SUT Journal of Mathematics
Vol. 58, No. 2 (2022), 97–115
DOI: 10.55937/sut/1670226368

Measure of departure from marginal point-symmetry for multi-way contingency tables

Ken Saito, Aki Ishii, Kiyotaka Iki, and Sadao Tomizawa

(Received April 27, 2022)

Abstract. To analyze two-way contingency tables with ordered categories, Iki and Tomizawa (2018) proposed a measure to distinguish two kinds of marginal asymmetry for the midpoint. However, that measure cannot necessarily judge all of the marginal asymmetries. We improve the measure and give a new measure for multi-way contingency tables. We also derive an approximate confidence interval for the measure.

AMS 2020 *Mathematics Subject Classification.* 62H17

Key words and phrases. Point symmetry, ordered categories, multi-way contingency table

§1. Introduction

Firstly, consider $R \times C$ rectangular contingency tables with ordered categories to call two-way tables. Let p_{ij} denote the probability that an observation will fall in the (i, j)th cell of the table $(i = 1, \dots, R;\ j = 1, \dots, C)$. Tomizawa [8] proposed the point-symmetry model for $R \times C$ contingency tables as follows:

$$p_{ij} = p_{i^* j^{**}} \ (i = 1, \dots, R;\ j = 1, \dots, C), \tag{1.1}$$

where $i^* = R + 1 - i$ and $j^{**} = C + 1 - j$. See Tomizawa [8], Tomizawa and Tahata [9], and Tahata and Tomizawa [7] for the details. Tomizawa [8] also proposed the marginal point-symmetry model defined by

$$\begin{aligned} p_{i\cdot} &= p_{i^*\cdot} \ (i = 1, \dots, R), \\ p_{\cdot j} &= p_{\cdot j^{**}} \ (j = 1, \dots, C), \end{aligned} \tag{1.2}$$

where

$$p_{i\cdot} = \sum_{j=1}^{C} p_{ij} \ \text{ and } \ p_{\cdot j} = \sum_{i=1}^{R} p_{ij}.$$

The model (1.2) indicates that the row marginal distribution is point-symmetric with respect to the midpoint and the column marginal distribution is also point-symmetric with respect to the midpoint. Let $[x]$ denote the maximum integer which is not larger than a real number x. For example, when $R = 4$, $\left[\frac{R}{2}\right] = 2$, and when $C = 7$, $\left[\frac{C}{2}\right] = 3$. The marginal point-symmetry model is also expressed as essentially

$$p_{i\cdot} = p_{i^*\cdot} \quad \left(i = 1, \ldots, \left[\frac{R}{2}\right]\right),$$
$$p_{\cdot j} = p_{\cdot j^{**}} \quad \left(j = 1, \ldots, \left[\frac{C}{2}\right]\right).$$

Secondly, suppose we have R^k contingency tables ($k \geq 2$) with ordered categories, to call multi-way tables. Let X_l ($l = 1, \ldots, k$) be random variables. Let $p_{\boldsymbol{i}}$ denote the probability that an observation will fall in the $\boldsymbol{i} = (i_1, \ldots, i_k)$th cell of the table ($i_n = 1, \ldots, R;\ n = 1, \ldots, k$). Wall and Lienert [11] proposed the point-symmetry model defined by

$$p_{\boldsymbol{i}} = p_{\boldsymbol{i}^*} \text{ for any } \boldsymbol{i} = (i_1, \ldots, i_k), \tag{1.3}$$

where $\boldsymbol{i}^* = (i_1^*, \ldots, i_k^*)$ and $i_t^* = R + 1 - i_t$.

The hth-order ($1 \leq h < k$) marginal probability is defined by $p_{\boldsymbol{i}}^{\boldsymbol{s}}$ that is $p_{\boldsymbol{i}}^{\boldsymbol{s}} = \Pr(X_{s_1} = i_1, \ldots, X_{s_h} = i_h)$, where $\boldsymbol{s} = (s_1, \ldots, s_h)$, $1 \leq s_1 < \cdots < s_h \leq k$ and $i_n = 1, \ldots, R$ ($n = 1, \ldots, h$). For a fixed $h = 1, \ldots, k-1$, Tahata and Tomizawa [6] proposed the marginal point-symmetry model defined by

$$p_{\boldsymbol{i}}^{\boldsymbol{s}} = p_{\boldsymbol{i}^*}^{\boldsymbol{s}} \text{ for any } \boldsymbol{s} = (s_1, \ldots, s_h). \tag{1.4}$$

When the model does not hold, we are interested in measuring the degree of departure from the model. Tomizawa et al. [10] proposed the measure from the point-symmetry model (1.1). For the measure from the marginal point-symmetry model (1.2), Yamamoto et al. [12] proposed the power-divergence type measure of $\psi^{(\lambda)}$. When the measure $\psi^{(\lambda)} = 1$, there are four types of complete asymmetry for $i = 1, \ldots, [R/2];\ j = 1, \ldots, [C/2]$, (i) $p_{i\cdot} = 0$ and $p_{\cdot j} = 0$, (ii) $p_{i^*\cdot} = 0$ and $p_{\cdot j} = 0$, (iii) $p_{i\cdot} = 0$ and $p_{\cdot j^{**}} = 0$, and (iv) $p_{i^*\cdot} = 0$ and $p_{\cdot j^{**}} = 0$. However, we cannot distinguish four types of complete asymmetry the type (i) to (iv). In some cases, it is important to know which type of asymmetry we have. In a clinical trial, when row and column variables denote conditions before treatment and after treatment, respectively, the type (i) denotes that treatment has no effect, but the type (ii) denotes that treatment has a remarkable effect.

