(s(\gamma^{Ni}_l)\otimes t(\gamma^{Ni}_l))$$
$$=\sum_{b\in\mathcal{B}}\Big\langle\omega_j\Psi_{2i+1}\Big(\sum_{0\le j_1\le N-2}1\otimes\gamma^{1}_{l+j_1+N(c-1)+1}\otimes\gamma^{N-1}_{l+j_1+N(c-1)+2}\otimes\cdots$$
$$\otimes\,\gamma^{1}_{l+j_1+N(i-2)+1}\otimes\gamma^{N-1}_{l+j_1+N(i-2)+2}\otimes\gamma^{1}_{l+j_1+N(i-1)+1}\otimes v_{l+j_1+N(i-1)+2}bv_{l+N-1}$$
$$\otimes\,\nu(\gamma^{j_1+1}_{l})\otimes\nu(\gamma^{1}_{l+1+j_1})\otimes\nu(\gamma^{N-1}_{l+2+j_1})\otimes\cdots\otimes\nu(\gamma^{1}_{l+j_1+N(c-2)+1})$$
$$\otimes\,\nu(\gamma^{N-1}_{l+j_1+N(c-2)+2})\otimes 1\Big),1\Big\rangle s(\gamma^{Ni}_l)b^*\gamma^{N-j_1-2}_{l+j_1+N(i-1)+2}t(\gamma^{Ni}_l)$$
$$=\sum_{0\le j_1\le N-2}\ \sum_{\substack{0\le p\le j_1\\ j+p\equiv N-2 \pmod e}}$$
$$\Big\langle\omega_j\Psi_{2i+1}\Big(1\otimes\gamma^{1}_{l+j_1+N(c-1)+1}\otimes\gamma^{N-1}_{l+j_1+N(c-1)+2}\otimes\cdots\otimes\gamma^{1}_{l+j_1+N(i-2)+1}$$
$$\otimes\,\gamma^{N-1}_{l+j_1+N(i-2)+2}\otimes\gamma^{1}_{l+j_1+N(i-1)+1}\otimes v_{l+j_1+N(i-1)+1}\gamma^{N-j_1-1+p}_{l+j_1+N(i-1)+2}v_{l+N-1}$$
$$\otimes\,\nu(\gamma^{j_1+1}_{l})\otimes\nu(\gamma^{1}_{l+1+j_1})\otimes\cdots\otimes\nu(\gamma^{N-1}_{l+j_1+N(c-3)+2})$$
$$\otimes\,\nu(\gamma^{1}_{l+j_1+N(c-2)+1})\otimes\nu(\gamma^{N-1}_{l+j_1+N(c-2)+2})\otimes 1\Big),1\Big\rangle$$
$$s(\gamma^{Ni}_l)(\gamma^{N-j_1-1+p}_{l+j_1+N(i-1)+2})^*\gamma^{N-j_1-2}_{l+j_1+N(i-1)+2}t(\gamma^{Ni}_l)$$
$$=\sum_{0\le j_1\le N-2}\ \sum_{\substack{0\le p\le j_1\\ j+p\equiv N-2 \pmod e}}\Big\langle\omega_j(v_{l+j_1+N(c-1)+1}\otimes\gamma^{p}_{l+N(c-1)+Ni+2}),1\Big\rangle$$
$$s(\gamma^{Ni}_l)\gamma^{j_1-p}_{l+N(i-1)+2+p}\gamma^{N-j_1-2}_{l+j_1+N(i-1)+2}t(\gamma^{Ni}_l)$$
$$=\sum_{N-j-2\le j_1\le N-2}\Big\langle\omega_j\Big(v_{l+j_1+N(c-1)+1}\otimes\gamma^{N-j-2}_{l+j_1+N(c-1)+Ni+2}\Big),1\Big\rangle$$
$$s(\gamma^{Ni}_l)\gamma^{j}_{l+Ni-j}t(\gamma^{Ni}_l)$$
$$=\sum_{j_1=N-j}^{N}\Big\langle\omega_j\Big(v_{l+N(c-1)+j_1-1}\otimes\gamma^{N-j-2}_{l+N(c-1)+Ni+j_1}\Big),1\Big\rangle\gamma^{j}_{l}.$$

Finally, we compute Δ_{2i+1}. We have

$$\begin{aligned}
&\Delta_{2i+1}\omega_j(s(\gamma_l^{Ni})\otimes t(\gamma_l^{Ni}))\\
&=\sum_{b\in\mathcal{B}}\Big\langle\omega_j\Psi_{2i+1}\Big(\sum_{0\leq j_1,\dots,j_i\leq N-2}1\otimes b\otimes\nu(\gamma_l^{j_1+1})\otimes\nu(\gamma_{l+1+j_1}^1)\otimes\cdots\\
&\qquad\otimes\nu(\gamma_{l+2i-2+j_1+\cdots+j_{i-1}}^{j_i+1})\otimes\nu(\gamma_{l+2i-1+j_1+\cdots+j_i}^1)\otimes 1\Big),1\Big\rangle\\
&\qquad\qquad s(\gamma_l^{Ni})b^*\gamma_{l+2i+j_1+\cdots+j_i}^{Ni-(2i+j_1+\cdots+j_i)}t(\gamma_l^{Ni})\\
&=\sum_{b\in\mathcal{B}}\Big\langle\omega_j\Psi_{2i+1}(1\otimes b\otimes\nu(\gamma_l^{N-1})\otimes\nu(\gamma_{l+N-1}^1)\otimes\cdots\\
&\qquad\otimes\nu(\gamma_{l+N(i-1)}^{N-1})\otimes\nu(\gamma_{l+Ni-1}^1)\otimes 1),1\Big\rangle s(\gamma_l^{Ni})b^*t(\gamma_l^{Ni})\\
&=\langle\omega_j\Psi_{2i+1}(1\otimes\gamma_{l+j}^{N-j-1}\otimes\nu(\gamma_l^{N-1})\otimes\nu(\gamma_{l+N-1}^1)\otimes\cdots\\
&\qquad\otimes\nu(\gamma_{l+N(i-1)}^{N-1})\otimes\nu(\gamma_{l+Ni-1}^1)\otimes 1),1\rangle s(\gamma_l^{Ni})(\gamma_{l+j}^{N-j-1})^*t(\gamma_l^{Ni})\\
&=\langle\omega_j(\sum_{p=1}^{N-j-1}\gamma_{l+j}^{p-1}\otimes\gamma_{l+j+Ni+p}^{N-j-1-p}),1\rangle s(\gamma_l^{Ni})\gamma_l^j t(\gamma_l^{Ni}).
\end{aligned}$$

