SUT Journal of Mathematics
Vol. 59, No. 1 (2023), 11–32
DOI: 10.55937/sut/1685793568

Asymptotic stability of soliton for discrete nonlinear Schrödinger equation on one-dimensional lattice

Masaya Maeda and Masafumi Yoneda

(Received October 28, 2022)

Abstract. In this paper we prove the asymptotic stability of solitons for a discrete nonlinear Schrödinger equation near the anti-continuous limit. Our novel insight is that the analysis of linearized operator, usually non-symmetric, can be reduced to a study of simple self-adjoint operator almost like the free discrete Laplacian restricted on the space of odd functions.

AMS 2020 *Mathematics Subject Classification.* 35Q55, 37K40.

Key words and phrases. discrete nonlinear Schrödinger equation, anti-continuous limit, soliton, asymptotic stability.

§1. Introduction

In this paper, we study the discrete nonlinear Schrödinger equation (DNLS) on $\mathbb{Z}$

$$\mathrm{i}\partial_t u = -\Delta_{\mathrm{d}} u - |u|^6 u, \ u : \mathbb{R} \times \mathbb{Z} \to \mathbb{C}, \tag{1.1}$$

where Δ_{d} is the discrete Laplacian given by

$$\Delta_{\mathrm{d}} f(x) = f(x+1) - 2f(x) + f(x-1).$$

Remark 1.1. We have chosen to work on the specific nonlinearity $-|u|^6 u$. However, it will be clear that the proof and the result of this paper will hold for general nonlinearity $g(|u|^2)u$ with smooth g satisfying $g(0) = g'(0) = g''(0) = 0$.

DNLS (1.1) appears in models in physics such as Bose-Einstein condensation in optical lattice [4] and photonic lattice [10]. The aim of this paper is to study the stability property of bound state solutions (solitons) $e^{\mathrm{i}\omega t}\phi_\omega$.

To state our result precisely, we set

$$\|u\|_{l^p_a} := \|e^{a|\cdot|}u\|_{l^p}, \quad (u,v) := \sum_{x\in\mathbb{Z}} u(x)\overline{v(x)}, \quad \langle u,v\rangle := \mathrm{Re}(u,v).$$

We will consider large and concentrated solitons given by the following Proposition.

Proposition 1.2. *There exist $\omega_0 > 0$, $C > 0$ and a C^∞-function $\omega \mapsto \phi_\omega$ from (ω_0, ∞) to l^2_{10} such that*

$$0 = -\Delta_{\mathrm{d}}\phi_\omega + \omega\phi_\omega - |\phi_\omega|^6\phi_\omega, \tag{1.2}$$

and

$$\sum_{j=0}^{2} \omega^j \|\partial_\omega^j \phi_\omega - \partial_\omega^j \left(\omega^{\frac{1}{6}}\delta_0\right)\|_{l^2_{10}} \le C\omega^{-\frac{5}{6}}, \tag{1.3}$$

$$\sum_{j=0}^{2} \omega^j \|P_0^\perp \partial_\omega^j \phi_\omega\|_{l^2_{10}} \le C\omega^{-\frac{5}{6}}, \tag{1.4}$$

where $P_0^\perp = 1 - (\cdot, \delta_0)\delta_0$ and $\delta_0(x) = 1$ if $x = 0$ and $\delta_0(x) = 0$ if $x \neq 0$.

Remark 1.3. If ϕ_ω satisfies (1.2), then $u(t,x) = e^{\mathrm{i}\omega t}\phi_\omega(x)$ is a solution of (1.1).

The main result of this paper is the asymptotic stability result for solitons $e^{\mathrm{i}\omega t}\phi_\omega$ given in Proposition 1.2 for ω sufficiently large. In particular, we prove the following:

Theorem 1.4. *There exists $\omega_1 \ge \omega_0$, where ω_0 is given in Proposition 1.2, such that for any $\omega_* > \omega_1$, there exist $\delta_0 > 0$ and $C > 0$ such that if $\epsilon := \|u - \phi_{\omega_*}\|_{l^2} < \delta_0$, then there exist $\theta \in C^\infty([0,\infty), \mathbb{R})$, $\omega_+ > \omega_1$ and $\xi_+ \in l^2$ such that*

$$\lim_{t\to\infty} \|u(t) - e^{\mathrm{i}\theta(t)}\phi_{\omega_+} - e^{\mathrm{i}t\Delta_{\mathrm{d}}}\xi_+\|_{l^2} = 0, \tag{1.5}$$

$$|\log\omega_* - \log\omega_+| + \|\xi_+\|_{l^2} \le C\epsilon. \tag{1.6}$$

Considering large solitons concentrated on a finite set (in this case $\{0\} \subset \mathbb{Z}$) is equivalent to studying the solitons near the so-called "anti-continuous limit".

Indeed, setting $u(t,x) = \omega^{\frac{1}{6}} v(\omega t, x)$, v satisfies

$$\mathrm{i}\partial_t v = -\epsilon\Delta_{\mathrm{d}} v - |v|^6 v, \tag{1.7}$$

where $\epsilon = \omega^{-1}$. By such rescaling, solitons given in Proposition 1.2 are rescaled to $e^{\mathrm{i}t}\psi_\epsilon(x)$ with $\psi_\epsilon \sim \delta_0$. In the anti-continuous limit $\epsilon \to 0$, (1.7) reduces to an infinite system of unrelated ordinarily differential equations and in particular possesses a solution $e^{\mathrm{i}t}\delta_0$. The first rigorous treatment for the existence of solitons (also called discrete breathers in the context of discrete nonlinear Klein-Gordon equations) branching from the above solution, was given by MacKay and Aubry [14] followed by [1, 11]. Further, it was shown by Weinstein [25] that for sufficiently large $\omega > 0$, ϕ_ω is a ground state, i.e. minimizer of energy under the restriction of l^2-norm, and thus orbitally stable.

Remark 1.5. In this paper, we have chosen to work on the rescaled setting to reduce the number of parameters. Indeed, even if we consider (1.7) we still need to consider a family of solitons $\psi_{\epsilon,\omega}$ with ω near 1.

The asymptotic stability problem of solitons near the anti-continuous limit is no less important than the problem of existence, but it has not yet been thoroughly investigated. The only asymptotic stability result we are aware is by Bambusi [2] who studied the asymptotic stability of nonlinear discrete Klein-Gordon equations (for linear/spectral stability see [19, 20]. Also, for asymptotic stability of bound states bifurcating from linear potential, see [13, 9, 18, 15, 16]). Our result, Theorem 1.4, is a DNLS version of Bambusi's result, with simplified proof, as explained below.

We now explain the outline of the proof of Theorem 1.4. We start from a standard strategy initiated by [3] for the asymptotic stability of solitons of nonlinear Schrödinger equations (see also [23] for small solitons). That is, we decompose the solution u near $\{e^{\mathrm{i}\theta}\phi_{\omega_*} \mid \theta \in \mathbb{R}\}$ as $u = e^{\mathrm{i}\theta}\phi_\omega + \xi$, with $i\xi$ orthogonal to $\{\mathrm{i}e^{\mathrm{i}\theta}\phi_\omega, e^{\mathrm{i}\theta}\partial_\omega\phi_\omega\}$. Then, the problem is to study the dynamics of θ, ω and ξ. Roughly, the equation of ξ, which we obtain by substituting the ansatz into DNLS (1.1), will have the form

$$\mathrm{i}\partial_t\xi = H_{\theta,\omega}\xi + O(\xi^2),$$

where

$$H_{\theta,\omega}\xi := \left(-\Delta_{\mathrm{d}} + \omega - 4\phi_\omega^6\right)\xi - 3e^{2\mathrm{i}\theta}\phi_\omega^6\overline{\xi}.$$

The "linearized operator" $H_{\theta,\omega}$ is not $\mathbb{C}$-linear due to the complex conjugate. Thus, it is natural to study the corresponding matrix $\mathbb{C}$-linear operator

$$\mathcal{H}_{\theta,\omega} := \begin{pmatrix} -\Delta_{\mathrm{d}} + \omega - 4\phi_\omega^6 & -3e^{2\mathrm{i}\theta}\phi_\omega^6 \\ 3e^{-2\mathrm{i}\theta}\phi_\omega^6 & \Delta_{\mathrm{d}} - \omega + 4\phi_\omega^6 \end{pmatrix}.$$

However, in general it is hard to study the spectral properties of the operator which are needed for the proof of asymptotic stability and one is forced to assume, for example, the nonexistence of embedded eigenvalues, edge resonances and internal modes (and if one admits internal modes, then one needs to assume the Fermi Golden Rule property), see e.g. [6, 8].