Iki and Tomizawa [3] proposed a measure using marginal average point-symmetry that is expanded marginal point-symmetry. That measure lets us distinguish the type (i) and type (iv) complete asymmetry. However, that measure cannot judge the type (ii) and type (iii) complete asymmetry.

This paper proposes a measure expanded to 1st-ordered marginal point-symmetry for multi-way tables. In Section 2, we propose an improved measure of Iki and Tomizawa [3] and give a large-sample confidence interval. In Section 3, we extend the measure to multi-way tables.

§2. Two-way tables

2.1. Measure

Consider the $R \times C$ contingency tables. Let

$$q_{i\cdot} = \frac{p_{i\cdot}}{\delta_1}, \quad q_{i^*\cdot} = \frac{p_{i^*\cdot}}{\delta_1} \quad \left(i = 1, \ldots, \left[\frac{R}{2}\right]\right),$$
$$q_{\cdot j} = \frac{p_{\cdot j}}{\delta_2} \text{ and } q_{\cdot j^{**}} = \frac{p_{\cdot j^{**}}}{\delta_2} \quad \left(j = 1, \ldots, \left[\frac{C}{2}\right]\right).$$

Assume that $\{p_{i\cdot} + p_{i^*\cdot} \neq 0\}$ and $\{p_{\cdot j} + p_{\cdot j^{**}} \neq 0\}$. We propose the measure to represent the degree of departure from marginal point-symmetry defined by

$$\gamma_{MPS} = \frac{\delta_1\gamma_1 + \delta_2\gamma_2}{\delta_1 + \delta_2}, \tag{2.1}$$

where

$$\delta_1 = \sum_{i=1}^{\left[\frac{R}{2}\right]} (p_{i\cdot} + p_{i^*\cdot}), \quad \delta_2 = \sum_{j=1}^{\left[\frac{C}{2}\right]} (p_{\cdot j} + p_{\cdot j^{**}})$$

and

$$\gamma_1 = \frac{4}{\pi} \sum_{i=1}^{\left[\frac{R}{2}\right]} (q_{i\cdot} + q_{i^*\cdot}) \left(\theta_{1(i)} - \frac{\pi}{4}\right) \tag{2.2}$$

with

$$\theta_{1(i)} = \arccos\left(\frac{p_{i\cdot}}{\sqrt{p_{i\cdot}^2 + p_{i^*\cdot}^2}}\right)$$

and

$$\gamma_2 = \frac{4}{\pi} \sum_{i=1}^{\left[\frac{C}{2}\right]} (q_{\cdot j} + q_{\cdot j^{**}}) \left(\theta_{2(j)} - \frac{\pi}{4}\right) \tag{2.3}$$

with

$$\theta_{2(j)} = \arccos\left(\frac{p_{\cdot j^{**}}}{\sqrt{p_{\cdot j}^2 + p_{\cdot j^{**}}^2}}\right),$$

We indicate that the sub-measure γ_1 in (2.2) represents the degree of departure from point-symmetry of row marginal distribution, and the sub-measure γ_2 in (2.3) represents the degree of departure from point-symmetry of column marginal distribution. The measure γ_{MPS} in (2.1), which is the weighted sum of the sub-measure γ_1 and γ_2, represents the degree of departure from marginal point-symmetry.

The ranges of $\{\theta_{1(i)}\}$ and $\{\theta_{2(j)}\}$ are $0 \leq \theta_{1(i)} \leq \frac{\pi}{2}$ and $0 \leq \theta_{2(j)} \leq \frac{\pi}{2}$. Thus, the submeasure γ_1 and γ_2 lie between -1 and 1. Therefore, the measure γ_{MPS} lies between -1 and 1.

The submeasure γ_1 has characteristics that (1) $\gamma_1 = 1$ if and only if $p_{i\cdot} = 0$ for $i = 1, \ldots, \left[\frac{R}{2}\right]$, and (2) $\gamma_1 = -1$ if and only if $p_{i^*\cdot} = 0$ for $i = 1, \ldots, \left[\frac{R}{2}\right]$. Similarly, the submeasure γ_2 has characteristics that (1) $\gamma_2 = 1$ if and only if $p_{\cdot j^{**}} = 0$ for $j = 1, \ldots, \left[\frac{C}{2}\right]$, and (2) $\gamma_2 = -1$ if and only if $p_{\cdot j} = 0$ for $j = 1, \ldots, \left[\frac{C}{2}\right]$. The measure γ_{MPS} has characteristics that (1) $\gamma_{MPS} = 1$ if and only if $\gamma_1 = \gamma_2 = 1$, and (2) $\gamma_{MPS} = -1$ if and only if $\gamma_1 = \gamma_2 = -1$.

Note that if the marginal point-symmetry model (1.2) holds, we have $\gamma_1 = 0$ and $\gamma_2 = 0$; but the converse dose not hold. Similarly, if the marginal point-symmetry model holds, then we have $\gamma_{MPS} = 0$; but the converse dose not hold. We also note that if the submeasure $\gamma_1 = 0$ and $\gamma_2 = 0$, then measure $\gamma_{MPS} = 0$; but the converse dose not hold.

For example, consider the artificial probabilities in Tables 1a, 1b and 1c. For Table 1a, since there is the structure of the type (iii) complete asymmetry that is $p_{i\cdot} = 0$ (i.e., $\gamma_1 = 1$) and $p_{\cdot j^{**}} = 0$ (i.e., $\gamma_2 = 1$), we see that the measure $\gamma_{MPS} = 1$. Also for Table 1b, since there is the structure of the type (ii) complete asymmetry that is $p_{i^*\cdot} = 0$ (i.e., $\gamma_1 = -1$) and $p_{\cdot j} = 0$ (i.e., $\gamma_2 = -1$), we see that the measure $\gamma_{MPS} = -1$. For Table 1c, since there is the structure of the type (i) complete asymmetry that is $p_{i\cdot} = 0$ (i.e., $\gamma_1 = 1$) and $p_{\cdot j} = 0$ (i.e., $\gamma_2 = -1$), we see that the measure $\gamma_{MPS} = 0$.