Hence we have

$$\Delta\omega_j(s(\gamma_l^{Ni})\otimes t(\gamma_l^{Ni}))=\sum_{k=1}^{2i+1}\Delta_k(s(\gamma_l^{Ni})\otimes t(\gamma_l^{Ni}))=\frac{Ni+N-j-1}{g_0}\gamma_l^j.$$

Therefore, we obtain

$$\Delta\omega_j=\frac{Ni+N-j-1}{g_0}\mu(\sum_{l=1}^{e}(\gamma_l^j,\gamma_l^{Ni})).$$

On the other hand, if $\operatorname{char}K\mid N$ and $N(i+1)\equiv N-1 \pmod e$, then it is easily shown that $\Delta\mu(\sum_{l=1}^e(v_l,\gamma_l^{Ni}))=0$ by the bilinear form $\langle\ ,\ \rangle$. $\square$

Corollary 4.7 (cf. [23]). *We set*

$$\begin{aligned}
x_{i,j}&=\sum_{l=1}^{e}(\gamma_l^j,\gamma_l^{Ni})\in\mathrm{HH}^{2i}(\Lambda),\\
y_{i',j'}&=\sum_{l=1}^{e}(\gamma_l^{j'+1},\gamma_l^{Ni'+1})\in\mathrm{HH}^{2i'+1}(\Lambda)
\end{aligned}$$

and $w_{i'',0} = \sum_{l=1}^{e}(v_l, \gamma_l^{Ni''+1}) \in \mathrm{HH}^{2i''+1}(\Lambda)$ (*if* $\mathrm{char}\, K \mid N$ *and* $Ni'' \equiv N-1 \pmod e$), *where* $i, i', i'' \geq 0$ *and* j *satisfies*

$$\begin{cases} 0 \leq j \leq N-1 & \textit{if } \mathrm{char}\, K \mid N \textit{ and } Ni \equiv N-1 \pmod e, \\ 0 \leq j \leq N-2 & \textit{otherwise.} \end{cases}$$

Then the bracket $[\ ,\ ]$ *on* $\mathrm{HH}^*(\Lambda)$ *is given as follows*:

$$[x_{i,j}, x_{i',j'}] = [y_{i,j}, y_{i',j'}] = [w_{i,0}, w_{i',0}] = [x_{i,j}, w_{i',0}] = [y_{i,j}, w_{i',0}] = 0,$$

$$[x_{i,j}, y_{i',j'}] = \begin{cases} -(Ni-j)x_{i+i',j+j'} \\ \quad \textit{if } j+j' \leq N-2, \textit{ and } \mathrm{char}\, K \nmid N \textit{ or } N(i+i') \not\equiv N-1 \pmod e, \\ jx_{i+i',j+j'} \\ \quad \textit{if } j+j' \leq N-2,\ \mathrm{char}\, K \mid N \textit{ and } N(i+i') \equiv N-1 \pmod e, \\ -(j+1)x_{i+i',j+j'} \\ \quad \textit{if } j+j' = N-1,\ \mathrm{char}\, K \mid N \textit{ and } N(i+i') \equiv N-1 \pmod e, \\ 0 \qquad \textit{otherwise.} \end{cases}$$

§5. Hochschild cohomology of self-injective Nakayama algebras 2: Case (b)

Let $N = me + t (m \geq 0, 0 \leq t \leq e-1, t \neq 1)$ and $g_0 = \gcd(N-1, e)$. Then $\mathrm{ord}(\nu) = \frac{e}{g_0}$. Throughout this section, for a self-injective Nakayama algebra $\Lambda = Z_e/J^N$ with the Nakayama automorphism ν, we assume that

$$\text{Case (b)}: \ \mathrm{char}\, K \mid \mathrm{ord}(\nu) = \frac{e}{g_0}.$$

In this case, $N \not\equiv 1 \pmod e$ and the Nakayama automorphism ν is not necessarily diagonalizable. In this section, we compute $\mathrm{HH}^*(\Lambda)^{\nu\uparrow}$ and BV differential on $\mathrm{HH}^*(\Lambda)^{\nu\uparrow}$ defined by Volkov [22].

We consider the chain map

$$\mathrm{Hom}_{\Lambda^e}(\Phi, \Lambda)\psi^{-1}\phi_\nu\psi\mathrm{Hom}_{\Lambda^e}(\Psi, \Lambda) : \mathrm{Hom}_{\Lambda^e}(\boldsymbol{P}, \Lambda) \to \mathrm{Hom}_{\Lambda^e}(\boldsymbol{P}, \Lambda),$$

where $\psi : \mathrm{Hom}_{\Lambda^e}(\mathrm{Bar}(\Lambda), \Lambda) \to (C^*(\Lambda), \delta_*)$ is an isomorphism given by

$$[\psi f](a_1 \otimes \cdots \otimes a_n) = f(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1),$$

for $f \in \mathrm{Hom}_{\Lambda^e}(\Lambda^{\otimes n+2}, \Lambda)$. This map induces an isomorphism of Hochschild cohomology. We denote the above chain map by F_ν.