For our problem, the first attempt is to use the fact $\phi_\omega = \omega^{\frac{1}{6}}\left(\delta_0 + O(\omega^{-1})\right)$ where the remainder is small and decaying exponentially, as proved in Proposition 1.2. So, we replace ϕ_ω by $\omega^{\frac{1}{6}}\delta_0$ and consider

$$\widetilde{\mathcal{H}}_{\theta,\omega} := \begin{pmatrix} -\Delta_{\mathrm{d}} + \omega - 4\omega\delta_0 & -3e^{2\mathrm{i}\theta}\omega\delta_0 \\ 3e^{-2\mathrm{i}\theta}\omega\delta_0 & \Delta_{\mathrm{d}} - \omega + 4\omega\delta_0 \end{pmatrix}.$$

It is possible to study the spectral properties and show decay estimates related to the operator $\widetilde{\mathcal{H}}_{\theta,\omega}$. However, one can reduce this operator one step further. Recall that we were assuming $\mathrm{i}\xi \in \{\mathrm{i}e^{\mathrm{i}\theta}\phi_\omega, e^{\mathrm{i}\theta}\partial_\omega\phi_\omega\}^\perp$ (here, we are considering real inner product). Applying the approximation $\phi_\omega \sim \omega^{\frac{1}{6}}\delta_0$ to the *orthogonality condition*, it will reduced to $\mathrm{i}\xi \in \{\mathrm{i}e^{\mathrm{i}\theta}\omega\delta_0, \frac{1}{6}e^{\mathrm{i}\theta}\omega^{-\frac{5}{6}}\delta_0\}^\perp = \{\delta_0, \mathrm{i}\delta_0\}^\perp$, which is equivalent to $\xi(0) = 0$. For ξ satisfying $\xi(0) = 0$ and $\eta = \overline{\xi}$, we have

$$\widetilde{\mathcal{H}}_{\theta,\omega}\begin{pmatrix}\xi \\ \eta\end{pmatrix} = \begin{pmatrix} -\Delta_{\mathrm{d}} + \omega & 0 \\ 0 & \Delta_d - \omega \end{pmatrix}\begin{pmatrix}\xi \\ \eta\end{pmatrix}.$$

Thus, $\widetilde{\mathcal{H}}_{\theta,\omega}$ reduces to a diagonal matrix and there will be no point considering matrix operator anymore. Thus, the task is now to study $\Delta_0 := P_0^\perp\Delta_{\mathrm{d}}P_0^\perp$ where $P_0^\perp$ is the projection on to the space $\{\xi \mid \xi(0) = 0\}$, see Proposition 1.2.

The analysis of Δ_0 further reduces to the analysis of Δ_{d} restricted on odd functions (see proof of Proposition 2.1, we also note that Bambusi [2] studies similar operator). Thus, we will obtain the Strichartz estimate for free and the Kato smoothing estimates, which do not hold for Δ_{d}, from a simple fact that the edge resonance of Δ_{d} is even. Therefore, we obtain the linear estimates needed (the idea using the Kato smoothing is due to [9]). The rest of the paper is more of less standard except tracing the ω dependence of the error terms carefully.

This paper is organized as follows: In Section 2, we prove the linear estimates for Δ_0. Section 3 will be devoted to the modulation argument and in particular we derive the equation of ξ (see (3.18) and (3.21)) and θ, ω (see, (3.27)). In Section 4, we prove Theorem 1.4 by bootstrapping argument (Proposition 4.1). Finally, in the Appendix, we prove Proposition 1.2.

In the following, we use $a \lesssim b$ by meaning $a \le Cb$ for some constant $C > 0$ not depending on important parameters. Also, if $a \lesssim b$ and $b \lesssim a$ we write $a \sim b$.

§2. Linear estimates

Recall that in Proposition 1.2, $P_0^\perp$ was given by

$$P_0^\perp u(x) = \begin{cases} u(x) & x \neq 0, \\ 0 & x = 0. \end{cases} \tag{2.1}$$

We set the restriction of Δ_{d} to $P_0^{\perp} l^2(\mathbb{Z}) = \{u \in l^2(\mathbb{Z}) \mid u(0) = 0\}$ by

$$\Delta_0 := P_0^{\perp}\Delta_{\mathrm{d}} : P_0^{\perp} l^2(\mathbb{Z}) \to P_0^{\perp} l^2(\mathbb{Z}).$$

In particular, for $u \in P_0^{\perp} l^2(\mathbb{Z})$, we have

$$(\Delta_0 u)(x) = \begin{cases} u(\pm 2) - 2u(\pm 1) & x = \pm 1, \\ u(x+1) - 2u(x) + u(x-1) & |x| \geq 2. \end{cases}$$

For an interval $I \subset \mathbb{R}$, we set

$$\mathrm{Stz}(I) := L^\infty(I, l^2(\mathbb{Z})) \cap L^6(I, l^\infty(\mathbb{Z})),$$
$$\mathrm{Stz}^*(I) := L^1(I, l^2(\mathbb{Z})) + L^{\frac{6}{5}}(I, l^1(\mathbb{Z})).$$

The linear estimates we use in this paper are the following:

Proposition 2.1 (Strichartz and Kato smoothing estimates)**.** *Let $I \subset \mathbb{R}$ be an interval with $0 \in I$. Let $u_0 : \mathbb{Z} \to \mathbb{C}$ and $f : \mathbb{R} \times \mathbb{Z} \to \mathbb{C}$. Then, we have*

$$\|e^{\mathrm{i}t\Delta_0} P_0^{\perp} u_0\|_{\mathrm{Stz}(I) \cap L^2(I, l^2_{-1})} \lesssim \|u_0\|_{l^2}, \tag{2.2}$$

$$\|\int_0^{\cdot} e^{\mathrm{i}(\cdot - s)\Delta_0} P_0^{\perp} f(s)\, ds\|_{\mathrm{Stz}(I)} \lesssim \|f\|_{\mathrm{Stz}^*(I) + L^2(I, l^2_1)}, \tag{2.3}$$

$$\|\int_0^{\cdot} e^{\mathrm{i}(\cdot - s)\Delta_0} P_0^{\perp} f(s)\, ds\|_{L^2(I, l^2_{-1})} \lesssim \|f\|_{L^2(I, l^2_1)}. \tag{2.4}$$

Proof. First, let $P_\pm u(x) = u(x)$ if $\pm x \geq 1$ and $P_\pm u(x) = 0$ if $\pm x \leq 0$. Then, we have $[\Delta_0, P_\pm] = 0$ so we get

$$e^{it\Delta_0}(P_+ + P_-) = (P_+ + P_-)e^{it\Delta_0}.$$

This implies $\Delta_0 = \Delta_+ \oplus \Delta_-$ where $\Delta_\pm := P_\pm \Delta_0 : P_\pm l^2 \to P_\pm l^2$. Thus, it suffices to show each estimate for $\Delta_\pm$ and we will only consider Δ_+. Further, for $u : \mathbb{N} \to \mathbb{C}$, we set $Tu : \mathbb{Z} \to \mathbb{C}$ by $Tu(x) = u(x)$ for $x \geq 1$, $Tu(0) = 0$ and $Tu(x) = -u(-x)$ for $x \leq -1$. Then, we have $T\Delta_+ = \Delta_{\mathrm{d}} T$.

Therefore, it suffices to show the estimates (2.2), (2.3) and (2.4) for Δ_{d} and u_0, f restricted to odd functions. Thus, we immediately have the Strichartz estimates by [24]. That is, we have

$$\|e^{\mathrm{i}t\Delta_0} P_0^{\perp} u_0\|_{\mathrm{Stz}(I)} \lesssim \|u_0\|_{l^2}, \quad \|\int_0^{\cdot} e^{\mathrm{i}(\cdot - s)\Delta_0} P_0^{\perp} f(s)\, ds\|_{\mathrm{Stz}(I)} \lesssim \|f\|_{\mathrm{Stz}^*(I)}.$$

Next, we show the Kato smoothness estimate

$$\|e^{\mathrm{i}t\Delta_0} P_0^{\perp} u_0\|_{L^2(I, l^2_{-1})} \lesssim \|u_0\|_{l^2}, \tag{2.5}$$

which does not hold for Δ_{d} with general u_0. To show (2.5), it suffices to show

$$\sup_{\substack{\|u\|_{l^2_1}\le 1\\ u:\mathrm{odd}}} \sup_{\mathrm{Im}\,\lambda\neq 0} \|(-\Delta_{\mathrm{d}}-\lambda)^{-1}u\|_{l^2_{-1}} \lesssim 1, \tag{2.6}$$

which is a sufficient condition for the Kato smoothness, see [12] or Corollary of Theorem XIII.25 of [21]. From the Fourier transform,

$$\left((-\Delta_{\mathrm{d}}-\lambda)^{-1}u\right)(x) = \frac{1}{2\pi}\sum_{y\in\mathbb{Z}}\int_{[0,2\pi]} \frac{e^{-\mathrm{i}\xi|x-y|}}{2-2\cos\xi-\lambda}u(y)d\xi. \tag{2.7}$$