2.2. Approximate confidence interval

Let n_{ij} denote the observed frequency in the (i, j)th cell of the table ($i = 1, \ldots, R$; $j = 1, \ldots, C$). Assuming that a multinomial distribution applies to the $R \times C$ table, we shall consider an approximate standard error and large-sample confidence interval for the measure γ_{MPS} and the submeasure γ_1 and γ_2 using the delta method, description of which are given by, for example, Bishop et al. [2]. The sample version of γ_{MPS}, i.e., $\widehat{\gamma}_{MPS}$, is given by γ_{MPS}

with $\{p_{ij}\}$ replaced by $\{\widehat{p}_{ij}\}$, where $\widehat{p}_{ij} = n_{ij}/N$ and $N = \sum\sum n_{ij}$. Using the delta method, $\sqrt{N}(\widehat{\gamma}_{MPS} - \gamma_{MPS})$ has asymptotically (as $N \to \infty$) a normal distribution with mean zero and variance

$$\sigma^2[\gamma_{MPS}] = \sum_{i=1}^{R}\sum_{j=1}^{C} p_{ij}\left(\frac{\partial \gamma_{MPS}}{\partial p_{ij}}\right)^2,$$

where

$$\begin{aligned}\frac{\partial \gamma_{MPS}}{\partial p_{ij}} =&(\delta_1+\delta_2)^{-2}\left\{(\delta_1+\delta_2)\left(\delta_1\frac{\partial \gamma_1}{\partial p_{ij}} + \delta_2\frac{\partial \gamma_2}{\partial p_{ij}}\right)\right\}\\ &+ (\delta_1+\delta_2)^{-2}\left\{(\gamma_1-\gamma_2)\left(\delta_2\frac{\partial \delta_1}{\partial p_{ij}} - \delta_1\frac{\partial \delta_2}{\partial p_{ij}}\right)\right\},\end{aligned}$$

with

$$\frac{\partial \delta_1}{\partial p_{ij}} = \begin{cases} 1 & (i=1,\dots,\left[\frac{R}{2}\right], \left[\frac{R+1}{2}\right]+1,\dots,R,\ j=1,\dots,C),\\ 0 & \text{otherwise,}\end{cases}$$

$$\frac{\partial \delta_2}{\partial p_{ij}} = \begin{cases} 1 & (i=1,\dots,R,\ j=1,\dots,\left[\frac{C}{2}\right], \left[\frac{C+1}{2}\right]+1,\dots,C),\\ 0 & \text{otherwise,}\end{cases}$$

$$\frac{\partial \gamma_1}{\partial p_{ij}} = \begin{cases} \dfrac{4}{\pi\delta_1}\left\{\arccos\left(\dfrac{p_{i\cdot}}{\sqrt{p_{i\cdot}^2+p_{i^*\cdot}^2}}\right) - \dfrac{p_{i^*\cdot}(p_{i\cdot}+p_{i^*\cdot})}{p_{i^*\cdot}^2+p_{i\cdot}^2}\right\} - \dfrac{\gamma_1+1}{\delta_1} & \\ \qquad (i=1,\dots,\left[\frac{R}{2}\right],\ j=1,\dots,C), & \\ \dfrac{4}{\pi\delta_1}\left\{\arccos\left(\dfrac{p_{i^*\cdot}}{\sqrt{p_{i\cdot}^2+p_{i^*\cdot}^2}}\right) + \dfrac{p_{i^*\cdot}(p_{i\cdot}+p_{i^*\cdot})}{p_{i^*\cdot}^2+p_{i\cdot}^2}\right\} - \dfrac{\gamma_1+1}{\delta_1} & \\ \qquad (i=\left[\frac{R+1}{2}\right]+1,\dots,R,\ j=1,\dots,C), & \\ 0 \qquad \text{otherwise,} & \end{cases}$$

and

$$\frac{\partial \gamma_2}{\partial p_{ij}} = \begin{cases} \dfrac{4}{\pi\delta_2}\left\{\arccos\left(\dfrac{p_{\cdot j^{**}}}{\sqrt{p_{\cdot j}^2+p_{\cdot j^{**}}^2}}\right) + \dfrac{p_{\cdot j^{**}}(p_{\cdot j}+p_{\cdot j^{**}})}{p_{\cdot j^{**}}^2+p_{\cdot j}^2}\right\} - \dfrac{\gamma_2+1}{\delta_2} & \\ \qquad (i=1,\dots,R,\ j=1,\dots,\left[\frac{C}{2}\right]), & \\ \dfrac{4}{\pi\delta_2}\left\{\arccos\left(\dfrac{p_{\cdot j}}{\sqrt{p_{\cdot j}^2+p_{\cdot j^{**}}^2}}\right) - \dfrac{p_{\cdot j^{**}}(p_{\cdot j}+p_{\cdot j^{**}})}{p_{\cdot j^{**}}^2+p_{\cdot j}^2}\right\} - \dfrac{\gamma_2+1}{\delta_2} & \\ \qquad (i=1,\dots,R,\ j=\left[\frac{C+1}{2}\right]+1,\dots,C), & \\ 0 \qquad \text{otherwise.} & \end{cases}$$

Let $\widehat{\sigma}^2[\gamma_{MPS}]$ denote $\sigma^2[\gamma_{MPS}]$ with $\{p_{ij}\}$ replaced by $\{\widehat{p}_{ij}\}$. Then, $\widehat{\sigma}[\gamma_{MPS}]/\sqrt{N}$ is an estimator of approximate standard error of $\widehat{\gamma}_{MPS}$, and

$$\left(\widehat{\gamma}_{MPS} - Z_{\frac{\alpha}{2}}\sqrt{\frac{\widehat{\sigma}^2[\gamma_{MPS}]}{N}},\ \ \widehat{\gamma}_{MPS} + Z_{\frac{\alpha}{2}}\sqrt{\frac{\widehat{\sigma}^2[\gamma_{MPS}]}{N}}\right)$$

is an approximate $(1-\alpha)$ confidence interval for γ_{MPS}. Here, $Z_{\alpha/2}$ is the upper $\alpha/2$ point of the standard normal distribution.