Lemma 5.1. *We set* $\mathrm{Hom}_{\Lambda^e}(P_n,\Lambda)^\nu = \{f \in \mathrm{Hom}_{\Lambda^e}(P_n,\Lambda) \mid F_\nu f = f\}$. *Then* $\mathrm{Hom}_{\Lambda^e}(P_n,\Lambda)^\nu$ *is the following*:

$$\mathrm{Hom}_{\Lambda^e}(P_{2i},\Lambda)^\nu = K\{f^i_{\phi_{j,l}} \mid 1 \le l \le g_0,\ 0 \le j \le N-1,\ j \equiv Ni \pmod{e}\},$$

$$\mathrm{Hom}_{\Lambda^e}(P_{2i-1},\Lambda)^\nu = \begin{cases} K\{g^i_{\phi_{j,l}} \mid 1 \le l \le g_0,\ -1 \le j \le N-2,\ j \equiv N(i-1) \pmod{e}\} \\ \qquad \text{if } \gcd(N,e)=1 \text{ and } Ni \equiv N-1 \pmod{e}, \\ K\{g^i_{\phi_{j,l}} \mid 1 \le l \le g_0,\ 0 \le j \le N-2,\ j \equiv N(i-1) \pmod{e}\} \\ \qquad \text{if } Ni \not\equiv N-1 \pmod{e}, \end{cases}$$

for $i \ge 0$, *where* $f_{\phi_{j,l}}$ *and* $g_{\phi_{j,l}}$ *are given by*

$$f^i_{\phi_{j,l}} = \mu\Big(\sum_{k=0}^{\frac{e}{g_0}-1} (\gamma^j_{l+k(N-1)}, \gamma^{Ni}_{l+k(N-1)})\Big),$$

$$g^i_{\phi_{j,l}} = \mu\Big(\sum_{k=0}^{\frac{e}{g_0}-1} (\gamma^{j+1}_{l+k(N-1)}, \gamma^{N(i-1)+1}_{l+k(N-1)})\Big).$$

In particular, if $e \mid N$, *that is,* $t=0$, *then*

$$\mathrm{Hom}_{\Lambda^e}(P_{2i},\Lambda)^\nu = K\{f^i_{\phi_{j,l}} \mid 1 \le l \le g_0,\ 0 \le j \le N-2,\ j \equiv Ni \pmod{e}\}.$$

Proof. For $f \in \mathrm{Hom}_{\Lambda^e}(P_{2i},\Lambda)$ and $s(\gamma^{Ni}_l) \otimes t(\gamma^{Ni}_l) \in P_{2i}$, we have

$$\begin{aligned}
&[F_\nu f](s(\gamma^{Ni}_l) \otimes t(\gamma^{Ni}_l)) \\
&= [\mathrm{Hom}_{\Lambda^e}(\Phi,\Lambda)\psi^{-1}\phi_\nu\psi\mathrm{Hom}_{\Lambda^e}(\Psi,\Lambda)f](s(\gamma^{Ni}_l) \otimes t(\gamma^{Ni}_l)) \\
&= s(\gamma^{Ni}_l)\nu^{-1}\Bigg(f\Psi\Bigg(\sum_{0 \le j_1,\dots,j_i \le N-2} 1 \otimes \nu(\gamma^{j_1+1}_l) \otimes \nu(\gamma^1_{l+1+j_1}) \otimes \nu(\gamma^{j_2+1}_{l+2+j_1}) \\
&\qquad \otimes \nu(\gamma^1_{l+3+j_1+j_2}) \otimes \cdots \otimes \nu(\gamma^{j_i+1}_{l+2i-2+j_1+\cdots+j_{i-1}}) \otimes \nu(\gamma^1_{l+2i-1+j_1+\cdots+j_i}) \otimes 1\Bigg)\Bigg) \\
&\qquad\qquad \gamma^{Ni-(2i+j_1+\cdots+j_i)}_{l+2i+j_1+\cdots+j_i} t(\gamma^{Ni}_l) \\
&= s(\gamma^{Ni}_l)\nu^{-1}(f(\nu(\gamma^{Ni}_l)) \otimes \nu(t(\gamma^{Ni}_l))))t(\gamma^{Ni}_l).
\end{aligned}$$

Similarly, for $g \in \mathrm{Hom}_{\Lambda^e}(P_{2c-1},\Lambda)$ and $s(R_{2c-1}) \otimes t(R_{2c-1}) \in P_{2c-1}$, we have

$$\begin{aligned}
&[F_\nu g](s(R_{2c-1}) \otimes t(R_{2c-1})) \\
&\quad = s(R_{2c-1})\nu^{-1}(g(\nu(s(R_{2c-1})) \otimes \nu(t(R_{2c-1}))))t(R_{2c-1}).
\end{aligned}$$

Therefore, we can determine the set $\mathrm{Hom}_{\Lambda^e}(P_n,\Lambda)^\nu$ as above. □

Lemma 5.2. *The following hold*:

$$[\psi\mathrm{Hom}(\Psi,\Lambda)](\mathrm{Hom}(P_n,\Lambda)^\nu)\subset C^n(\Lambda)^\nu,$$
$$[\mathrm{Hom}(\Phi,\Lambda)\psi^{-1}](C^n(\Lambda)^\nu)\subset \mathrm{Hom}_{\Lambda^{\mathrm{e}}}(P_n,\Lambda)^\nu.$$

In particular, for $n\geq 0$, $H^n(\mathrm{Hom}_{\Lambda^{\mathrm{e}}}(\boldsymbol{P},\Lambda)^\nu)\cong \mathrm{HH}^n(\Lambda)^{\nu\uparrow}$.