By the residue theorem, we can rewrite the r.h.s. of (2.7). Take $R>0$ and set

$$A\cup B\cup C\cup D = [0,2\pi]\cup[2\pi-\mathrm{i}R,-\mathrm{i}R]\cup[-\mathrm{i}R,0]\cup[2\pi,2\pi-\mathrm{i}R],$$

where $[a,b]:=\{a+t(b-a)\mid t\in[0,1]\}$ for $a,b\in\mathbb{C}$. Then we have

$$\int_{A\cup B\cup C\cup D}\frac{e^{-\mathrm{i}\xi|x-y|}}{2-2\cos\xi-\lambda}d\xi = 2\pi i\frac{e^{-\mathrm{i}\mu|x-y|}}{2\sin\mu},$$

where $\cos\mu = 1-\frac{\lambda}{2}$ with $\mathrm{Im}\,\mu\le 0$. By

$$\int_{C\cup D}\frac{e^{-\mathrm{i}\xi|x-y|}}{2-2\cos\xi-\lambda}u(y)d\xi = 0,$$

and

$$\lim_{R\to\infty}\int_{B}\frac{e^{-\mathrm{i}\xi|x-y|}}{2-2\cos\xi-\lambda}u(y)d\xi = 0,$$

we have

$$\left((-\Delta_{\mathrm{d}}-\lambda)^{-1}u\right)(x) = \mathrm{i}\sum_{y\in\mathbb{Z}}\frac{e^{-\mathrm{i}\mu|x-y|}}{2\sin\mu}u(y).$$

Using the fact that u is odd, we can further rewrite

$$\begin{aligned}\left((-\Delta_{\mathrm{d}}-\lambda)^{-1}u\right)(x) &= \mathrm{i}\sum_{y\in\mathbb{Z}}\frac{e^{-\mathrm{i}\mu|x-y|}}{2\sin\mu}u(y)\\ &= \mathrm{i}\sum_{y>0}\frac{e^{-\mathrm{i}\mu|x-y|}}{2\sin\mu}u(y) + \mathrm{i}\sum_{y<0}\frac{e^{-\mathrm{i}\mu|x-y|}}{2\sin\mu}u(y)\\ &= \mathrm{i}\sum_{y>0}\frac{e^{-\mathrm{i}\mu|x-y|}-e^{-\mathrm{i}\mu|x+y|}}{2\sin\mu}u(y).\end{aligned} \tag{2.8}$$

We now decompose (2.8) as below and estimate the contribution of each term.

$$\sum_{y>0} \frac{e^{-\mathrm{i}\mu|x-y|}-e^{-\mathrm{i}\mu|x+y|}}{\sin\mu}u(y) \tag{2.9}$$
$$= \sum_{0<y<x} \frac{e^{\mathrm{i}(x-y)\mu}-e^{(x+y)\mu}}{\sin\mu}u(y) + \sum_{y\geq x} \frac{e^{\mathrm{i}(-x+y)\mu}-e^{\mathrm{i}(x+y)\mu}}{\sin\mu}u(y).$$

For the first term of the r.h.s. of (2.9), by Euler's formula and the finite geometric series, we have

$$\left|\frac{e^{\mathrm{i}\mu(x-y)}-e^{\mathrm{i}\mu(x+y)}}{\sin\mu}\right| = |e^{\mathrm{i}\mu(x-y)}|\left|\frac{1-e^{2\mathrm{i}\mu x}}{\sin\mu}\right| \tag{2.10}$$
$$= |2\mathrm{i}e^{\mathrm{i}\mu}e^{\mathrm{i}\mu(x-y)}|\left|\frac{e^{2\mathrm{i}\mu x}-1}{e^{2\mathrm{i}\mu}-1}\right| = |2\mathrm{i}e^{\mathrm{i}\mu}e^{\mathrm{i}\mu(x-y)}||\sum_{k=0}^{x-1} e^{\mathrm{i}\mu k}| \lesssim \langle x\rangle\,,$$

where we have used the fact $x > 0$, $x-y>0$ and $\operatorname{Im}\mu < 0$. Then, by the Hölder inequality and (2.10),

$$\left|\sum_{0<y<x} \frac{e^{-\mathrm{i}y\mu}-e^{\mathrm{i}y\mu}}{\sin\mu}u(y)\right| \lesssim \langle x\rangle^{\frac{3}{2}} \|u\|_{l^2}.$$

Therefore,

$$\|e^{-|x|}\sum_{0<y<x} \frac{e^{-\mathrm{i}\mu(x-y)}-e^{\mathrm{i}\mu(x+y)}}{2\mathrm{i}\sin\mu}u(y)\|_{l^2} \lesssim \|u\|_{l^2} \leq \|u\|_{l^2_1}.$$

Next, we consider the second term of the r.h.s. of (2.9) . From (2.10), exchanging the role of x and y, we have

$$\left|\frac{e^{\mathrm{i}(-x+y)\mu}-e^{\mathrm{i}(x+y)\mu}}{2\mathrm{i}\sin\mu}\right| \lesssim \langle y\rangle\,.$$

Thus,

$$\left\|\sum_{y\geq x} \frac{e^{\mathrm{i}(-x+y)\mu}-e^{\mathrm{i}(x+y)\mu}}{2\mathrm{i}\sin\mu}u(y)\right\|_{l^2_{-1}} \lesssim \|e^{-|x|}\sum_{y\geq x}\langle y\rangle\, u(y)\|_{l^2} \lesssim C\|u\|_{l^2_1}.$$

Therefore, we have (2.6).

Next the estimate

$$\|\int_0^{\cdot} e^{\mathrm{i}(\cdot-s)\Delta_0}P_0^{\perp}f(s)\,ds\|_{\mathrm{Stz}(I)} \lesssim \|f\|_{L^2(I,l^2_1)}$$

follows from the dual of (2.5) and the Christ-Kiselev lemma [5, 22].

Finally, we prove (2.4) by a parallel argument of Lemma 8.7 of [7]. The following formula was proved in Lemma 4.5 of [17]:

$$\begin{aligned}(2.11)\qquad & 2\int_0^t e^{-\mathrm{i}(t-s)\Delta_0}P_0^{\perp}f(s)\,ds\\ &=\frac{\mathrm{i}}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\mathrm{i}t\lambda}(R(\lambda-\mathrm{i}0)+R(\lambda+\mathrm{i}0))P(\mathcal{F}_t^{-1}f)(\lambda)\,d\lambda\\ &\quad+\int_0^{\infty} e^{-\mathrm{i}(t-s)\Delta_0}P_0^{\perp}f(s)\,ds-\int_{-\infty}^0 e^{-\mathrm{i}(t-s)\Delta_0}P_0^{\perp}f(s)\,ds,\end{aligned}$$

where $R(\lambda)=(-\Delta_0-\lambda)^{-1}$ and $\mathcal{F}_t^{-1}$ is the inverse Fourier transform with respect to the t variable. For the first term of the r.h.s. of (2.11), by the Plancherel theorem, we have

$$\begin{aligned}&\|\int_{\mathbb{R}} e^{-\mathrm{i}t\lambda}(R(\lambda-\mathrm{i}0)+R(\lambda+\mathrm{i}0))P_0^{\perp}(\mathcal{F}_t^{-1}f)(\lambda)\,d\lambda\|_{L^2 l^2_{-1}}\\ &\lesssim\max_{\pm}\|R(\lambda\pm\mathrm{i}0)P_0^{\perp}(\mathcal{F}_t^{-1}f)(\lambda)\|_{L^2_\lambda l^2_{-1}}\\ &\lesssim\max_{\pm}\sup_{\lambda\in\mathbb{R}}\|R(\lambda\pm\mathrm{i}0)\|_{l^2_1\to l^2_{-1}}\|\mathcal{F}_t^{-1}f\|_{L^2_\lambda l^2_1}\lesssim\|f\|_{L^2 l^2_1},\end{aligned}$$

where we have used (2.6) in the third inequality. Here, we note that the operator norm $\|\cdot\|_{l^2_1\to l^2_{-1}}$ is given by the supremum of all odd functions in l^2_1 as in (2.6). The second and third term can be estimated by using (2.5) and its dual. Therefore, we have the conclusion. □

§3. Modulation argument

We set $\phi[\theta,\omega]:=e^{\mathrm{i}\theta}\phi_\omega$. Then, by (1.2), we have

$$(3.1)\qquad \mathrm{i}\omega\partial_\theta\phi[\theta,\omega]=-\Delta\phi[\theta,\omega]-|\phi[\theta,\omega]|^6\phi[\theta,\omega].$$

Further, differentiating (3.1) with respect to θ and ω, we obtain

$$(3.2)\qquad \mathcal{H}[\theta,\omega]\partial_\theta\phi[\theta,\omega]=\mathrm{i}\omega\partial_\theta^2\phi[\theta,\omega],$$

$$(3.3)\qquad \mathcal{H}[\theta,\omega]\partial_\omega\phi[\theta,\omega]=\mathrm{i}\partial_\theta\phi[\theta,\omega]+\mathrm{i}\omega\partial_\theta\partial_\omega\phi[\theta,\omega],$$

where

$$\begin{aligned}(3.4)\qquad \mathcal{H}[\theta,\omega]u&:=-\Delta u+\mathcal{V}[\theta,\omega]u\\ &:=-\Delta_{\mathrm{d}}u-4|\phi[\theta,\omega]|^6u-3|\phi[\theta,\omega]|^4\phi[\theta,\omega]^2\overline{u}.\end{aligned}$$

Remark 3.1. The operator $\mathcal{H}[\theta,\omega]$ is not $\mathbb{C}$-linear but only $\mathbb{R}$-linear due to the complex conjugate in the last term of (3.4).