As for $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$, $\sqrt{N}(\widehat{\gamma}_k - \gamma_k)$ asymptotically has (as $n \to \infty$) a normal distribution with mean zero and variance

$$\sigma^2[\gamma_k] = \sum_{i=1}^{R}\sum_{j=1}^{C} p_{ij}\left(\frac{\partial \gamma_k}{\partial p_{ij}}\right)^2,$$

and

$$\left(\widehat{\gamma}_k - Z_{\frac{\alpha}{2}}\sqrt{\frac{\widehat{\sigma}^2[\gamma_k]}{N}},\ \ \widehat{\gamma}_k + Z_{\frac{\alpha}{2}}\sqrt{\frac{\widehat{\sigma}^2[\gamma_k]}{N}}\right)$$

is an approximate $(1-\alpha)$ confidence interval for γ_k $(k = 1, 2)$.

§3. Multi-way tables

Consider the $R_1 \times \cdots \times R_k$ contingency tables $(k \geq 2)$. Let X_l $(l = 1, \ldots, k)$ be lth random variables. Let $p_{\boldsymbol{i}}$ denote the probability that an observation will fall in the $\boldsymbol{i} = (i_1, \ldots, i_k)$th cell of the table $(i_t = 1, \ldots, R_t;\ t = 1, \ldots, k)$.

3.1. Measure

Let

$$\delta_j = \sum_{i=1}^{\left[\frac{R_j}{2}\right]}\left(p_i^{[j]} + p_{i^*}^{[j]}\right),$$

where 1st-order marginal probability of the jth dimension is

$$p_i^{[j]} = \Pr(X_j = i)\ \ (i = 1, \ldots, R_j,\ j = 1, \ldots, k).$$

Assume that $\{p_i^{[j]} + p_{i^*}^{[j]} \neq 0\}$. We propose a measure to represent the degree of departure from the marginal point-symmetry defined by

$$\Gamma_{MPS} = \frac{\sum_{j=1}^{k}(y_j\delta_j\Gamma_j + (1-y_j)\delta_j\Gamma_{j^*})}{\sum_{l=1}^{k}\delta_l},$$

where

$$\Gamma_j = \frac{4}{\pi} \sum_{i=1}^{\left[\frac{R_j}{2}\right]} \left(q_i^{[j]} + q_{i^*}^{[j]}\right)\left(\theta_{j(i)} - \frac{\pi}{4}\right)$$

with

$$\theta_{j(i)} = \arccos\left(\frac{p_i^{[j]}}{\sqrt{\left(p_i^{[j]}\right)^2 + \left(p_{i^*}^{[j]}\right)^2}}\right)$$

and

$$\Gamma_{j^*} = \frac{4}{\pi} \sum_{i=1}^{\left[\frac{R_j}{2}\right]} \left(q_i^{[j]} + q_{i^*}^{[j]}\right)\left(\theta_{j(i^*)} - \frac{\pi}{4}\right)$$

with

$$\theta_{j(i^*)} = \arccos\left(\frac{p_{i^*}^{[j]}}{\sqrt{\left(p_i^{[j]}\right)^2 + \left(p_{i^*}^{[j]}\right)^2}}\right)$$

and

$$q_i^{[j]} = \frac{p_i^{[j]}}{\delta_j}, \quad q_{i^*}^{[j]} = \frac{p_{i^*}^{[j]}}{\delta_j} \quad \left(i = 1, \ldots, \left[\frac{R_j}{2}\right], \quad j = 1, \ldots, k\right),$$

where $\boldsymbol{y} = (y_1, \ldots, y_j, \ldots, y_k)$ and y_j is equal to 0 or 1 ($j = 1, \ldots, k$). For example, when we consider γ_{MPS} with $k = 2$, we see $\boldsymbol{y} = (1, 0)$. Similarly, when we consider the measure of Iki and Tomizawa [3], we see $\boldsymbol{y} = (1, 1)$.

We point out that the sub-measure Γ_j and Γ_{j^*} represent the degree of departure from point-symmetry of jth marginal distribution. The measure Γ_{MPS}, being the weighted sum of all of significant Γ_j or Γ_{j^*} ($j = 1, \ldots, k$), represents the degree of departure from marginal point-symmetry.

