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Slash indecomposability of Brauer-friendly modules

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Abstract. Ishioka-Kunugi [9] gives an equivalent condition for Scott modules to be Brauer indecomposable. This paper generalizes the equivalent condition to that for Brauer-friendly modules to be slash indecomposable.

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§1. Introduction

Let p be a prime number and $\mathcal{O}$ a complete discrete valuation ring with algebraically closed residue field k of characteristic p. In the modular representation theory of finite groups, the following Broué's conjecture is one of the most important problems and has been studied by many researchers.

Conjecture (Broué's conjecture)**.** *Let G be a finite group, b a block of $\mathcal{O}G$ with a defect group P, and c the Brauer correspondent of b in $\mathcal{O}N_G(P)$. If P is abelian, then the block algebras $\mathcal{O}Gb$ and $\mathcal{O}N_G(P)c$ are derived equivalent.*

It is known that the conjecture holds in many groups and constructing a stable equivalence of Morita type between the block algebras $\mathcal{O}Gb$ and $\mathcal{O}N_G(P)c$ can be used to prove the correctnesses. In Theorem 1.1 and Theorem 1.4, we review the gluing principle of constructing stable equivalences of Morita type for principal blocks and general blocks.

First, we consider the case where b is the principal block of $\mathcal{O}G$. In this case, M. Broué introduced the following method which is useful for constructing a stable equivalence of Morita type.

Theorem 1.1 (Broué's gluing principle [6, 6.3. Theorem])**.** *Let G and H be finite groups having a common Sylow p-subgroup P such that $\mathcal{F}_P(G) = \mathcal{F}_P(H)$ and b and c the principal blocks of $\mathcal{O}G$ and $\mathcal{O}H$, respectively. For any* *subgroup Q of P, let b_Q and c_Q be the principal blocks of $kC_G(Q)$ and $kC_H(Q)$, respectively, and $M = S(G \times H, \Delta P)$ the Scott $\mathcal{O}(G \times H)$-module with vertex ΔP. Then the following are equivalent.*

(i) The bimodule M and its dual M^* induce a stable equivalence of Morita type between $\mathcal{O}Gb$ and $\mathcal{O}Hc$.

(ii) The bimodule $Br_Q(M)$ and its dual $Br_Q(M)^*$ induce a Morita equivalence between $kC_G(Q)b_Q$ and $kC_H(Q)c_Q$, for each non-trivial subgroup Q of P.

In [11], R. Kessar, N. Kunugi, and N. Mitsuhashi introduced the Brauer indecomposability, which plays a key role when we apply the principle to principal blocks..

Definition 1.2 ([11])**.** *Let M be an indecomposable $\mathcal{O}G$-module. We say that M is Brauer indecomposable if $Res^{N_G(Q)/Q}_{QC_G(Q)/Q}(Br_Q(M))$ is indecomposable or 0, for any p-subgroup Q of G.*

In [9], H. Ishioka and N. Kunugi gave an equivalent condition for Scott modules to be Brauer indecomposable as follows.

Theorem 1.3 ([9, Theorem 1.3])**.** *Let G be a finite group and P a p-subgroup of G. Let $M = S(G, P)$ and suppose that $\mathcal{F} = \mathcal{F}_P(G)$ is saturated. Then the following conditions are equivalent.*

(i) M is Brauer indecomposable.

(ii) $Res^{N_G(Q)}_{QC_G(Q)}(S(N_G(Q), N_P(Q)))$ is indecomposable, for each fully $\mathcal{F}$-normalized subgroup Q of P.

If these conditions are satisfied, then $Br_Q(M) \cong S(N_G(Q), N_P(Q))$ for each fully $\mathcal{F}$-normalized subgroup Q of P.

Next, we consider the case where b is a general block of $\mathcal{O}G$. M. Linckelmann has generalized Broué's gluing principle to general blocks as follows.

Theorem 1.4 (Linckelmann's gluing principle [12, Theorem 1.2])**.** *Let G and H be finite groups and b and c blocks of $\mathcal{O}G$ and $\mathcal{O}H$, respectively, with a common defect group P. Let $i \in (\mathcal{O}Gb)^{\Delta P}$ and $j \in (\mathcal{O}Hc)^{\Delta P}$ be almost source idempotents. For any subgroup Q of P, denote by e_Q and f_Q the unique blocks of $kC_G(Q)$ and $kC_H(Q)$, respectively, satisfying $Br_{\Delta Q}(i)e_Q \neq 0$ and $Br_{\Delta Q}(j)f_Q \neq 0$. Denote by $\hat{e}_Q$ and $\hat{f}_Q$ the unique blocks of $\mathcal{O}C_G(Q)$ and $\mathcal{O}C_H(Q)$ lifting e_Q and f_Q, respectively. Suppose that $\mathcal{F}_{(P,\hat{e}_P)}(G, b) = \mathcal{F}_{(P,\hat{f}_P)}(H, c)$, and write $\mathcal{F} = \mathcal{F}_{(P,\hat{e}_P)}(G, b)$. Let V be an $\mathcal{F}$-stable indecomposable endo-permutation $\mathcal{O}P$-module with vertex P, viewed as an $\mathcal{O}\Delta P$-module* *through the canonical isomorphism $\Delta P \cong P$. Let M be an indecomposable direct summand of the $\mathcal{O}Gb$-$\mathcal{O}Hc$-bimodule*

$$\mathcal{O}Gi \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta P}^{P\times P}(V) \otimes_{\mathcal{O}P} j\mathcal{O}H.$$