It is easy to check that $\mathcal{H}[\theta,\omega]$ is symmetric with respect to the real inner-product $\langle\cdot,\cdot\rangle$. That is, we have $\langle \mathcal{H}[\theta,\omega]u,v\rangle = \langle u, \mathcal{H}[\theta,\omega]v\rangle$. We set

$$\Omega(u,v) := \langle \mathrm{i}u, v\rangle\ ,\ \ u,v\in l^2.$$

Remark 3.2. Ω is the symplectic form associated to discrete NLS (1.1).

We set

$$\mathbf{H}_{\mathrm{c}}[\theta,\omega] := \{u\in l^2 \mid \Omega(u,\partial_\theta\phi[\theta,\omega]) = \Omega(u,\partial_\omega\phi[\theta,\omega]) = 0\}$$

and

$$\mathcal{T}_\omega(r) := \{u\in l^2 \mid \inf_{\theta\in\mathbb{R}} \|u-\phi[\omega,\theta]\|_{l^2} < r\}.$$

Lemma 3.3 (Modulation)**.** *For $\omega_* > \omega_0$, where ω_0 is given in Proposition 1.2, there exist $\delta>0$, $\theta\in C^\infty(\mathcal{T}_{\omega_*}(\delta),\mathbb{R})$ and $\omega\in C^\infty(\mathcal{T}_{\omega_*}(\delta),\mathbb{R})$ such that*

$$\xi(u) := u-\phi[\theta(u),\omega(u)] \in \mathbf{H}_{\mathrm{c}}[\theta(u),\omega(u)]. \tag{3.5}$$

Proof. Set

$$\mathcal{F}(\theta,\omega,u) := \begin{pmatrix}\Omega(u-\phi[\theta,\omega],\partial_\theta\phi[\theta,\omega])\\ \Omega(u-\phi[\theta,\omega],\partial_\omega\phi[\theta,\omega])\end{pmatrix}. \tag{3.6}$$

Then, we have $\mathcal{F}(\theta,\omega,\phi[\theta,\omega])=0$ and

$$\begin{aligned} D_{(\theta,\omega)}\mathcal{F}(\theta,\omega,u) = &\ q'(\omega)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\\ &+\begin{pmatrix}\Omega(u-\phi[\theta,\omega],\partial_\theta^2\phi[\theta,\omega]) & \Omega(u-\phi[\theta,\omega],\partial_\theta\partial_\omega\phi[\theta,\omega])\\ \Omega(u-\phi[\theta,\omega],\partial_\theta\partial_\omega\phi[\theta,\omega]) & \Omega(u-\phi[\theta,\omega],\partial_\omega^2\phi[\theta,\omega])\end{pmatrix},\end{aligned}$$

where $q(\omega)=\frac{1}{2}\|\phi[\theta,\omega]\|_{l^2}^2$. Since $\partial_\theta\phi[\theta,\omega]=\mathrm{i}\phi[\theta,\omega]$, we have

$$q'(\omega) = \langle\partial_\omega\phi[\theta,\omega],\phi[\theta,\omega]\rangle = \Omega(\partial_\omega\phi[\theta,\omega],\mathrm{i}\phi[\theta,\omega]) = \Omega(\partial_\omega\phi[\theta,\omega],\partial_\theta\phi[\theta,\omega]).$$

Thus, from Proposition 1.2, for large ω, we have

$$q'(\omega)\sim\omega^{-\frac{2}{3}}>0. \tag{3.7}$$

Therefore, $D_{(\theta,\omega)}\mathcal{F}(\theta,\omega,\phi[\theta,\omega])$ is invertible. By the implicit function theorem we have the conclusion. □

Lemma 3.4. *There exists $\omega_1 > \omega_0$ such that for $\omega_* > \omega_1$, if $u \in \mathcal{T}_{\omega_*}(\delta)$, we have*

$$\|\xi(u)\|_{l^2} \lesssim \inf_{\theta\in\mathbb{R}} \|u - \phi[\theta, \omega_*]\|_{l^2}. \tag{3.8}$$

Here, $\delta > 0$ is the constant (depending on ω_*) given in Lemma 3.3.

Proof. Fix $u \in \mathcal{T}_{\omega_*}(0, \delta)$ and set $\epsilon = \inf_{\theta\in\mathbb{R}} \|u - \phi[\theta, \omega_*]\|_{l^2}$. We take θ_0 to satisfy $\epsilon = \|u - \phi[\theta_0, \omega_*]\|_{l^2}$ and set $v := u - \phi[\theta_0, \omega_*]$ and for $s \in [0, 1]$,

$$u[s] := \phi[\theta_0, \omega_*] + sv,\ \omega[s] := \omega(u[s]),\ \theta[s] := \theta(u[s]),$$
$$A[s] := \|u[s] - \phi[\omega[s], \theta[s]]\|_{l^2}.$$

Notice that we have $u[0] = \phi[\theta_0, \omega_*]$, $\theta[0] = \theta_0$, $\omega[0] = \omega_*$, $u[1] = u$ and (3.8) is equivalent to $A[1] \lesssim \epsilon$. So, to show (3.8) we prove the following claim.

Claim 3.5. There exist $\omega_1 > \omega_0$ and $C_0 > 0$ such that if $s \in (0, 1)$, $\omega_* > \omega_1$ and if

$$|\omega[s] - \omega_*| \le C_0 \omega_*^{5/6} \epsilon, \tag{3.9}$$

$$A[s] \le C_0 \epsilon, \tag{3.10}$$

we have (3.9) and (3.10) with C_0 replaced by $C_0/2$.

Proof of Claim 3.5. We assume (3.9) and (3.10) for all $\tau \in [0, s]$ for some $s \in (0, 1)$. We take $C_0 = \omega_1^{\frac{1}{12}}$. In this proof, when we use $\lesssim$ or $\sim$, the implicit constant will not depend on ω_1, ω_* nor s.

From (3.9), we have

$$\omega[s] \sim \omega_*. \tag{3.11}$$

From the fundamental theorem of calculus, we have

$$\begin{aligned} A[s] &\le \epsilon + \|\phi[\theta_s, \omega_s] - \phi[\theta_0, \omega_*]\|_{l^2} \\ &\le \epsilon + \int_0^s \left(\|\partial_\theta \phi[\theta_\tau, \omega_\tau]\|_{l^2} |D_u \theta(u_\tau) v| + \|\partial_\omega \phi[\theta_\tau, \omega_\tau]\|_{l^2} |D_u \omega(u_\tau) v|\right) d\tau. \end{aligned} \tag{3.12}$$

Differentiating $\mathcal{F}(\theta(u_\tau), \omega(u_\tau), u_\tau) = 0$ with respect to τ, where $\mathcal{F}$ is the function given in (3.6), we have

$$\begin{pmatrix} D_u\theta(u_\tau)v \\ D_u\omega(u_\tau)v \end{pmatrix} = \left(D_{(\theta,\omega)}\mathcal{F}(\theta(u_\tau), \omega(u_\tau), u_\tau)\right)^{-1} \begin{pmatrix} \Omega(v, \partial_\theta\phi[\theta_\tau, \omega_\tau]) \\ \Omega(v, \partial_\omega\phi[\theta_\tau, \omega_\tau]) \end{pmatrix}. \tag{3.13}$$

The determinant of $D_{(\theta,\omega)}\mathcal{F}$ can be explicitly written as

$$\begin{aligned} &\det\left(D_{(\theta,\omega)}\mathcal{F}\right)(\theta_\tau, \omega_\tau, u_\tau) \\ &= (q'(\omega_\tau))^2 + \Omega(u_\tau - \phi[\theta_\tau, \omega_\tau], \partial_\theta^2\phi[\theta_\tau, \omega_\tau])\Omega(u_\tau - \phi[\theta_\tau, \omega]_\tau, \partial_\omega^2\phi[\theta_\tau, \omega_\tau]) \\ &\quad - \Omega(u_\tau - \phi[\theta_\tau, \omega_\tau], \partial_\theta\partial_\omega\phi[\theta_\tau, \omega_\tau])^2. \end{aligned}$$

By (3.7) and (3.11), we have $q'(\omega_\tau)^2 \sim \omega_*^{-\frac{4}{3}} = \omega_*^{-\frac{16}{12}}$ and

$$|\det\left(D_{\theta,\omega}\mathcal{F}\right)(\theta_\tau,\omega_\tau,u_\tau) - (q'(\omega_\tau))^2| \lesssim A[\tau]\omega_\tau^{\frac{1}{3}-2} \lesssim \epsilon\omega_*^{-\frac{19}{12}}.$$

Thus, we have

$$\det\left(D_{\theta,\omega}\mathcal{F}\right)(\theta_\tau,\omega_\tau,u_\tau) \sim \omega_*^{-\frac{4}{3}}. \tag{3.14}$$

Computing the r.h.s. explicitly and using Proposition 1.2, (3.11) and (3.14), we have

$$|D_u\theta(u_\tau)v| + \omega_*^{-1}|D_u\omega(u_\tau)v| \lesssim \omega_*^{-\frac{1}{6}}\left(1 + A(s)\omega_*^{-\frac{1}{6}}\right)\epsilon.$$

Substituting this bound in (3.12) and $\omega_s = \omega_* + \int_0^t D_u\omega(u_\tau)v\,d\tau$, we have

$$\begin{aligned} A[s] &\lesssim \left(1 + A(s)\omega_*^{-\frac{1}{6}}\right)\epsilon \lesssim (1 + \omega_1^{-\frac{1}{12}}\epsilon)\epsilon \lesssim \epsilon, \\ |\omega[s] - \omega_*| &\lesssim \omega_*^{\frac{5}{6}}\left(1 + A(s)\omega_*^{-\frac{1}{6}}\right)\epsilon \lesssim \omega_*^{\frac{5}{6}}(1 + \omega_1^{-\frac{1}{12}}\epsilon)\epsilon \lesssim \omega_*^{\frac{5}{6}}\epsilon. \end{aligned}$$

Therefore, we have the conclusion. □

By Claim 3.5 and the continuity argument, we obtain (3.9) and (3.10) with $s = 1$ and in particular (3.8). □

Recall that $P_0^\perp$ is given in (2.1). For large ω, the two spaces $P_0^\perp l^2$ and $\mathbf{H}_{\mathrm{c}}[\theta,\omega]$ become similar.

