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On the Petersson norm of the Hilbert-Siegel cusp form under the Ikeda lift

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Abstract. In this paper, we show an arithmeticity of the ratio of the Petersson norm of the Hilbert-Siegel cusp form determined by the Ikeda lift from a Hilbert cusp form to that of the Hilbert cusp form. This is a generalization of the algebraicity of the ratio of the Petersson norm of the Siegel cusp form under the Ikeda lift of an elliptic cusp form to that of the elliptic cusp form due to Y. Choie and W. Kohnen.

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Introduction

Choie and Kohnen [1] investigated the algebraicity of the ratio of the Petersson norm of the Siegel cusp form under the Saito-Kurokawa lift and the Ikeda lift to that of an elliptic cusp form (cf. [2]). Katsurada and Kawamura [8] expressed the Petersson norm of the Siegel cusp form $F(Z)$ defined by the Ikeda lift of an elliptic cusp form $f(z)$ in terms of some L-functions associated with $f(z)$. This leads to the arithmeticity of the ratio of the Petersson norm of $F(Z)$ to that of $f(z)$. Recently, Ikeda and Yamana [6] established the lift from Hilbert cusp forms to Hilbert-Siegel cusp forms and determined the standard L-functions attached to this lift explicitly (cf. [5]). It seems meaningful to ask for an expression of the Petersson norm of the Hilbert-Siegel cusp form $\mathcal{F}(Z)$ given by the Ikeda lift from a Hilbert cusp form $f(z)$ by means of some L-functions attached to $f(z)$. As a first step to this, we would like to propose to elucidate the arithmeticity of the ratio of the Petersson norm of $\mathcal{F}(Z)$ to that of $f(z)$.

The purpose of this paper is to prove the arithmeticity of the ratio of the Petersson norm of the Hilbert-Siegel cusp form $\mathcal{F}(Z)$ determined by the Ikeda lift from a Hilbert cusp form $f(z)$. Using arithmetic properties of Fourier coefficients of $\mathcal{F}(Z)$ due to [4], [6] and [12] and those of critical values of L-functions attached to $f(z)$ by [10], we may deduce the arithmeticity of the ratio of these Petersson norms (cf. [4], [6], [9], [10] and [12]). This arithmetic property of the Fourier coefficients of $\mathcal{F}(Z)$ must be required for an application to the arithmeticity theorem of critical values of the standard L-function associated with $\mathcal{F}(Z)$.

We mention that our results give a generalization of Choie and Kohnen [1].

§1. The statement of result

We denote by $\mathbb{Z}$, $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ the ring of rational integers, the rational number field, the real number field and the complex number field, respectively. Let F be a totally real algebraic number field of finite degree g. Throughout this paper, we assume that F has narrow class number one. We denote by $\mathfrak{o}$, $\mathfrak{d}$ and D_F the maximal order of F, the different of F relative to $\mathbb{Q}$ and the discriminant of F, respectively.

For an associative ring R with identity element, we denote by $R^\times$ the group of all its invertible elements and by R_n^m the module of all $m \times n$ matrices with entries in R. We put $R^m = R_1^m$, $M_m(R) = R_m^m$ and $\mathrm{GL}_m(R) = (M_m(R))^\times$ for simplicity. Denote by I_m the identity matrix of $M_m(R)$. When R is commutative, $\mathrm{SL}_m(R)$ denotes the special linear group of degree m over R. Furthermore, define the symplectic group $\mathrm{Sp}(m, R)$ of degree m over R by

$$\mathrm{Sp}(m,R) = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_{2m}(R) \;\middle|\; {}^t g \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} g = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} \right\}. \tag{1.1}$$

We denote by $S_l(\mathrm{Sp}(m,\mathfrak{o}))$ the space of all Hilbert-Siegel cusp forms of weight l with respect to $\mathrm{Sp}(m,\mathfrak{o})$. When $m = 1$, we put the space $S_l(\mathrm{Sp}(1,\mathfrak{o}))$ as $S_l(\mathrm{SL}_2(\mathfrak{o}))$.

Let f be a Hecke eigen form of $S_l(\mathrm{SL}_2(\mathfrak{o}))$. Denote by $L(s, f)$ (resp. $L(s, \pi(f))$) the Hecke L-function (resp. the automorphic L-function) attached to f (resp. $\pi(f)$) defined in [10, (2.25)] (resp. [3, (6.43)] and [7, pp. 349-350]) in the sense of Shimura[10] (resp. Jacquet-Langlands [7]), where $\pi(f)$ means the automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ associated with f and $\mathbb{A}_F$ denotes the adele ring of F. Observe that

$$\begin{aligned} L(s,f) &= \prod_{\wp} \left(1 - a(\wp)N(\wp)^{-s} + N(\wp)^{l-1-2s}\right)^{-1}, \\ L(s,\pi(f)) &= \prod_{\wp} \left(\left(1 - \alpha_\wp N(\wp)^{-s}\right)\left(1 - \alpha_\wp^{-1} N(\wp)^{-s}\right)\right)^{-1} \end{aligned} \tag{1.2}$$

$$\text{and} \quad L(s,\pi(f)) = L\left(s+\frac{l-1}{2}, f\right),$$

where $\wp$ runs over all prime ideals in F, $a(\wp)$ is the Hecke eigen value of f at $\wp$ and $\alpha_\wp$ and $\alpha_\wp^{-1}$ are the Satake parameters of f at $\wp$.