The ranges of $\{\theta_{j(i)}\}$ and $\{\theta_{j(i^*)}\}$ are $0 \leq \theta_{j(i)} \leq \frac{\pi}{2}$ and $0 \leq \theta_{j(i^*)} \leq \frac{\pi}{2}$. Thus, the submeasure Γ_j and Γ_{j^*} lie between -1 and 1. Therefore, the measure Γ_{MPS} lies between -1 and 1. The submeasure Γ_j satisfies that (1) $\Gamma_j = 1$ if and only if $p_i^{[j]} = 0$ for $i = 0, \ldots, \left[\frac{R_j}{2}\right]$ and (2) $\Gamma_j = -1$ if and only if $p_{i^*}^{[j]} = 0$ for $i = 0, \ldots, \left[\frac{R_j}{2}\right]$. Similarly, the submeasure Γ_{j^*} satisfies that (1) $\Gamma_{j^*} = 1$ if and only if $p_{i^*}^{[j]} = 0$ for $i = 0, \ldots, \left[\frac{R_j}{2}\right]$ and (2) $\Gamma_{j^*} = -1$ if and only if $p_i^{[j]} = 0$ for $i = 1, \ldots, \left[\frac{R_j}{2}\right]$. The measure Γ_{MPS} satisfied that (1) $\Gamma_{MPS} = 1$ if and only if all of significant Γ_j or Γ_{j^*} is equal to 1 ($j = 1, \ldots, k$), and (2) $\Gamma_{MPS} = -1$ if and only if all of significant Γ_j or Γ_{j^*} is equal to -1. (3) When y_j corresponds to j digit of binary number, e.g. $\boldsymbol{y} = (1, 0, 0)$ correspond to

100, Γ_{MPS} of correspondent ones complement of $\boldsymbol{y}$ is obtained by changing the sign of Γ_{MPS} of $\boldsymbol{y}$.

3.2. Approximate confidence interval

We give an approximate standard error and large-sample confidence interval for the measure Γ_{MPS} using the delta method. Let $n_{i_1\cdots i_k}$ denote the observed frequency in the $(i_1,\ldots,i_k)$th cell of the table $(i_t = 1,\ldots,R_t;\ t = 1,\ldots,k)$. Let

$$N = \sum_{i_1=1}^{R_1}\cdots\sum_{i_k=1}^{R_k} n_{i_1\cdots n_k}.$$

We estimate Γ_{MPS} by $\widehat{\Gamma}_{MPS}$ is given by Γ_{MPS} with $\{p_{i_1\cdots i_k}\}$ replaced by $\{\widehat{p}_{i_1\cdots i_k}\}$, where $\widehat{p}_{i_1\cdots i_k} = n_{i_1\cdots i_k}/N$. Using the delta method, as $N \to \infty$, $\sqrt{N}(\widehat{\Gamma}_{MPS} - \Gamma_{MPS})$ asymptotically has a normal distribution with mean zero and variance

$$\sigma^2[\Gamma_{MPS}] = \sum_{i_1=1}^{R_1}\cdots\sum_{i_k=1}^{R_k} p_{i_1\cdots i_k}\left(\frac{\partial\Gamma_{MPS}}{\partial p_{i_1\cdots i_k}}\right)^2,$$

where

$$\begin{aligned}\frac{\partial\Gamma_{MPS}}{\partial p_i^{[j]}} &= \left(\sum_{l=1}^{k}\delta_l\right)^{-2}\left[\left\{\sum_{j=1}^{k}\left(y_j\delta_j\frac{\partial\Gamma_j}{\partial p_i^{[j]}} + (1-y_j)\delta_j\frac{\partial\Gamma_{j^*}}{\partial p_i^{[j]}}\right)\right\}\left(\sum_{l=1}^{k}\delta_l\right)\right] \\ &\quad + \left(\sum_{l=1}^{k}\delta_l\right)^{-2}\left[\sum_{j=1}^{k}(y_j\Gamma_j + (1-y_j)\Gamma_{j^*})\left(\sum_{\substack{l=1\\ l\neq j}}^{k}\delta_l\frac{\partial\delta_j}{\partial p_i^{[j]}} - \delta_j\sum_{\substack{l=1\\ l\neq j}}^{k}\frac{\partial\delta_l}{\partial p_i^{[j]}}\right)\right]\end{aligned}$$

with

$$\frac{\partial\delta_j}{\partial p_i^{[j]}} = \begin{cases}1 & \left(i = 1,\ldots,\left[\frac{R_j}{2}\right],\left[\frac{R_j+1}{2}\right]+1,\ldots,R_j\right),\\ 0 & \text{otherwise,}\end{cases}$$

$$\frac{\partial \Gamma_j}{\partial p_i^{[j]}} = \begin{cases} \dfrac{4}{\pi \delta_j} \left\{ \arccos \left(\dfrac{p_i^{[j]}}{\sqrt{\left(p_i^{[j]}\right)^2 + \left(p_{i^*}^{[j]}\right)^2}} \right) - \dfrac{p_{i^*}^{[j]} \left(p_i^{[j]} + p_{i^*}^{[j]}\right)}{\left(p_{i^*}^{[j]}\right)^2 + \left(p_i^{[j]}\right)^2} \right\} - \dfrac{\Gamma_j + 1}{\delta_j} & \left(i = 1, \ldots, \left[\frac{R_j}{2}\right]\right), \\ \dfrac{4}{\pi \delta_j} \left\{ \arccos \left(\dfrac{p_{i^*}^{[j]}}{\sqrt{\left(p_i^{[j]}\right)^2 + \left(p_{i^*}^{[j]}\right)^2}} \right) + \dfrac{p_{i^*}^{[j]} \left(p_i^{[j]} + p_{i^*}^{[j]}\right)}{\left(p_{i^*}^{[j]}\right)^2 + \left(p_i^{[j]}\right)^2} \right\} - \dfrac{\Gamma_j + 1}{\delta_j} & \left(i = \left[\frac{R_j+1}{2}\right] + 1, \ldots, R\right), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\frac{\partial \Gamma_{j^*}}{\partial p_i^{[j]}} = \begin{cases} \dfrac{4}{\pi \delta_j} \left\{ \arccos \left(\dfrac{p_{i^*}^{[j]}}{\sqrt{\left(p_i^{[j]}\right)^2 + \left(p_{i^*}^{[j]}\right)^2}} \right) + \dfrac{p_{i^*}^{[j]} \left(p_i^{[j]} + p_{i^*}^{[j]}\right)}{\left(p_{i^*}^{[j]}\right)^2 + \left(p_i^{[j]}\right)^2} \right\} - \dfrac{\Gamma_{j^*} + 1}{\delta_j} & \left(i = 1, \ldots, \left[\frac{R_j}{2}\right]\right), \\ \dfrac{4}{\pi \delta_j} \left\{ \arccos \left(\dfrac{p_i^{[j]}}{\sqrt{\left(p_i^{[j]}\right)^2 + \left(p_{i^*}^{[j]}\right)^2}} \right) - \dfrac{p_{i^*}^{[j]} \left(p_i^{[j]} + p_{i^*}^{[j]}\right)}{\left(p_{i^*}^{[j]}\right)^2 + \left(p_i^{[j]}\right)^2} \right\} - \dfrac{\Gamma_{j^*} + 1}{\delta_j} & \left(i = \left[\frac{R_j+1}{2}\right] + 1, \ldots, R\right), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\widehat{\sigma}^2[\Gamma_{MPS}]$ denote $\sigma^2[\Gamma_{MPS}]$ with $\{p_{i_1 \cdots i_k}\}$ replaced by $\{\widehat{p}_{i_1 \cdots i_k}\}$. Then, $\widehat{\sigma}[\Gamma_{MPS}]/\sqrt{N}$ is an estimator of the approximate standard error of $\widehat{\Gamma}_{MPS}$, and