Lemma 3.6. $P_0^\perp|_{\mathbf{H}_{\mathrm{c}}[\theta,\omega]}$ *is invertible. Moreover,*

$$Q[\theta,\omega] := \left(P_0^\perp\big|_{\mathbf{H}_{\mathrm{c}}[\theta,\omega]}\right)^{-1} : P_0^\perp l^2 \to \mathbf{H}_{\mathrm{c}}[\theta,\omega]$$

is given by

$$\begin{aligned} &Q[\theta,\omega]u \\ &\quad = u + e^{\mathrm{i}\theta}\left(-\phi_\omega(0)^{-1}\Omega(u,\partial_\theta\phi[\theta,\omega]) + \mathrm{i}\partial_\omega\phi_\omega(0)^{-1}\Omega(u,\partial_\omega\phi[\theta,\omega])\right)\delta_0, \end{aligned} \tag{3.15}$$

and for $u \in P_0^\perp l^2$*, we have*

$$\|u - Q[\theta,\omega]u\|_{l^2_1} \lesssim \omega^{-1}\|u\|_{l^2_{-1}}. \tag{3.16}$$

Proof. Since $\left(P_0^\perp u\right)(x) = u(x)$ for $x \neq 0$, the only possible form of the inverse of $P_0^\perp|_{\mathbf{H}_c[\theta,\omega]}$ is

$$Q[\theta,\omega]u = u + e^{\mathrm{i}\theta} q(u)\delta_0, \tag{3.17}$$

with $q(u) \in \mathbb{C}$. Substituting (3.17) into $\Omega(Q[\theta,\omega]u, \partial_X \phi[\theta,\omega]) = 0$ for $X = \theta, \omega$, we have

$$\begin{aligned} \operatorname{Re} q(u) &= -\phi_\omega(0)^{-1}\Omega(u, \partial_\theta\phi[\theta,\omega]), \\ \operatorname{Im} q(u) &= \partial_\omega\phi_\omega(0)^{-1}\Omega(u, \partial_\omega\phi[\theta,\omega]). \end{aligned}$$

Since $v = Q[\theta,\omega]P_0^\perp u$ is the unique element of l^2 satisfying $v(x) = u(x)$ for $x \neq 0$ and $v \in \mathbf{H}_c[\theta,\omega]$, we see $v = u$ for $u \in \mathbf{H}_c[\theta,\omega]$. Finally, (3.16) follows from Proposition 1.2 and (3.15). ☐

In the following, we write $\theta(t) := \theta(u(t))$, $\omega(t) := \omega(u(t))$ and $\xi(t) := \xi(u(t))$, where $u(t)$ is the solution of (1.1). Substituting $u = \phi[\theta,\omega] + \xi$ into the equation, we have

$$\mathrm{i}\dot{\xi} + \mathrm{i}\partial_\theta\phi[\theta,\omega](\dot{\theta} - \omega) + \mathrm{i}\partial_\omega\phi[\theta,\omega]\dot{\omega} = \mathcal{H}[\theta,\omega]\xi - f[\theta,\omega,\xi] - |\xi|^6\xi, \tag{3.18}$$

where

$$f[\theta,\omega,\xi] := \sum_{0\le a\le 4, 0\le b\le 3, 2\le a+b\le 6} A_{a,b}\phi[\theta,\omega]^{4-a}\overline{\phi[\theta,\omega]}^{3-b}\xi^a\overline{\xi}^b, \tag{3.19}$$

for some $A_{a,b} \in \mathbb{N}$. We set

$$\eta(t) := P_0^\perp \xi(t). \tag{3.20}$$

Notice that from Lemma 3.6, we have $\xi(t) = Q[\theta(t),\omega(t)]\eta(t)$.

Applying $P_0^\perp$ to (3.18), we have

$$\begin{aligned} \mathrm{i}\dot{\eta} = &-\Delta_0\eta - P_0^\perp\Delta_{\mathrm{d}}(1 - Q[\theta,\omega])\eta + P_0^\perp\mathcal{V}[\theta,\omega]\xi \\ &- \mathrm{i}P_0^\perp\partial_\theta\phi[\theta,\omega](\dot{\theta} - \omega) - \mathrm{i}P_0^\perp\partial_\omega\phi[\theta,\omega]\dot{\omega} - P_0^\perp f(\theta,\omega,\xi) - P_0^\perp\left(|\xi|^6\xi\right). \end{aligned} \tag{3.21}$$

We next seek for the equation for $\dot{\theta} - \omega$ and $\dot{\omega}$. First, taking the inner product of (3.18) with $\partial_\omega\phi[\theta,\omega]$, we have

$$\begin{aligned} &\Omega(\dot{\xi}, \partial_\omega\phi[\theta,\omega]) - q'(\omega)(\dot{\theta} - \omega) \\ &= \langle \mathcal{H}[\theta,\omega]\xi, \partial_\omega\phi[\theta,\omega]\rangle - \left\langle f[\theta,\omega,\xi] + |\xi|^6\xi, \partial_\omega\phi[\theta,\omega]\right\rangle. \end{aligned} \tag{3.22}$$

Now, since $\frac{d}{dt}\Omega(\xi, \partial_\omega\phi[\theta,\omega]) = 0$,

$$\Omega(\dot{\xi}, \partial_\omega\phi[\theta,\omega]) = -\Omega(\xi, \partial_\omega^2\phi[\theta,\omega])\dot{\omega} - \Omega(\xi, \partial_\theta\partial_\omega\phi[\theta,\omega])\dot{\theta}. \tag{3.23}$$

From (3.3), we have

$$\begin{aligned}\langle \mathcal{H}[\theta,\omega]\xi, \partial_\omega \phi[\theta,\omega]\rangle &= \langle \xi, \mathrm{i}\partial_\theta \phi[\theta,\omega] + \mathrm{i}\omega\partial_\theta\partial_\omega\phi[\theta,\omega]\rangle \\ &= -\omega\Omega(\xi, \partial_\theta\partial_\omega\phi[\theta,\omega]).\end{aligned} \tag{3.24}$$

Combining (3.22), (3.23) and (3.24), we have

$$\begin{aligned}&\left(q'(\omega) + \Omega(\xi, \partial_\theta\partial_\omega\phi[\theta,\omega])\right)\left(\dot\theta - \omega\right) + \Omega(\xi, \partial_\omega^2\phi[\theta,\omega])\dot\omega \\ &= \left\langle f[\theta,\omega,\xi] + |\xi|^6\xi, \partial_\omega\phi[\theta,\omega]\right\rangle.\end{aligned} \tag{3.25}$$

Similarly, taking the innerproduct $\langle (3.18), \partial_\theta\phi[\theta,\omega]\rangle$, we have

$$\begin{aligned}&\left(q'(\omega) - \Omega(\xi, \partial_\theta\partial_\omega\phi[\theta,\omega])\right)\dot\omega - \Omega(\xi, \partial_\theta^2\phi[\theta,\omega])(\dot\theta - \omega) \\ &= -\left\langle f[\theta,\omega,\xi] + |\xi|^6\xi, \partial_\theta\phi[\theta,\omega]\right\rangle.\end{aligned} \tag{3.26}$$

Combining (3.25) and (3.26), we have

$$A[\theta,\omega,\eta]\begin{pmatrix}\dot\theta-\omega \\ \omega^{-1}\dot\omega\end{pmatrix} = \begin{pmatrix}\left\langle f[\theta,\omega,\xi] + |\xi|^6\xi, \partial_\omega\phi[\theta,\omega]\right\rangle \\ -\omega^{-1}\left\langle f[\theta,\omega,\xi] + |\xi|^6\xi, \partial_\theta\phi[\theta,\omega]\right\rangle\end{pmatrix}, \tag{3.27}$$

where

$$\begin{aligned}&A[\theta,\omega,\eta] \\ &:= \begin{pmatrix} q'(\omega) + \Omega(Q[\theta,\omega]\eta, \partial_\theta\partial_\omega\phi[\theta,\omega]) & \omega\Omega(Q[\theta,\omega]\eta, \partial_\omega^2\phi[\theta,\omega]) \\ -\omega^{-1}\Omega(Q[\theta,\omega]\eta, \partial_\theta^2\phi[\theta,\omega]) & q'(\omega) - \Omega(Q[\theta,\omega]\eta, \partial_\theta\partial_\omega\phi[\theta,\omega])\end{pmatrix}.\end{aligned} \tag{3.28}$$

Here, we have multiplied ω^{-1} to (3.26) to adjust the scale.