Next we remember the definition of the standard L-function attached to a Hecke eigen form in $S_l(\mathrm{Sp}(m,\mathfrak{o}))$. Let $\mathcal{F}$ be a Hecke eigen form of $S_l(\mathrm{Sp}(m,\mathfrak{o}))$. We denote by $L(s,\mathcal{F},\mathrm{st})$ the standard L-function determined by the Satake parameters of $\mathcal{F}$ at all prime ideals in F. More precisely, define

$$\begin{aligned} L(s,\mathcal{F},\mathrm{st}) = \zeta_F(s)\prod_{\wp}\Bigg(&\left(1-\alpha_{1\wp}N(\wp)^{-s}\right)\left(1-\alpha_{1\wp}^{-1}N(\wp)^{-s}\right) \\ &\cdots\left(1-\alpha_{m\wp}N(\wp)^{-s}\right)\left(1-\alpha_{m\wp}^{-1}N(\wp)^{-s}\right)\Bigg)^{-1}, \end{aligned} \tag{1.3}$$

where $\zeta_F(s)$ means the Dedekind L-function of F, $\wp$ runs over all prime ideals $\wp$ in F and $\alpha_{1\wp},\alpha_{1\wp}^{-1},\ldots,\alpha_{m\wp},\alpha_{m\wp}^{-1}$ denote the Satake parameters of $\mathcal{F}$ at $\wp$. From here, we make clear arithmeticity properties of the Petersson norm of the Ikeda lift from a Hilbert cusp form to the Hilbert-Siegel cusp form formulated by the Ikeda-Yamana [6].

Let n and k be even positive integers. Take a normalized Hecke eigen form f of $S_{2k-n}(\mathrm{SL}_2(\mathfrak{o}))$. Let $\mathcal{F}=\Psi(f)$ be the cusp form of $S_k(\mathrm{Sp}(n,\mathfrak{o}))$ which is the Ikeda lift of f given in [5] and [6]. We refer to Theorem 3.2, Theorem 3.3 in [5] and Corollary 10.1 in [6] in this regard. Then, Ikeda and Yamana [6] proved that $\mathcal{F}$ is a Hecke eigen form and its standard L-function $L(s,\mathcal{F},\mathrm{st})$ attached to $\mathcal{F}$ is equal to

$$L(s,\mathcal{F},\mathrm{st}) = \zeta_F(s)\prod_{j=1}^{n} L(s+k-j,f). \tag{1.4}$$

Remark 1. Here we explain some differences of notation in this paper and that in [6, pp.1162-1164]. Put $n=2n'$ and $k=2k'$. Furthermore, put $\widetilde{n}=n'$ and $\widetilde{k}=2k'-n'$. Then, notation $\widetilde{n}$ and $\widetilde{k}$ coincide with notation n and k given in [6, pp.1162-1164].

Remark 2. We mention that the relation (1.4) can be formulated in terms of several automorphic L-functions $L(s,\pi(f))$. Using the relation (1.2), we can derive (1.4) from the corresponding relation in [6, p.1164]. Shimura [10] investigated the arithmetic property of the Hecke L-functions $L(s,f)$ at critical points. Since we need to apply results of them for the proof of our main theorem, we adopt the formulation (1.4) using L-functions $L(s,f)$.

We may deduce the following theorem.

Theorem. *Under the above notation and assumptions, suppose that* $k > 2n+1$. *Then,*

$$\left(\frac{\pi^{ng/2}\langle f,f\rangle^{n/2}}{D_F^{1/2}\langle \mathcal{F},\mathcal{F}\rangle}\right)^{\sigma} = \frac{\pi^{ng/2}\langle f^{\sigma},f^{\sigma}\rangle^{n/2}}{D_F^{1/2}\langle \mathcal{F}^{\sigma},\mathcal{F}^{\sigma}\rangle} \in \mathbb{Q}(f) \quad \text{for all } \sigma\in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}), \tag{1.5}$$

where $\langle f,f\rangle$ *(resp.* $\langle \mathcal{F},\mathcal{F}\rangle$*) denotes the Petersson inner product of* f *(resp.* $\mathcal{F}$*) given in [10] (resp. [9]),* f^{σ} *(resp.* $\mathcal{F}^{\sigma}$*) means the action of* σ *on* f *(resp.* $\mathcal{F}$*) defined in [10] (resp. [13]) and* $\mathbb{Q}(f)$ *denotes the field generated by the Fourier coefficients of* f *over* $\mathbb{Q}$.