$$\left(\widehat{\Gamma}_{MPS} - Z_{\frac{\alpha}{2}} \sqrt{\frac{\widehat{\sigma}^2[\Gamma_{MPS}]}{N}}, \quad \widehat{\Gamma}_{MPS} + Z_{\frac{\alpha}{2}} \sqrt{\frac{\widehat{\sigma}^2[\Gamma_{MPS}]}{N}} \right)$$

is an approximate $1 - \alpha$ confidence interval for Γ_{MPS}. Here, $Z_{\alpha/2}$ is the upper $\alpha/2$ point of the standard normal distribution.

§4. Examples

4.1. Example 1 (Two-way table)

Consider the data in Tables 2a and 2b taken directly from Agresti [1]. These data describe the results of a randomized, double-blind clinical trial comparing an active hypnotic drug with a placebo in patients with insomnia. The outcome variable is a patient's reported time to fall asleep going to bed, measured using four categories (<20 minutes, 20-30 minutes, 30-60 minutes, and >60 minutes).

We see from Table 3a that for the data in Table 2a, the estimated value of the sub-measure γ_1 is 0.545, and the confidence interval for γ_1 does not include zero. Also, Table 3a shows that the estimated value of the sub-measure γ_2 is 0.584, and the confidence interval for γ_2 does not include zero. Since the importance of sub-measure γ_1 and γ_2 are almost the same, the measure γ_{MPS} is estimated to lie between γ_1 and γ_2, and the confidence interval for γ_{MPS} does not include zero.

As for Table 3b for the data in Table 2b, the estimated value of the sub-measure γ_1 is 0.512 and the confidence interval for γ_1 does not include zero. From Table 3b, the estimated value of the sub-measure γ_2 is 0.000, and the confidence interval for γ_2 contains zero. Iki and Tomizawa [3] considered the structure of the column is the average column point-symmetry.

In addition, when we compare the data in Tables 2a and 2b using the estimated sub-measure γ_1, the degree of departure toward $p_{i\cdot} = 0$ (then $p_{i^*\cdot} > 0$) for $i = 1, 2$ is almost same for the data in Tables 2a and 2b. However, when we compare using the estimated submeasure γ_2, for patient's reported time after treatment, the degree of departure toward $p_{\cdot j} = 0$ (then $p_{\cdot j^{**}} > 0$) for $j = 1, 2$ is greater in Active treatment than in Placebo treatment. Therefore, patient's reported time after treatment in Active treatment would tend to be shorter than that in Placebo treatment.

4.2. Example 2 (Three-way table)

Consider the data in Tables 4a and 4b taken from the 2016 General Social Survey [5] conducted by the National Opinion Research Center at the University of Chicago. These describe the cross classifications of subject's opinions regarding government spending on Education, Environment, and Assistance to the poor in 1984 and 2016. The common response categories are 'too little', 'about right', and 'too much'.

When $\boldsymbol{y} = (1, 1, 1)$, the measure takes 1 when Education, Environment, and Assistance to the poor are all too much and takes -1 when all are too little. When $\boldsymbol{y} = (1, 1, 0)$, it takes 1 when Education and Environment are too much, and Assistance to the poor is too little and takes -1 when Education and Environment are too little, and Assistance to the poor is too much. When $\boldsymbol{y} = (1, 0, 1)$, it takes 1 when Education and Assistance to the poor are too much, and Environment is too little, and the measure takes -1 when Education and Assistance to the poor are too little, and Environment is too much. When $\boldsymbol{y} = (1, 0, 0)$, it takes 1 when Education is too much, and Environment and Assistance to the poor are too little, and it takes -1 when Education is too little, and Environment and Assistance to the poor are too much. By changing the value of $\boldsymbol{y}$, we can see where the frequencies are concentrated in the three-way contingency table.

No apparent difference in the measure values for any $\boldsymbol{y}$ in Table 5 indicates that the trend in answers has not changed between 1984 and 2016. The measures of $\boldsymbol{y} = (1, 1, 1)$ are -0.820 in Table 5a and -0.857 in Table 5b, respectively, which indicates that many people believe that government spending is not sufficient on the environment, education, and assistance to the poor.