§4. Proof of main theorem

We set $X_T := \mathrm{Stz}(0,T) \cap L^2((0,T), l^2_{-1})$.

Proposition 4.1. *There exists $\omega_1 > \omega_0$ such that for $\omega_* > \omega_1$, there exist $\epsilon_0 \in (0,1)$ and $C_0 > 1$ with $C_0\epsilon_0 < 1$ such that for $T > 0$, if $\epsilon := \inf_{\theta\in\mathbb{R}} \|u(0) - \phi[\omega_*, \theta]\|_{l^2} < \epsilon_0$ and*

$$\|\xi\|_{\mathrm{Stz}\cap L^2 l^2_{-1}(0,T)} \le C_0\epsilon, \tag{4.1}$$

$$\|\omega^{-1}\dot\omega\|_{L^1\cap L^\infty(0,T)} + \|\dot\theta - \omega\|_{L^1\cap L^\infty(0,T)} \le C_0\epsilon, \tag{4.2}$$

then the above holds with C_0 replaced by $C_0/2$.

In the following, we assume (4.1) and (4.2). Further, when we use $\lesssim$ or $\sim$, the implicit constant will not depend on C_0, ϵ, ω_* nor ω_1. Since $\sup_{t\in(0,T)}|\omega(t)-\omega_*|\leq\|\dot\omega\|_{L^1(0,T)}\leq C_0\epsilon$, assuming $\omega_1>2$ if necessary, we have

$$\omega(t)\sim\omega_* \text{ for all } t\in(0,T). \tag{4.3}$$

Further, since we have set $\eta=P_0^\perp\xi$ in Section 3, from (2.1), we have

$$\|\eta\|_{\mathrm{Stz}\cap L^2((0,T),l^2_{-1})}\leq C_0\epsilon. \tag{4.4}$$

We start with the estimate of η.

Lemma 4.2. *Under the assumption of Proposition 4.1, we have*

$$\|\eta\|_{X_T}\lesssim\|\eta(0)\|_{l^2}+C_0\omega_1^{-\frac{5}{6}}\epsilon+(C_0\epsilon)^7. \tag{4.5}$$

Proof. From (3.21) and Proposition 2.1, we have

$$\begin{aligned}\|\eta\|_{X_T}\lesssim{}&\|\eta(0)\|_{l^2}+\|P_0^\perp\Delta_{\mathrm{d}}(1-Q[\theta,\omega])\eta\|_{L^2((0,T),l_1^2)}\\&+\|P_0^\perp\mathcal{V}[\theta,\omega]Q[\theta,\omega]\eta\|_{L^2((0,T),l_1^2)}\\&+\|P_0^\perp\partial_\theta\phi[\theta,\omega](\dot\theta-\omega)\|_{L^2((0,T),l_1^2)}+\|P_0^\perp\partial_\omega\phi[\theta,\omega]\dot\omega\|_{L^2((0,T),l_1^2)}\\&+\|P_0^\perp f(\theta,\omega,\xi)\|_{L^2((0,T),l_1^2)}+\|P_0^\perp\left(|\xi|^6\xi\right)\|_{L^1((0,T),l^2)}\\&+\|\int_0^\cdot e^{\mathrm{i}\Delta_0(\cdot-s)}P_0^\perp\left(|\xi|^6\xi(s)\right)\,ds\|_{L^2((0,T),l^2_{-1})}.\end{aligned} \tag{4.6}$$

By $\|P_0^\perp\|_{l_1^2\to l_1^2}\leq1$, $\|\Delta_{\mathrm{d}}\|_{l_1^2\to l_1^2}\lesssim1$ and Lemma 3.6, we have

$$\|P_0^\perp\Delta_{\mathrm{d}}(1-Q[\theta,\omega])\eta\|_{L^2((0,T),l_1^2)}\lesssim\omega_1^{-1}\|\eta\|_{X_T}\lesssim C_0\omega_1^{-\frac{5}{6}}\epsilon. \tag{4.7}$$

By (1.4) and (3.4), we have $\|P_0^\perp\mathcal{V}[\theta,\omega]Q[\theta,\omega]\|_{l^2_{-1}\to l_1^2}\lesssim\omega_1^{-5}$. Thus,

$$\|P_0^\perp\mathcal{V}[\theta,\omega]Q[\theta,\omega]\eta\|_{L^2((0,T),l_1^2)}\lesssim\omega_1^{-5}\|\eta\|_{X_T}\lesssim C_0\omega_1^{-\frac{5}{6}}\epsilon. \tag{4.8}$$

For the terms in the second line of (4.6), by (1.4), we have

$$\begin{aligned}&\|P_0^\perp\partial_\theta\phi[\theta,\omega](\dot\theta-\omega)\|_{L^2((0,T),l_1^2)}+\|P_0^\perp\partial_\omega\phi[\theta,\omega]\dot\omega\|_{L^2((0,T),l_1^2)}\\&\lesssim\omega_1^{-\frac{5}{6}}\|\dot\theta-\omega\|_{L^2(0,T)}+\omega_1^{-\frac{5}{6}}\|\omega^{-1}\dot\omega\|_{L^2(0,T)}\lesssim C_0\omega^{-\frac{5}{6}}\epsilon.\end{aligned} \tag{4.9}$$

For the first term of the third line of (4.6), by (1.4) and (3.19),

$$\begin{aligned}&\|P_0^\perp f(\theta,\omega,\xi)\|_{L^2((0,T),l^2{}_1)}\\&\lesssim\sum_{j=2}^6\omega_1^{-\frac{5}{6}(7-j)}\|\xi\|_{L^\infty((0,T),l^2)}^{j-1}\|\xi\|_{L^2((0,T),l^2_{-1})}\lesssim C_0\omega_1^{-\frac{5}{6}}\epsilon.\end{aligned} \tag{4.10}$$

For the second term in the third line, since $\mathrm{Stz}(0,T) \hookrightarrow L^7((0,T), l^{14}(\mathbb{Z}))$, we have

$$\|P_0^\perp \left(|\xi|^6\xi\right)\|_{L^1((0,T),l^2)} \le \|\xi\|_{X_T}^7 \le (C_0\epsilon)^7. \tag{4.11}$$

Finally, for the last term of (4.6), by Proposition 2.1, we have

$$\begin{aligned}
&\|\int_0^\cdot e^{\mathrm{i}\Delta_0(\cdot-s)}P_0^\perp\left(|\xi|^6\xi(s)\right)\,ds\|_{L^2((0,T),l^2_{-1})}\\
&\le \int_0^T \|e^{\mathrm{i}\Delta_0(\cdot-s)}P_0^\perp\left(|\xi|^6\xi(s)\right)\|_{L^2((0,T),l^2_{-1})}\,ds\\
&\lesssim \int_0^T \|P_0^\perp\left(|\xi|^6\xi(x)\right)\|_{l^2}\,ds = \|P_0^\perp\left(|\xi|^6\xi\right)\|_{L^1((0,T),l^2)}.
\end{aligned}$$

Thus, from (4.11), we have

$$\|\int_0^\cdot e^{\mathrm{i}\Delta_0(\cdot-s)}P_0^\perp\left(|\xi|^6\xi(s)\right)\,ds\|_{L^2((0,T),l^2_{-1})} \lesssim (C_0\epsilon)^7. \tag{4.12}$$

Combining (4.6)–(4.12), we have (4.5). □

Lemma 4.3. *Under the assumption of Proposition 4.1, we have*

$$|\dot\theta-\omega| + |\omega^{-1}\dot\omega| \lesssim \omega^{-\frac13}\|\eta\|_{l^2_{-1}}^2 + \|\eta\|_{l^2_{-1}}^7. \tag{4.13}$$

Proof. First, from Proposition 1.2 and Lemma 3.6 we have

$$\begin{aligned}
&|\Omega(Q[\theta,\omega]\eta, \partial_\theta\partial_\omega\phi[\theta,\omega])|\\
&\le |\Omega(\eta, P_0^\perp\partial_\theta\partial_\omega\phi[\theta,\omega])| + |\Omega((1-Q[\theta,\omega])\eta, \partial_\theta\partial_\omega\phi[\theta,\omega])| \lesssim \omega^{-\frac{11}{6}}\|\eta\|_{l^2_{-1}}.
\end{aligned}$$

Similarly, we have

$$|\omega\Omega(Q[\theta,\omega]\eta, \partial_\omega^2\phi[\theta,\omega])| + |\omega^{-1}\Omega(Q[\theta,\omega]\eta, \partial_\theta^2\phi[\theta,\omega])| \lesssim \omega^{-\frac{11}{6}}\|\eta\|_{l^2_{-1}}.$$

By (3.7), we see that if $\|\eta\|_{l^2} \lesssim 1$, $A[\theta,\omega]$, defined in (3.28), is invertible and we have

$$\|A[\theta,\omega,\eta]^{-1}\|_{\mathbb{C}^2\to\mathbb{C}^2} \lesssim \omega^{\frac23}. \tag{4.14}$$