Proof. We will check that $[\psi\mathrm{Hom}_{\Lambda^{\mathrm{e}}}(\Psi,\Lambda)](\mathrm{Hom}(P_n,\Lambda)^\nu)\subset C^n(\Lambda)^\nu$. For $f^i_{\phi_{j,l'}}\in \mathrm{Hom}_{\Lambda^{\mathrm{e}}}(P_{2i},\Lambda)^\nu$ and $\gamma_l^{j_1}\otimes\gamma_{l+j_1}^{j_2}\otimes\cdots\otimes\gamma^{j_{2i}}_{l+\sum_{k=1}^{2i-1}j_k}\in\Lambda^{\otimes 2i}$, where $l\equiv l'$ (mod g_0), $j_{2k-1}+j_{2k}\geq N$ for $1\leq k\leq i$, and $\sum_{k=1}^{2i}j_k\leq Ni+N-1$, we have

$$\begin{aligned}
&[\phi_\nu\psi\mathrm{Hom}(\Psi,\Lambda)(f^i_{\phi_{j,l'}})](\gamma_l^{j_1}\otimes\gamma_{l+j_1}^{j_2}\otimes\cdots\otimes\gamma^{j_{2i}}_{l+\sum_{k=1}^{2i-1}j_k})\\
&=\nu^{-1}\left([\psi(f^i_{\phi_{j,l'}}\Psi)](\nu(\gamma_l^{j_1})\otimes\nu(\gamma_{l+j_1}^{j_2})\otimes\cdots\otimes\nu(\gamma^{j_{2i}}_{l+\sum_{k=1}^{2i-1}j_k}))\right)\\
&=\nu^{-1}\left(f^i_{\phi_{j,l'}}(s(\nu(\gamma_l^{j_1}))\otimes\nu(\gamma_{l+Ni}^{\sum_{k=1}^{2i-1}j_k-Ni}))\right)\\
&=\nu^{-1}\left(s(\nu(\gamma_l^{j_1}))\sum_{k=0}^{\frac{e}{g_0}-1}\gamma^j_{l'+k(N-1)}\nu(\gamma_{l+Ni}^{\sum_{k=1}^{2i-1}j_k-Ni})\right)\\
&=s(\gamma_l^{j_1})\nu^{-1}(\sum_{k=0}^{\frac{e}{g_0}-1}\gamma^j_{l'+k(N-1)})\gamma_{l+Ni}^{\sum_{k=1}^{2i-1}j_k-Ni}\\
&=s(\gamma_l^{j_1})(\sum_{k=0}^{\frac{e}{g_0}-1}\gamma^j_{l'+k(N-1)})\gamma_{l+Ni}^{\sum_{k=1}^{2i-1}j_k-Ni}\\
&=[\psi\mathrm{Hom}_{\Lambda^{\mathrm{e}}}(\Psi,\Lambda)(f^i_{\phi_{j,l'}})](\gamma_l^{j_1}\otimes\gamma_{l+j_1}^{j_2}\otimes\cdots\otimes\gamma^{j_{2i}}_{l+\sum_{k=1}^{2i-1}j_k}).
\end{aligned}$$

For other elements in $\Lambda^{\otimes 2i}$ of the form $p_1\otimes\cdots\otimes p_{2i}$, where $p_1,\dots,p_{2i}$ are paths in Λ, we have

$$\begin{aligned}
[\phi_\nu\psi\mathrm{Hom}(\Psi,\Lambda)(f^i_{\phi_{j,l'}})](p_1\otimes\cdots\otimes p_{2i})&=[\psi\mathrm{Hom}(\Psi,\Lambda)(f^i_{\phi_{j,l'}})](p_1\otimes\cdots\otimes p_{2i})\\
&=0.
\end{aligned}$$

Hence, $[\psi\mathrm{Hom}(\Psi,\Lambda)](\mathrm{Hom}_{\Lambda^{\mathrm{e}}}(P_{2i},\Lambda)^\nu)\subset C^{2i}(\Lambda)^\nu$. By a similar computation, we have $\phi_\nu\psi\mathrm{Hom}(\Psi,\Lambda)(P_{2i+1},\Lambda)^\nu=\psi\mathrm{Hom}_{\Lambda^{\mathrm{e}}}(\Psi,\Lambda)$ for $i\geq 0$.

Next, we check that $[\mathrm{Hom}(\Phi,\Lambda)\psi^{-1}](C^n(\Lambda)^\nu)\subset\mathrm{Hom}_{\Lambda^{\mathrm{e}}}(P_n,\Lambda)^\nu$. For $f\in C^{2i}(\Lambda)^\nu$ and $s(\gamma_l^{Ni})\otimes t(\gamma_l^{Ni})\in P_{2i}$, we have

$$\begin{aligned}
&[F_\nu(\mathrm{Hom}(\Phi,\Lambda)(\psi^{-1}(f)))](s(\gamma_l^{Ni})\otimes t(\gamma_l^{Ni}))\\
&=[(\mathrm{Hom}(\Phi,\Lambda)\psi^{-1}\phi_\nu\psi\mathrm{Hom}(\Psi,\Lambda))(\mathrm{Hom}(\Phi,\Lambda)(\psi^{-1}f))](s(\gamma_l^{Ni})\otimes t(\gamma_l^{Ni}))
\end{aligned}$$

$$= [\psi^{-1}(\phi_\nu(\psi((\psi^{-1}f)\Phi\Psi)))]\Phi(s(\gamma_l^{Ni})\otimes t(\gamma_l^{Ni}))$$