As mentioned in Section 3.1, Property (3), comparing $\boldsymbol{y} = (1, 1, 1)$ and $\boldsymbol{y} = (0, 0, 0)$ in Table 5a, the measures estimate only change sign. This is a natural result if we note that when $\boldsymbol{y} = (0, 0, 0)$, the measure is 1 when Education, Environment, and Assistance to the poor are too little.

4.3. Example 3 (Three-way table)

Consider the data in Tables 6a and 6b obtained from Japan Meteorological Agency. These are obtained from the daily atmospheric temperatures at Sapporo, Tokyo, and Naha in Japan in 2010 and 2016, using three levels, 'below normal', 'normal', and 'above normal'. $\boldsymbol{y} = (1, 1, 0)$ and $\boldsymbol{y} = (1, 0, 1)$ are greatly different between 2010 and 2016. Comparing $\boldsymbol{y} = (1, 1, 0)$ and $\boldsymbol{y} = (1, 1, 1)$, we can see that in 2010, the average temperature in Naha was slightly below normal on many days. On the other hand, in 2016, there were considerably more days with above-normal temperatures. Next, comparing $\boldsymbol{y} = (1, 0, 1)$ and $\boldsymbol{y} = (1, 1, 1)$, we can see that there were more days in 2010 than in 2016 when the temperature in Tokyo was slightly above normal. Finally, comparing the measures for $\boldsymbol{y} = (1, 1, 1)$ for 2010 and 2016 shows that 2016 has somewhat higher values, indicating that the overall temperature is higher in 2016 for these three cities.

§5. Concluding remarks

We proposed a new measure to distinguish two kinds of complete asymmetry for the midpoint. Since the measure Γ_{MPS} always ranges between -1

and 1 independent of the dimension k and the sample size N, Γ_{MPS} is useful for comparing the degrees of departure from marginal point-symmetry in several tables. Our measure is the extension of the measure given by Iki and Tomizawa [3]. Since sub-measures Γ_j and Γ_{j^*} depend only on the marginal frequency of jth dimension, one can easily calculate our measure even though k increases.

Acknowledgement

The authors would like to thank a referee for the helpful comments.

References

[1] Agresti, A. (2002). *Categorical Data Analysis.*, 2nd ed. New York, Wiley.

[2] Bishop, Y. M., Fienberg, S. E., and Holland, P. W. (1975). *Discrete Multivariate Analysis: Theory and Practice.* Cambridge: The MIT Press.

[3] Iki, K. and Tomizawa, S. (2018). Measure of departure from marginal average point-symmetry for two-way contingency tables. *Journal of Modern Applied Statistical Methods* **17**, 2-20.

[4] Japan Meteorological Agency (2022). Kako no kishou data download (download historical weather data). `https://www.data.jma.go.jp/gmd/risk/obsdl/index.php` (in Japanese). (Accessed on 07/02/2022).

[5] Smith, T. W., Davern, M., Freese, J., and Hout, M. (2018). General Social Surveys, 1972-2016 [machine-readable data file] /Principal Investigator, Smith, T. W.; Co-Principal Investigator, Davern, M.; Co-Principal Investigator, Freese, J.; Co-Principal Investigator, Hout, M.; Sponsored by National Science Foundation. – NORC ed. – Chicago: NORC at the University of Chiago [producer]; Storrs, CT: The Roper Center for Public Opinion Research, University of Connecticut [distributor]. 1 data file (62,466 logical records) + 1 codebook (3,689.). – (National Data Program for the Social Sciences, No. 24).

[6] Tahata, K. and Tomizawa, S. (2008). Orthogonal decomposition of point-symmetry for multiway tables. *Advances in Statistical Analysis* **92**, 255–269.

[7] Tahata, K. and Tomizawa, S. (2014). Symmetry and asymmetry models and decompositions of models for contingency tables. *SUT Journal of Mathematics* **50**, 131–165.

[8] Tomizawa, S. (1985). The decompositions for point symmetry models in two-way contingency tables. *Biometrical Journal* **27**, 895–905.

[9] Tomizawa, S. and Tahata, K. (2007). The analysis of symmetry and asymmetry: orthogonality of decomposition of symmetry into quasi-symmetry and marginal symmetry for multi-way tables. *Journal de la Société Française de Statistique* **148**, 3–36.

[10] Tomizawa, S., Yamamoto, K., and Tahata, K. (2007). An entropy measure of departure from point-symmetry for two-way contingency tables. *Symmetry: Culture and Science; Journal of International Symmetry Association* **18**, 279–297.

[11] Wall, K. D. and Lienert, G. A. (1976). A test for point-symmetry in j-dimensional contingency-cubes. *Biometrical Journal* **18**, 259–264.

[12] Yamamoto, K., Tahata, K., Suzuki, M., and Tomizawa, S. (2011). Measure of departure from marginal point-symmetry for two-way contingency tables. *Statistica* **71**, 367–380.

Table 1: Artificial probabilities

(a) $\gamma_{MPS} = 1$

	Y				
X	(1)	(2)	(3)	(4)	Total
(1)	0	0	0	0	0
(2)	0	0	0	0	0
(3)	0.3	0.2	0	0	0.5
(4)	0.2	0.3	0	0	0.5
Total	0.5	0.5	0	0	1

(b) $\gamma_{MPS} = -1$

	Y				
X	(1)	(2)	(3)	(4)	Total
(1)	0	0	0.3	0.2	0.5
(2)	0	0	0.2	0.3	0.5
(3)	0	0	0	0	0
(4)	0	0	0	0	0
Total	0	0	0.5	0.5	1

(c) $\gamma_{MPS} = 0$

	Y				
X	(1)	(2)	(3)	(4)	Total
(1)	0	0	0	0	0
(2)	0	0	0	0	0
(3)	0	0	0.3	0.2	0.5
(4)	0	0	0.2	0.3	0.5
Total	0	0	0.5	0.5	1

Table 2: Insomniac patients reported time (in minutes) to fall asleep after going to bed from Agresti [1].