Next,

$$\begin{aligned}
&|\langle f[\theta,\omega,\xi], \partial_\omega\phi[\theta,\omega]\rangle|\\
&\lesssim \sum_{j=2}^6 \left\langle |\phi[\theta,\omega]|^{7-j}|\xi|^{j-1}\left(|\eta| + |(1-Q[\theta,\omega])\eta|\right), |\partial_\omega\phi[\theta,\omega]|\right\rangle\\
&\lesssim \sum_{j=2}^6 \left(\omega^{\frac{-46+6j}{6}} + \omega^{-\frac{4+j}{6}}\right)\|\eta\|_{l^2_{-1}}^j \lesssim \omega^{-1}\|\eta\|_{l^2_{-1}}^2.
\end{aligned} \tag{4.15}$$

Similarly, we have

$$|\omega^{-1}\left\langle f[\theta,\omega,\xi],\partial_\theta\phi[\theta,\omega]\right\rangle| \lesssim \omega^{-1}\|\eta\|_{l^2_{-1}}^2. \tag{4.16}$$

Finally,

$$|\left\langle |\xi|^6\xi,\partial_\omega\phi[\theta,\omega]\right\rangle| + |\omega^{-1}\left\langle |\xi|^6\xi,\partial_\theta\phi[\theta,\omega]\right\rangle| \lesssim \omega^{-\frac{5}{6}}\|\eta\|_{l^2_{-1}}^7. \tag{4.17}$$

Therefore, from (3.27) and (4.14)–(4.17), we obtain (4.13). □

Proof of Proposition 4.1. By Lemma 3.6, (3.8) and (4.5), we have

$$\|\xi\|_{X_T} \le C(1 + C_0\omega_1^{-\frac{5}{6}} + C_0(C_0\epsilon)^6)\epsilon$$

for some $C > 0$. Thus, taking $C_0 = 4C$ and ω_1 sufficiently large and ϵ_0 sufficiently small so that $C(C_0\omega_1^{-\frac{5}{6}} + C_0(C_0\epsilon_0)^6) \le \frac{1}{4}C_0$, we have (4.1) with C_0 replaced by $C_0/2$.

Next, from (4.3) and (4.13), we have

$$\|\dot\theta - \omega\|_{L^1\cap L^\infty(0,T)} + \|\omega^{-1}\dot\omega\|_{L^1\cap L^\infty} \le \widetilde{C}(\omega_1^{-\frac{1}{3}} + C_0^6\epsilon^6)C_0\epsilon$$

for some $\widetilde{C} > 0$. Thus, taking ω_1 sufficiently large and ϵ_0 sufficiently small so that $\widetilde{C}(\omega_1^{-\frac{1}{3}} + C_0^6\epsilon^6) \le \frac{1}{2}$, we have (4.2) with C_0 replaced by $C_0/2$. □

Proof of Theorem 1.4. By Proposition 4.1, we have (4.1) and (4.2) with $T = \infty$. In particular, this estimate implies the convergence of ω in (1.6) and the bound on the first term in the inequality of (1.6). Further, since $\|\xi\|_{\mathrm{Stz}} < \infty$, by standard argument we see that there exists ξ_+ such that $\|\xi(t) - e^{it\Delta}\xi_+\|_{l^2} \to 0$ as $t \to \infty$. Therefore, we have (1.5) and the bound on the second term in the inequality of (1.6). □

§A. Proof of Proposition 1.2

In this Appendix, we prove Proposition 1.2. We start from the following lemma.

Lemma A.1. *For sufficiently large $\omega > 0$, $-\Delta_0 + \omega$ is invertible on $P_0^\perp l^2_{10}$, where $\Delta_0 = P_0^\perp \Delta P_0^\perp$. Further, we have $\|(-\Delta_0 + \omega)^{-1}\|_{l^2_{10}\to l^2_{10}} \lesssim \omega^{-1}$.*

Proof. We first show $\|(-\Delta_{\mathrm{d}} + \omega)^{-1}\|_{l^2_{10}\to l^2_{10}} \lesssim \omega^{-1}$. For such estimate, it suffices to show

$$\|(-\Delta_{\mathrm{d}} + \omega)^{-1}u\|_{l^2_{10}} \lesssim \omega^{-1}\|u\|_{l^2_{10}} \tag{A.1}$$

for compactly supported u. Since $(-\Delta_{\mathrm{d}}+\omega)^{-1}$ is invertible on l^2 for $\omega>4$, we set

$$v := (-\Delta_{\mathrm{d}}+\omega)^{-1}u.$$

Then, we have $(-\Delta_{\mathrm{d}}+\omega)v=u$. Moreover, $(-\Delta_{\mathrm{d}}+\omega)v(x)=0$ for $|x|\geq R$ for some R, because u is supported on a bounded set. Recall that the general solution of $(-\Delta_{\mathrm{d}}+\omega)v=0$ is given by $c_+b(\omega)^x+c_-b(\omega)^{-x}$, where

$$b(\omega)=(2+\omega+\sqrt{\omega^2+4\omega})/4\sim\omega.$$

Since $b(\omega)>1$ and $v\in l^2$, we have

$$v(x)\sim b(\omega)^{-|x|}=e^{-|x|\log b(\omega)}.$$

Therefore, taking $\omega>0$ sufficiently large, we have $v\in l^2_{10}$. Now, applying $\cosh(10x)$ to $(-\Delta_{\mathrm{d}}+\omega)v=u$, we have

$$(-\Delta_{\mathrm{d}}+\omega)\widetilde{v}=\widetilde{u}+[\cosh(10x),\Delta_{\mathrm{d}}]\cosh(10x)^{-1}\widetilde{v},$$

where $\widetilde{v}=\cosh(10x)v$ and $\widetilde{u}=\cosh(10x)u$. Thus, from $\|(-\Delta_{\mathrm{d}}+\omega)^{-1}\|_{l^2\to l^2}\lesssim\omega^{-1}$, we have

$$\|\widetilde{v}\|_{l^2}\lesssim\omega^{-1}\left(\|\widetilde{u}\|_{l^2}+\|[\cosh(10x),\Delta_{\mathrm{d}}]\cosh(-10x)\widetilde{v}\|_{l^2}\right).$$

Since

$$\begin{aligned}&[\cosh(10x),\Delta_{\mathrm{d}}]\cosh(10x)^{-1}\widetilde{v}\\&=\left(\cosh(10x)\cosh(10(x-1))^{-1}-1\right)\widetilde{v}(x-1)\\&\quad+\left(\cosh(10x)\cosh(10(x+1))^{-1}-1\right)\widetilde{v}(x+1),\end{aligned}$$

and

$$\begin{aligned}&\|\cosh(10x)\cosh(10(x-1))^{-1}-1\|_{l^\infty}\\&\qquad+\|\cosh(10x)\cosh(10(x+1))^{-1}-1\|_{l^\infty}\lesssim 1,\end{aligned}$$

we have

$$\|\widetilde{v}\|_{l^2}\lesssim\omega^{-1}\left(\|\widetilde{u}\|_{l^2}+\|\widetilde{v}\|_{l^2}\right).$$

Therefore, by $\|\widetilde{v}\|_{l^2}\sim\|v\|_{l^2_{10}}$ and $\|\widetilde{u}\|_{l^2}\sim\|u\|_{l^2_{10}}$, we have (A.1) for sufficiently large ω.

Next, we consider $-\Delta_0+\omega$. For $u\in P_0^\perp l^2_{10}$, we look for $v\in P_0^\perp l^2_{10}$ satisfying $(-\Delta_0+\omega)v=u$. Since this can be written as $P_0^\perp(-\Delta_{\mathrm{d}}+\omega)v=u$, we have

$$(-\Delta_{\mathrm{d}}+\omega)v=u-(v(1)+v(-1))\delta_0. \tag{A.2}$$

Setting $\Phi_\omega[v] := (-\Delta_d + \omega)^{-1}(u - (v(1) + v(-1))\delta_0)$, we see that Φ_ω is a contraction mapping on

$$\overline{B_{l^2_{10}(0,C\omega^{-1})}} := \{w \in l^2_{10} \mid \|w\|_{l^2_{10}} \leq C\omega^{-1}\}$$

for $C > 0$ satisfying $\|(-\Delta_d+\omega)^{-1}\|_{l^2_{10}\to l^2_{10}} \leq C\omega^{-1}$ and sufficiently large $\omega > 0$. Thus, we find $v \in l^2_{10}$ satisfying (A.2) and $\|v\|_{l^2_{10}} \lesssim \omega^{-1}$.