$$= \sum_{0\le j_1,j_2,\dots,j_i\le N-2} s(\gamma_l^{Ni})\Big([(\phi_\nu(\psi((\psi^{-1}f)\Phi\Psi)))]$$

$$(\gamma_l^{j_1}\otimes\gamma_{l+j_1}^1\otimes\cdots\otimes\gamma^{j_i}_{l+2i-1+\sum_{k=1}^{i-1}j_k}\otimes\gamma^1_{l+2i-1+\sum_{k=1}^{i}j_k})\Big)\gamma^{Ni-(2i+\sum_{k=1}^i j_k)}_{l+2i+\sum_{k=1}^i j_k}$$

$$= \sum_{0\le j_1,j_2,\dots,j_i\le N-2} s(\gamma_l^{Ni})\Big(\nu^{-1}\big([\psi((\psi^{-1}f)\Phi\Psi)]$$

$$(\nu(\gamma_l^{j_1})\otimes\nu(\gamma_{l+j_1}^1)\otimes\cdots\otimes\nu(\gamma^{j_i}_{l+2i-1+\sum_{k=1}^{i-1}j_k})\otimes\nu(\gamma^1_{l+2i-1+\sum_{k=1}^{i}j_k}))\big)\Big)$$

$$\times\gamma^{Ni-(2i+\sum_{k=1}^i j_k)}_{l+2i+\sum_{k=1}^i j_k}$$

$$= \sum_{0\le j_1,j_2,\dots,j_i\le N-2} s(\gamma_l^{Ni})\Big(\nu^{-1}\Big([\psi^{-1}f](s(\nu(\gamma_l^{j_1}))\otimes\nu(\gamma_l^{j_1})\otimes\nu(\gamma^1_{l+j_1})\otimes$$

$$\cdots\otimes\nu(\gamma^{j_i}_{l+2i-1+\sum_{k=1}^{i-1}j_k})\otimes\nu(\gamma^1_{l+2i-1+\sum_{k=1}^{i}j_k})\otimes\nu(\gamma^{Ni-(2i+\sum_{k=1}^i j_k)}_{l+2i+\sum_{k=1}^i j_k}))\Big)\Big)$$

$$\times t(\gamma_l^{Ni})$$

$$= \sum_{0\le j_1,j_2,\dots,j_i\le N-2} s(\gamma_l^{Ni})\Big(\nu^{-1}\Big(s(\nu(\gamma_l^{j_1}))f(\nu(\gamma_l^{j_1})\otimes\nu(\gamma^1_{l+j_1})\otimes\cdots$$

$$\otimes\,\nu(\gamma^{j_i}_{l+2i-1+\sum_{k=1}^{i-1}j_k})\otimes\nu(\gamma^1_{l+2i-1+\sum_{k=1}^{i}j_k}))\nu(\gamma^{Ni-(2i+\sum_{k=1}^i j_k)}_{l+2i+\sum_{k=1}^i j_k})\Big)\Big)t(\gamma_l^{Ni})$$

$$= \sum_{0\le j_1,j_2,\dots,j_i\le N-2} s(\gamma_l^{Ni})\Big(s(\gamma_l^{j_1})\nu^{-1}\Big(f(\nu(\gamma_l^{j_1})\otimes\nu(\gamma^1_{l+j_1})\otimes\cdots$$

$$\otimes\,\nu(\gamma^{j_i}_{l+2i-1+\sum_{k=1}^{i-1}j_k})\otimes\nu(\gamma^1_{l+2i-1+\sum_{k=1}^{i}j_k}))\Big)\gamma^{Ni-(2i+\sum_{k=1}^i j_k)}_{l+2i+\sum_{k=1}^i j_k}\Big)t(\gamma_l^{Ni})$$

$$= \sum_{0\le j_1,j_2,\dots,j_i\le N-2} s(\gamma_l^{Ni})f(\gamma_l^{j_1}\otimes\gamma^1_{l+j_1}\otimes\cdots\otimes\gamma^{j_i}_{l+2i-1+\sum_{k=1}^{i-1}j_k}\otimes\gamma^1_{l+2i-1+\sum_{k=1}^{i}j_k})$$

$$\times\gamma^{Ni-(2i+\sum_{k=1}^i j_k)}_{l+2i+\sum_{k=1}^i j_k}$$

$$= [\mathrm{Hom}(\Phi,\Lambda)\psi^{-1}f](s(\gamma_l^{Ni})\otimes t(\gamma_l^{Ni})).$$

Hence, $[\mathrm{Hom}(\Phi,\Lambda)\psi^{-1}](C^{2i}(\Lambda)^\nu)\subset\mathrm{Hom}_{\Lambda^{\mathrm{e}}}(P_{2i},\Lambda)^\nu$ holds.

Similarly, we have $[\mathrm{Hom}(\Phi,\Lambda)\psi^{-1}](C^{2i+1}(\Lambda)^\nu)\subset\mathrm{Hom}_{\Lambda^{\mathrm{e}}}(P_{2i+1},\Lambda)^\nu$ for $i\ge 0$. Therefore, $[\mathrm{Hom}(\Phi,\Lambda)\psi^{-1}](C^{n}(\Lambda)^\nu)\subset\mathrm{Hom}_{\Lambda^{\mathrm{e}}}(P_{n},\Lambda)^\nu$ for $n\ge 0$.