(a) Active treatment

	Follow-up				
Initial	< 20	20-30	30-60	> 60	Total
< 20	7	4	1	0	12
20-30	11	5	2	2	20
30-60	13	23	3	1	40
> 60	9	17	13	8	47
Total	40	49	19	11	119

(b) Placebo treatment

	Follow-up				
Initial	< 20	20-30	30-60	> 60	Total
< 20	7	4	2	1	14
20-30	14	5	1	0	20
30-60	6	9	18	2	35
> 60	4	11	14	22	51
Total	31	29	35	25	120

Table 3: The estimated measures, approximate standard errors, and approximate 95% confidence interval for measures are applied to Tables 2a and 2b.

(a) For Table 2a

Measure	Estimated measure	Standard error	Confidence interval
γ_{MPS}	0.564	0.056	(+0.454, +0.675)
γ_1	0.545	0.087	(+0.375, +0.714)
γ_2	0.584	0.082	(+0.424, +0.745)

(b) For Table 2b

Measure	Estimated measure	Standard error	Confidence interval
γ_{MPS}	0.256	0.053	(+0.152, +0.361)
γ_1	0.512	0.089	(+0.337, +0.688)
γ_2	0.000	0.115	(−0.226, +0.226)

Table 4: Opinions regarding government on “Education”, “Environment”, and “Assistance to the poor” in 1984 and 2016 from the 2016 General Social Survey [5].

(a) Opinions about government spending in 1984

		Assistance to the poor		
Education	Environment	too little	about right	too much
too little	too little	152	34	14
	about right	45	20	8
	too much	19	2	2
about right	too little	34	19	4
	about right	18	26	7
	too much	5	3	2
too much	too little	4	4	5
	about right	9	1	6
	too much	2	2	1

(b) Opinions about government spending in 2016

		Assistance to the poor		
Education	Environment	too little	about right	too much
too little	too little	612	110	30
	about right	134	55	11
	too much	51	11	11
about right	too little	85	30	6
	about right	46	43	9
	too much	9	11	5
too much	too little	12	8	3
	about right	16	16	8
	too much	13	8	13

Table 5: The estimated measures, approximate standard errors, and approximate 95% confidence interval for measures are applied to Tables 4a and 4b.

(a) For Table 4a

$\boldsymbol{y}$	Estimated measure	Standard error	Confidence interval
(1,1,1)	−0.820	0.075	(−0.968, −0.673)
(1,1,0)	−0.277	0.025	(−0.326, −0.229)
(1,0,1)	−0.301	0.027	(−0.353, −0.249)
(1,0,0)	0.242	0.040	(+0.165, +0.319)
For the complement of (1,1,1)			
(0,0,0)	0.820	0.075	(+0.673, +0.968)

(b) For Table 4b

$\boldsymbol{y}$	Estimated measure	Standard error	Confidence interval
(1,1,1)	−0.857	0.044	(−0.943, −0.772)
(1,1,0)	−0.274	0.012	(−0.298, −0.250)
(1,0,1)	−0.338	0.015	(−0.368, −0.309)
(1,0,0)	0.245	0.022	(+0.201, +0.289)

Table 6: The daily atmospheric temperatures at Sapporo, Tokyo, and Naha in Japan in 2010 and 2016 [4].

(a) The daily atmospheric temperatures in 2010

		Naha		
Sapporo	Tokyo	below normal	normal	above normal
below normal	below normal	19	4	5
	normal	5	2	3
	above normal	35	12	45
normal	below normal	4	1	3
	normal	1	0	1
	above normal	11	3	11
above normal	below normal	49	4	16
	normal	8	0	6
	above normal	41	11	62

(b) The daily atmospheric temperatures in 2016

		Naha		
Sapporo	Tokyo	below normal	normal	above normal
below normal	below normal	6	6	29
	normal	2	0	12
	above normal	8	4	63
normal	below normal	4	1	7
	normal	1	1	3
	above normal	3	0	15
above normal	below normal	35	5	31
	normal	6	0	24
	above normal	21	7	71

Table 7: The estimated measures, approximate standard errors, and approximate 95% confidence interval for measures are applied to Tables 6a and 6b.

(a) For Table 6a

$\boldsymbol{y}$	Estimated measure	Standard error	Confidence interval
(1,1,1)	0.213	0.039	(+0.137, +0.290)
(1,1,0)	0.268	0.036	(+0.198, +0.337)
(1,0,1)	−0.097	0.030	(−0.156, −0.039)
(1,0,0)	−0.043	0.036	(−0.113, +0.027)

(b) For Table 6b

$\boldsymbol{y}$	Estimated measure	Standard error	Confidence interval
(1,1,1)	0.378	0.047	(+0.286, +0.470)
(1,1,0)	−0.027	0.026	(−0.078, +0.024)
(1,0,1)	0.205	0.030	(+0.146, +0.264)
(1,0,0)	−0.200	0.040	(−0.278, −0.122)

Ken Saito
Department of Information Sciences, Graduate School of Science and Technology,
Tokyo University of Science, Noda, Chiba, 278-8510, Japan

Aki Ishii
Department of Information Sciences, Faculty of Science and Technology,
Tokyo University of Science, Noda, Chiba, 278-8510, Japan

Kiyotaka Iki
Faculty of Economics, Nihon University, Chiyoda, Tokyo, 101-8360, Japan

Sadao Tomizawa
Department of Information Science, Meisei University, Hino, Tokyo, 191-8506, Japan