Finally, (A.2) at $x = 0$ implies $(2+\omega)v(0) = 0$ so we have $v \in P_0^\perp l^2_{10}$. Thus, we have the conclusion. □

Proof of Proposition 1.2. Substituting $\phi_\omega = \omega^{\frac{1}{6}}(\delta_0 + \varphi_\omega)$ into (1.2), we have

$$\omega^{-1}(-\Delta_{\rm d}\delta_0 - \Delta_{\rm d}\varphi_\omega) + \varphi_\omega - \sum_{n=1}^{6} {}_7C_n\delta_0\varphi_\omega^n - \varphi_\omega^7 = 0, \tag{A.3}$$

where ${}_7C_n$ are the binomial coefficients. We further decompose $\varphi_\omega = A_\omega\delta_0 + \psi_\omega$, where $\psi_\omega = P_0^\perp\varphi_\omega$ and $A_\omega \in \mathbb{R}$. Then, applying $P_0 := 1 - P_0^\perp$, we have

$$(-6 + 2\omega^{-1})A_\omega = -2\omega^{-1} + \omega^{-1}(\psi_\omega(1) + \psi_\omega(-1)) + \sum_{n=2}^{7} {}_7C_nA_\omega^n.$$

Thus, for $\omega > 0$ sufficiently large and for given ψ (with $\|\psi\|_{l^\infty} \lesssim 1$), we can solve $A_\omega = A_\omega(\psi)$. Further, we have

$$|A_\omega(\psi)| \lesssim \omega^{-1}(1 + |\psi(1)| + |\psi(-1)|), \tag{A.4}$$

$$|A_\omega(\psi_1) - A_\omega(\psi_2)| \lesssim \omega^{-1}(|\psi_1(1) - \psi_2(1)| + |\psi_1(-1) - \psi_2(-1)|). \tag{A.5}$$

Now, applying $P_0^\perp$ to (A.3), we have

$$\psi_\omega = (-\Delta_0 + \omega)^{-1}\left((A_\omega[\psi_\omega] + 1)(\delta_1 + \delta_{-1}) + \omega\psi_\omega^7\right), \tag{A.6}$$

where we have used $P_0^\perp\varphi_\omega^7 = \psi_\omega^7$. Setting the r.h.s. of (A.6) by $\Phi_\omega[\psi_\omega]$, we see that it suffices to show Φ_ω is a contraction mapping on

$$\overline{B_{P_0^\perp l^2_{10}(0,C\omega^{-1})}} := \{w \in P_0^\perp l^2_{10} \mid \|w\|_{l^2_{10}} \leq C\omega^{-1}\}$$

for some $C > 0$. However, from Lemma A.1, (A.4) and (A.5), we have

$$\|\Phi_\omega[0]\|_{l^2_{10}} \lesssim \omega^{-1}\|\delta_1 + \delta_{-1}\|_{l^2_{10}} \lesssim \omega^{-1}, \tag{A.7}$$

$$\begin{aligned} \|\Phi_\omega[\psi_1] - \Phi_\omega[\psi_2]\|_{l^2_{10}} &\lesssim \omega^{-1}\left(|A_\omega[\psi_1] - A_\omega[\psi_2]| + \omega\|\psi_1^7 - \psi_2^7\|_{l^2_{10}}\right) \\ &\lesssim \omega^{-1}\left(\omega^{-1} + \omega\|\psi_1\|^6_{l^2_{10}} + \omega\|\psi_1\|^6_{l^2_{10}}\right)\|\psi_1 - \psi_2\|_{l^2_{10}} \\ &\lesssim \omega^{-2}\|\psi_1 - \psi_2\|_{l^2_{10}}, \end{aligned} \tag{A.8}$$

where we have used

$$|\psi(1)| + |\psi(-1)| \le 2\|\psi\|_{l^\infty} \le 2\|\psi\|_{l^2_{10}} \le 2C\omega^{-1}$$

for $\psi \in B_{P_0^\perp l^2_{10}}(0, C\omega^{-1})$.

Thus, from (A.7) and (A.8) and since the implicit constant is independent of ω, taking C sufficiently large so that $\omega\|\Phi_\omega[0]\|_{l^2_{10}} < C/2$, we have the existence of the fixed point ψ_ω. Thus, we have the solution

$$\phi_\omega = \omega^{1/6}\left(\delta_0 + A_\omega[\psi_\omega]\delta_0 + \psi_\omega\right)$$

of (1.2) satisfying the estimates of (1.3) and (1.4) with $j = 0$.

We next estimate the derivative of ϕ_ω. We set $a(\omega) := A_\omega(\psi_\omega)$. Then, $a(\omega)$ satisfies

$$(-6 + 2\omega^{-1})a(\omega) = -2\omega^{-1} + \omega^{-1}\left(\psi_\omega(1) + \psi_\omega(-1)\right) + \sum_{n=2}^{7} {}_7C_n a(\omega)^n. \tag{A.9}$$

Differentiating (A.9) by ω, we have

$$\begin{aligned}&(-6 + 2\omega^{-1} - \sum_{n=2}^{7} n\,{}_7C_n a(\omega)^{n-1})a'(\omega)\\ &= \omega^{-1}\left(\partial_\omega\psi_\omega(1) + \partial_\omega\psi_\omega(-1)\right) + \omega^{-2}\left(2a(\omega) + 2 - \psi_\omega(1) - \psi_\omega(-1)\right).\end{aligned}$$

Therefore, by $|a(\omega)| + |\psi_\omega(1)| + |\psi_\omega(-1)| \lesssim \omega^{-1}$, we have

$$|a'(\omega)| \lesssim \omega^{-1}\left(|\partial_\omega\psi_\omega(1)| + |\partial_\omega\psi_\omega(-1)|\right) + \omega^{-2}. \tag{A.10}$$

Next, differentiating (A.6) with respect to ω, we have

$$\partial_\omega\psi_\omega = (-\Delta_0 + \omega)^{-1}\left(-\psi_\omega + a'(\omega)\left(\delta_{-1} + \delta_1\right) + 7\psi_\omega^6\partial_\omega\psi_\omega + \psi_\omega^7\right).$$

Thus, using (A.10), we have

$$\begin{aligned}&\|\partial_\omega\psi_\omega\|_{l^2_{10}}\\ &\lesssim \omega^{-1}\left(\omega^{-1} + \omega^{-1}\left(|\partial_\omega\psi_\omega(1)| + |\partial_\omega\psi_\omega(-1)|\right) + \omega^{-2} + \omega^{-5}\|\partial_\omega\psi_\omega\|_{l^2_{10}}\right).\end{aligned}$$

Taking ω sufficiently large, we have

$$\|\partial_\omega\psi_\omega\|_{l^2_{10}} \lesssim \omega^{-2}. \tag{A.11}$$

From (A.10) and (A.11), we have $|a'(\omega)| \lesssim \omega^{-2}$. From (A.10) and (A.11), we can deduce the estimate for (1.3) and (1.4) with $j = 1$.

Finally, for the case $j=2$, differentiating (A.9) twice by ω, we have

$$\begin{aligned}&(-6+2\omega^{-1}-\sum_{n=2}^{7} n{}_7C_n a(\omega)^{n-1})a''(\omega)\\ &=\omega^{-1}\left(\partial_\omega^2\psi_\omega(1)+\partial_\omega^2\psi_\omega(-1)\right)+2\omega^{-2}\left(2a'(\omega)+2-\partial_\omega\psi_\omega(1)-\partial_\omega\psi_\omega(-1)\right)\\ &\quad+2\omega^{-3}\left(2a(\omega)-\psi_\omega(1)-\psi_\omega(-1)\right)+\sum_{n=2}^{7} n(n-1){}_7C_n a(\omega)^{n-2}\left(a'(\omega)\right)^2,\end{aligned}$$

which provides the estimate

$$|a''(\omega)|\lesssim \omega^{-1}\left(|\partial_\omega^2\psi_\omega(1)|+|\partial_\omega^2\psi_\omega(-1)|\right)+\omega^{-3}. \tag{A.12}$$

Next, differentiating (A.6) twice with respect to ω, we have

$$\begin{aligned}\partial_\omega^2\psi_\omega&=(-\Delta_0+\omega)^{-1}\\ &\quad\left(-2\partial_\omega\psi_\omega+a''(\omega)\left(\delta_{-1}+\delta_1\right)+42\psi_\omega^5(\partial_\omega\psi_\omega)^2+7\psi_\omega^6\partial_\omega^2\psi_\omega+14\psi_\omega^6\partial_\omega\psi_\omega\right).\end{aligned}$$

From (A.12) and the previous estimates, we have

$$\begin{aligned}\|\partial_\omega^2\psi_\omega\|_{l^2_{10}}\lesssim \omega^{-1}\Big(\omega^{-2}+\omega^{-1}&\left(|\partial_\omega^2\psi_\omega(1)|+|\partial_\omega^2\psi_\omega(-1)|\right)\\ &+\omega^{-3}+\omega^{-8}+\omega^{-5}\|\partial_\omega\psi_\omega\|_{l^2_{10}}\Big).\end{aligned}$$

Therefore, we have

$$\|\partial_\omega^2\psi_\omega\|_{l^2_{10}}\lesssim\omega^{-3},\quad |a''(\omega)|\lesssim\omega^{-3}.$$

This completes the proof. □

Acknowledgments

M. M. was supported by JSPS KAKENHI Grant Numbers JP17H02853, JP19K03579 and G19KK0066A. M. Y. was supported by JST SPRING, Grant Number JPMJSP2109. We thank the anonymous referee for the helpful comments.

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Masaya Maeda
Department of Mathematics and Informatics, Faculty of Science, Chiba University,
Chiba 263-8522, Japan.
E-mail: `maeda@math.s.chiba-u.ac.jp`

Masafumi Yoneda
Graduate School of Science and Engineering, Chiba University,
Chiba 263-8522, Japan.
E-mail: `cama6585@chiba-u.jp`