Since $F_\nu:\mathrm{Hom}_{\Lambda^{\mathrm{e}}}(\boldsymbol{P},\Lambda)^\nu\to\mathrm{Hom}_{\Lambda^{\mathrm{e}}}(\boldsymbol{P},\Lambda)^\nu$ is equal to $\mathrm{id}_{\mathrm{Hom}_{\Lambda^{\mathrm{e}}}(\boldsymbol{P},\Lambda)^\nu}$, for $n\ge 0$, we have $H^n(\mathrm{Hom}_{\Lambda^{\mathrm{e}}}(\boldsymbol{P},\Lambda)^\nu)\cong\mathrm{HH}^n(\Lambda)^{\nu\uparrow}$. □

From now on, we give a basis of $\mathrm{HH}^n(\Lambda)^{\nu\uparrow}$ for $n\ge 0$ and determine the ring structure of $\mathrm{HH}^*(\Lambda)^{\nu\uparrow}$.

Theorem 5.3. *A basis of* $\mathrm{HH}^0(\Lambda)^{\nu\uparrow}$ *is given by*

$$\{\sum_{l=1}^{g_0} f^0_{\phi_{j,l}} \mid 0 \le j \le N-2 \text{ and } j \equiv 0 \pmod{e}\}.$$

For $i \ge 1$, *a basis of* $\mathrm{HH}^{2i}(\Lambda)^{\nu\uparrow}$ *is given by*

$$\{\sum_{l=1}^{g_0} f^i_{\phi_{j,l}} \mid 0 \le j \le N-2 \text{ and } j \equiv Ni \pmod{e}\}.$$

For $i \ge 1$, *a basis of* $\mathrm{HH}^{2i-1}(\Lambda)^{\nu\uparrow}$ *is given by*

$$\{g^i_{\phi_{j,1}} \mid 0 \le j \le N-2 \text{ and } j \equiv N(i-1) \pmod{e}\}.$$

Proof. By a direct computation, we have

$$\phi^{*\nu}_{2i+1}(f^i_{\phi_{j,l}}) = \begin{cases} g^{i+1}_{\phi_{j,l-1}} - g^{i+1}_{\phi_{j,l}} & \text{if } 1 \le j \le N-2, \\ & \text{or } j = 0 \text{ and } \gcd(N,e) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

for $j (0 \le j \le N-1)$ satisfying $Ni \equiv j \pmod{e}$, and

$$\begin{aligned} &\phi^{*\nu}_{2i}(g^i_{\phi_{j,l}}) \\ &= \begin{cases} \displaystyle\sum_{k=1}^{t} f^i_{\phi_{N-1,l+k}} & \text{if } \gcd(N,e) = 1,\ N(i-1)+1 \equiv 0 \pmod{e}, \\ & \ j = -1 \text{ and } t \neq 0, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

for $j (0 \le j \le N-2)$ satisfying $N(i-1) \equiv j \pmod{e}$.

Since we already assumed that $\operatorname{char} K \mid \operatorname{ord}(\nu) = \frac{e}{g_0}$, we note that $\operatorname{char} K \nmid N$ if $\gcd(N,e) = 1$. Hence, we have

$$\begin{aligned} &\operatorname{Ker} \phi^{*\nu}_{2i+1} \\ &= \begin{cases} \displaystyle\bigoplus_{\substack{0 \le j \le N-2 \\ j \equiv Ni \pmod{e}}} K \sum_{l=1}^{g_0} f^i_{\phi_{j,l}} \oplus \bigoplus_{l=1}^{g_0} K f^i_{\phi_{N-1,l}} & \text{if } Ni \equiv N-1 \pmod{e}, \\ \displaystyle\bigoplus_{\substack{0 \le j \le N-2 \\ j \equiv Ni \pmod{e}}} K \sum_{l=1}^{g_0} f^i_{\phi_{j,l}} & \text{otherwise,} \end{cases} \end{aligned}$$

$$\operatorname{Im}\phi^{*\nu}_{2i+1} = \bigoplus_{\substack{0\le j\le N-2\\ j\equiv Ni \pmod e}} \bigoplus_{l=2}^{g_0} K(g^{i+1}_{\phi_{j,l}} - g^{i+1}_{\phi_{j,1}}),$$

$$\operatorname{Ker}\phi^{*\nu}_{2i} = \bigoplus_{\substack{1\le j\le N-2\\ j\equiv N(i-1) \pmod e}} \bigoplus_{l=1}^{g_0} K g^{i}_{\phi_{j,l}},$$

$$\operatorname{Im}\phi^{*\nu}_{2i} = \begin{cases} \displaystyle\bigoplus_{l=1}^{g_0} K f^{i}_{\phi_{N-1,l}} & \text{if } \gcd(N,e)=1 \text{ and } Ni\equiv N-1 \pmod e,\\ 0 & \text{otherwise.}\end{cases}$$

Therefore, for each $\mathrm{HH}^n(\Lambda)^{\nu\uparrow}$ $(n\ge 0)$, we obtain a basis as claimed. $\square$

Next, in order to consider the ring structure, we recall the Yoneda product in $\mathrm{HH}^*(\Lambda)$. For $[\phi]\in \mathrm{HH}^n(\Lambda)$ and $[\psi]\in\mathrm{HH}^m(\Lambda)$ $(n,m\ge 0)$, there exists $\sigma_i (0\le i\le n)$ such that the following diagram commute:

$$\begin{tikzcd}
\cdots \arrow[r] & P_{m+n} \arrow[r, "d_{m+n}"] \arrow[d, "\sigma_n"] & P_{m+n-1} \arrow[r] \arrow[d, "\sigma_{n-1}"] & \cdots \arrow[r] & P_{m+1} \arrow[r, "d_{m+1}"] \arrow[d, "\sigma_1"] & P_m \arrow[d, "\sigma_0"] \arrow[dr, "\psi"] & & \\
\cdots \arrow[r] & P_n \arrow[r, "d_n"] & P_{n-1} \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r, "d_0"] & \Lambda \arrow[r] & 0
\end{tikzcd}$$

Figure 2: The commutative diagram for the Yoneda product

Then $[\phi]\times[\psi]$ is defined by $[\phi\sigma_n]\in\mathrm{HH}^{n+m}(\Lambda)$.