§2. Proof of Theorem

Define $A_1(n_1,n_2,f)$ and $A_n(\mathcal{F})$ by

$$A_1(n_1,n_2,f) = \frac{L(n_1,f)L(n_2,f)}{\pi^{(n_1+n_2)g}\langle f,f\rangle} \text{ and } A_n(\mathcal{F}) = \frac{L(k-n,\mathcal{F},\mathrm{st})}{\pi^{(2kn+k-3n(n+1)/2)g}\langle \mathcal{F},\mathcal{F}\rangle} \tag{2.1}$$

for f and $\mathcal{F}$ in Theorem. From (1.4) and (2.1), the ratio $\dfrac{\pi^{ng/2}\langle f,f\rangle^{n/2}}{D_F^{1/2}\langle \mathcal{F},\mathcal{F}\rangle}$ can be transformed into

$$\begin{aligned}
&\frac{\pi^{ng/2}\pi^{(2kn+k-3n(n+1)/2)g}A_n(\mathcal{F})\langle f,f\rangle^{n/2}}{D_F^{1/2}\zeta_F(k-n)\prod_{j=1}^{n}L(2k-n-j,f)}\\
&= \frac{\pi^{(k-n)g}\pi^{(2kn-3n^2/2)g}A_n(\mathcal{F})\langle f,f\rangle^{n/2}}{D_F^{1/2}\zeta_F(k-n)\prod_{\substack{j=1\\ j:\mathrm{odd}}}^{n-1}L(2k-n-j,f)L(2k-n-(j+1),f)}\\
&= \frac{\pi^{(k-n)g}\pi^{(2kn-3n^2/2)g}A_n(\mathcal{F})\langle f,f\rangle^{n/2}}{D_F^{1/2}\zeta_F(k-n)}\\
&\quad\times \prod_{\substack{j=1\\ j:\mathrm{odd}}}^{n-1}\frac{1}{\left(\pi^{(4k-2n-2j-1)g}A_1(2k-n-j,2k-n-j-1,f)\langle f,f\rangle\right)}\\
&= \frac{\pi^{(k-n)g}A_n(\mathcal{F})}{D_F^{1/2}\zeta_F(k-n)\prod_{\substack{j=1\\ j:\mathrm{odd}}}^{n-1}\left(A_1(2k-n-j,2k-n-j-1,f)/\pi^{g}\right)}.
\end{aligned} \tag{2.2}$$

Define

$$A(m,f)=\frac{L(m,f)}{\left(2\pi\sqrt{-1}\right)^{gm}}\ \text{and}\ E(f)=\frac{\pi^{(2k-n)g}\langle f,f\rangle}{\left(2\pi\sqrt{-1}\right)^{g(-1+(2k-n))}}. \tag{2.3}$$

Then we have

$$A_1(n_1,n_2,f)=\left(2\sqrt{-1}\right)^{g(1-(2k-n)+n_1+n_2)}\pi^g\frac{A(n_1,f)A(n_2,f)}{E(f)}. \tag{2.4}$$

When j is odd, we have

$$\begin{aligned}(2.5)\quad & A_1(2k-n-j,2k-n-j-1,f)/\pi^g\\ &=2^{g(2k-n-2j)}(-1)^{g(k-n/2-j)}\frac{A(2k-n-j,f)A(2k-n-j-1,f)}{E(f)}.\end{aligned}$$

Hence, by virtue of [10, (4.16)], we see that

$$\frac{A_1(2k-n-j,2k-n-j-1,f)}{\pi^g}\ \text{belongs to}\ \mathbb{Q}(f) \tag{2.6}$$

and

$$\left(\frac{A_1(2k-n-j,2k-n-j-1,f)}{\pi^g}\right)^{\sigma}=\frac{A_1(2k-n-j,2k-n-j-1,f^{\sigma})}{\pi^g}$$

for every $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, where $\overline{\mathbb{Q}}$ means the algebraic closure of $\mathbb{Q}$. Observe that

$$\frac{\pi^{(k-n)g}}{D_F^{1/2}\zeta_F(k-n)}\ \text{lies in}\ \mathbb{Q}\quad(\text{cf. [10, Proposition 3.1]}). \tag{2.7}$$

Noting [4, Theorem 13.5, Theorem 9.4 and it's Note], [10], [11] and [12, Proposition 8.1, Proposition 8.6 and Proposition 8.9], and using [6, Corollary 10.1], we can assume that the Fourier coefficients of $\mathcal{F}$ belong to the totally real algebraic number field $\mathbb{Q}(f)$. Therefore, by [9, Theorem 1.2], we have

$$(A_n(\mathcal{F}))^{\sigma}=A_n(\mathcal{F}^{\sigma})\ \text{for every}\ \sigma\in\mathrm{Aut}(\mathbb{C}/\mathbb{Q}). \tag{2.8}$$

Combining (2.2), (2.6), (2.7) and (2.8) with [10, (4.16)], we conclude the proof of our theorem.

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