If $\psi=\sum_{l=1}^{g_0} f^i_{\phi_j,l}\in\mathrm{HH}^{2i}(\Lambda)^{\nu\uparrow}$, then $\sigma_k : P_{2i+k}\to P_k$ is given by

$$\sigma_k(s(R_{2i+k})\otimes t(R_{2i+k})) = \sigma_k(s(R_{2i+k})\sum_{l=1}^{e}\gamma^j_l\otimes t(R_{2i+k})),$$

for $R_{2i+k}\in AP(2i+k)$. If $\psi = g^1_{\phi_0,1}\in\mathrm{HH}^1(\Lambda)$, then $\sigma_1 : P_2\to P_1$ is given by

$$\sigma_1(s(\gamma^N_l)\otimes t(\gamma^N_l)) = \sum_{\substack{0\le k\le N-1\\ l+k\equiv 1 \pmod{g_0}}}\sum_{k'=0}^{k}\gamma^{k'}_l\otimes\gamma^{N-k'-1}_{l+k'+1},$$

for $\gamma^N_l\in AP(2)$. Hence, we have $[g^1_{\phi_0,1}]^2=0$ and $[g^{i+1}_{\phi_j,1}] = [g^1_{\phi_0,1}]\times[\sum_{l=1}^{g_0} f^i_{\phi_j,l}]$. We leave out the notation [ ] and $\times$ if there is no confusion. Now, we can determine the ring structure of $\mathrm{HH}^*(\Lambda)^{\nu\uparrow}$.

Theorem 5.4. *Suppose that $N \le e$, and either* $\operatorname{char} K \nmid N$ *or* $\operatorname{char} K \mid N$ *and* $\gcd(N, e) \ne 1$. *Let $x = \sum_{l=1}^{g_0} f^0_{\phi_0,l}$, $y = g^1_{\phi_0,1}$ and $z_j = \sum_{l=1}^{g_0} f^{i_j}_{\phi_j,l}$, where $i_j(>0)$ is the smallest integer such that $Ni_j \equiv j \pmod e$ if there exists such an integer i_j. Then $\mathrm{HH}^*(\Lambda)^{\nu\uparrow}$ is generated by $\{x, y, z_j \mid 0 \le j \le N-2\}$, and the relations and the degree of elements are as follows*:

$$\textit{degree}: \quad \deg x = 0,\ \deg y = 1,\ \deg z_j = 2i_j.$$

$$\textit{relations}: \begin{cases} y^2 = 0,\ z_j^{[\frac{N}{j}]} = 0, \\ z_j^a = z_{aj} \text{ if } 1 \le aj \le N-2 \\ \qquad\qquad \text{and } i_{aj} = ai_j \text{ for } 1 \le j \le N-2, \\ z_{j_1} \cdots z_{j_n} = 0 \text{ if } j_1 + \cdots + j_n \ge N-1, \\ z_{j_1} \cdots z_{j_n} = z_{j'_1} \cdots z_{j'_{n'}} \text{ if } j_1 + \cdots + j_n = j'_1 + \cdots + j'_{n'}, \\ \qquad \text{where } 0 \le j_1, \dots, j_n, j'_1, \dots, j'_{n'} \le N-2. \end{cases}$$

Theorem 5.5. *Suppose that $N > e$, and either* $\operatorname{char} K \nmid N$, *or* $\operatorname{char} K \mid N$ *and* $\gcd(N, e) \ne 1$. *Let $x_0 = \sum_{l=1}^{g_0} f^0_{\phi_0,l}$, $x_1 = \sum_{l=1}^{g_0} f^0_{\phi_e,l}$ $y = g^1_{\phi_0,1}$ and $z_r = \sum_{l=1}^{g_0} f^{i_r}_{\phi_r,l}$, where $i_r(>0)$ is the smallest integer such that $Ni_r \equiv r \pmod e$ if there exists such an integer i_r for every $0 \le r \le e-1$. Then $\mathrm{HH}^*(\Lambda)^{\nu\uparrow}$ is generated by $\{x_0, x_1, y, z_r \mid 0 \le r \le e-1\}$ and the relations, and the degree of elements are as follows*:

$$\textit{degree}: \quad \deg x_k = 0 (k = 0, 1),\ \deg y = 1,\ \deg z_r = 2i_r.$$

$$\textit{relations}: \begin{cases} x_1^m = 0 \text{ if } N \equiv 0 \pmod e, \\ x_1^{m+1} = 0 \text{ if } N \not\equiv 0 \pmod e, \\ y^2 = 0,\ z_r^{[\frac{N}{r}]} = 0, \\ z_r^a = z_{ar} \text{ if } 1 \le ar \le e-1 \text{ and } i_{ar} = ai_r \text{ for } 1 \le r \le e-1, \\ z_{j_1} \cdots z_{j_n} = 0 \text{ if } j_1 + \cdots + j_n \ge N-1, \\ z_{j_1} \cdots z_{j_n} = z_{j'_1} \cdots z_{j'_{n'}} \text{ if } j_1 + \cdots + j_n = j'_1 + \cdots + j'_{n'}, \\ \qquad \text{where } 0 \le j_1, \dots, j_n, \text{ and } j'_1, \dots, j'_{n'} \le e-1. \end{cases}$$

Corollary 5.6. *For a self-injective Nakayama algebra Λ, $\mathrm{HH}^*(\Lambda)$ is isomorphic to $\mathrm{HH}^*(\Lambda)^{\nu\uparrow}$ as algebras.*

Theorem 5.7. *For $i \ge 0$ and $g^{i+1}_{\phi_j,1} \in \mathrm{HH}^{2i+1}(\Lambda)^{\nu\uparrow}$, $\Delta g^{i+1}_{\phi_j,1} = \frac{N-1}{g_0} \sum_{l=1}^{g_0} f^i_{\phi_j,l}$ holds, where $0 \le j \le N-2$ and $j \equiv Ni \pmod e$. For $i \ge 1$ and $\sum_{l=1}^{g_0} f^i_{\phi_j,1} \in \mathrm{HH}^{2i}(\Lambda)^{\nu\uparrow}$, $\Delta \sum_{l=1}^{g_0} f^i_{\phi_j,l} = 0$ holds, where $0 \le j \le N-2$ and $j \equiv Ni \pmod e$.*

Proof. By the proof of Theorem 4.6, for $g^{i+1}_{\phi_j,1} \in \mathrm{HH}^{2i+1}(\Lambda)^{\nu\uparrow}$, we have

$$\begin{aligned} &\Delta g^{i+1}_{\phi_j,1}(s(\gamma_l^{Ni}) \otimes t(\gamma_l^{Ni})) \\ &= \sum_{k=1}^{2i+1} \Delta_k g^{i+1}_{\phi_j,1}(s(\gamma_l^{Ni}) \otimes t(\gamma_l^{Ni})) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{c=1}^{i} \Big\langle g^{i+1}_{\phi_{j,1}} \Big(\sum_{p=1}^{N-j-1} \gamma^{p-1}_{l+N(c-1)} \otimes \gamma^{N-j-p-1}_{l+N(c-1)+Ni+p} v_{l+Nc-1} \Big), 1 \Big\rangle s(\gamma_l^{Ni}) \gamma_l^j t(\gamma_l^{Ni}) \\
&\quad + \sum_{c=1}^{i} \sum_{j_1=N-j}^{N} \Big\langle g^{i+1}_{\phi_{j,1}} \Big(v_{l+N(c-1)-1+j_1} \otimes \gamma^{N-j-2}_{l+N(c-1)+Ni+j_1} \Big), 1 \Big\rangle s(\gamma_l^{Ni}) \gamma_l^j t(\gamma_l^{Ni}) \\
&\qquad + \langle g^{i+1}_{\phi_{j,1}} (\sum_{p=1}^{N-j-1} \gamma^{p-1}_{l+j} \otimes \gamma^{N-j-p-1}_{l+j+Ni+p}), 1 \rangle s(\gamma_l^{Ni}) \gamma_l^j t(\gamma_l^{Ni}) \\
&= \sum_{c=1}^{i} \sum_{p=1}^{N} \Big\langle g^{i+1}_{\phi_{j,1}} \Big(v_{l+N(c-1)+p-1} \otimes \gamma^{N-j-2}_{l+N(c-1)+Ni+p} \Big), 1 \Big\rangle s(\gamma_l^{Ni}) \gamma_l^j t(\gamma_l^{Ni}) \\
&\qquad + \langle g^{i+1}_{\phi_{j,1}} (\sum_{p=1}^{N-j-1} \gamma^{p-1}_{l+Ni} v_{l+Ni+p-1} \otimes \gamma^{N-j-p-1}_{l+j+Ni+p}), 1 \rangle s(\gamma_l^{Ni}) \gamma_l^j t(\gamma_l^{Ni}) \\
&= \sum_{p=1}^{Ni+N-j-1} \langle g^{i+1}_{\phi_{j,1}} (v_{l+p-1} \otimes \gamma^{N-j-2}_{l+Ni+p}), 1 \rangle s(\gamma_l^{Ni}) \gamma_l^j t(\gamma_l^{Ni}) \\
&= \frac{Ni+N-j-1}{g_0} \gamma_l^j \\
&= \frac{N-1}{g_0} \gamma_l^j .
\end{aligned}
$$

By a similar computation, we obtain the second statement. □

Corollary 5.8. *The Gerstenhaber braket* $[\ ,\ ]$ *on* $\mathrm{HH}^*(\Lambda)^{\nu\uparrow}$ *is the zero map.*

Proof. Suppose that $N \leq e$, and either $\operatorname{char} K \nmid N$ or $\operatorname{char} K \mid N$ and $\gcd(N, e) \neq 1$. Then, for any generators a, b in Theorem 5.4, we have $[a, b] = 0$. On the other hand, suppose that $N > e$, and either $\operatorname{char} K \nmid N$, or $\operatorname{char} K \mid N$ and $\gcd(N, e) \neq 1$. Then, for any generators a, b in Theorem 5.5, we have $[a, b] = 0$. □

Finally, we give the non trivial Batalin-Vilkovisky algebra structure on cohomology of Hochschild complex related to the Nakayama automorphism for the self-injective Nakayama algebra in [9, Example 5.3].

Example 5.9 (cf. [9, Example 5.3])**.** Suppose that $\operatorname{char} K = 2$, $e = 2$ and $N = 4$. Then, $g_0 = 1$, $\operatorname{ord}(\nu) = 2$ and $\mathrm{HH}^*(\Lambda)^{\nu\uparrow} = K[x_1, y, z_0]/(x_1^2, y^2)$, where $\deg x_1 = 0$, $\deg y = 1$ and $\deg z_0 = 2$. Moreover, the bracket $[\ ,\ ] = 0$ and BV-differential Δ is given by

$$
\begin{aligned}
&\Delta(1) = \Delta(x_1) = \Delta(z_0) = \Delta(x_1 z_0) = \Delta(z_0^2) = 0, \\
&\Delta(y) = 1, \quad \Delta(y x_1) = x_1, \quad \Delta(y z_0) = z_0.
\end{aligned}
$$

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