Suppose that M has ΔP as a vertex as an $\mathcal{O}[G\times H]$-module. Then for any non-trivial subgroup Q of P, there is a canonical $kC_G(Q)e_Q$-$kC_H(Q)f_Q$-module M_Q satisfying $End_k(M_Q) \cong \mathrm{Br}_{\Delta Q}(\mathrm{End}_{\mathcal{O}}(\hat{e}_Q M \hat{f}_Q))$. Moreover, if for all non-trivial subgroups Q of P the bimodule M_Q induces a Morita equivalence between $kC_G(Q)e_Q$ and $kC_H(Q)f_Q$, then M and its dual M^* induce a stable equivalence of Morita type between $\mathcal{O}Gb$ and $\mathcal{O}Hc$.

In [3], E. Biland defined Brauer-friendly modules and generalized slash functors. Brauer-friendly modules are generalizations of (endo-)p-permutation modules. The module M which appears in the theorem above is a Brauer-friendly module, and the module M_Q which appears in the theorem can be represented as $Sl_{(\Delta Q,\hat{e}_Q\otimes\hat{f}_Q)}(M)$ by using a $(\Delta Q, \hat{e}_Q \otimes \hat{f}_Q)$-slash functor $Sl_{(\Delta Q,\hat{e}_Q\otimes\hat{f}_Q)}$. For Brauer-friendly modules, slash indecomposability can be defined in the similar way as Brauer indecomposability. For the same reason as in Broué's gluing principle, slash indecomposability plays an important role in Linckelmann's gluing principle.

In this study we generalize Ishioka-Kunugi's equivalent condition to an equivalent condition for Brauer-friendly modules to be slash indecomposable.

In Section 2 and 3, we review the definitions of subpairs, fusion systems, and Brauer functors and we review the theory of Brauer-friendly modules and slash functors that E. Biland defined in [3]. In Section 4, we prove a generalization of lemmas of [9, Section 2] for p-permutation modules, Scott modules to Brauer-friendly modules, Brauer-friendly Scott modules. In Section 5, we give an equivalent condition for Brauer-friendly modules to be slash indecomposable, which generalizes the equivalent condition for Scott modules to be Brauer indecomposable.

§2. Notation

Throughout this paper, we use the following notation and terminology. Basically, we use the same notation and terminology as in [3].

Let p be a prime number, $\mathcal{O}$ a complete discrete valuation ring with algebraically closed residue field k of characteristic p. We fix a finite group G and a block b of $\mathcal{O}G$. Throughout this paper, RG-modules mean finitely generated RG-lattices, for $R \in \{\mathcal{O}, k\}$. For any $x \in \mathcal{O}G$, we denote by $\overline{x}$ its image by the natural map $\mathcal{O}G \twoheadrightarrow kG$. We denote by ${}_{\mathcal{O}G}\mathbf{Mod}$ the category of all $\mathcal{O}G$-modules. We set $\Delta G = \{(g, g) \mid g \in G\}$. We write $\overline{N}_G(H) = N_G(H)/H$

for a subgroup H of G. For any G-set X and any subgroup H of G, we set $X^H = \{x \in X \mid h \cdot x = x, h \in H\}$. For any indecomposable $\mathcal{O}G$-module M, we denote by $\mathrm{vtx}(M)$ a vertex of M and $\mathrm{s}(M)$ a source of M. For any two $\mathcal{O}G$-modules M and N, we write $M \mid N$ if M is isomorphic to a direct summand of N. For any $\mathcal{O}G$-module M and any subgroup H of G, the relative trace map $\mathrm{Tr}_H^G : M^H \to M^G$ is defined by $\mathrm{Tr}_H^G(m) = \sum_{x \in G/H} x \cdot m$. For any $\mathcal{O}G$-module M and any p-subgroup P of G, the Brauer construction of M with respect to P is the $k\overline{N}_G(P)$-module defined by

$$\mathrm{Br}_P(M) = M^P/(\sum_{Q<P} \mathrm{Tr}_Q^P(M^Q) + J(\mathcal{O})M^P).$$

We denote by $\mathrm{br}_P^M : M^P \twoheadrightarrow \mathrm{Br}_P(M)$ the natural map. In particular, we write $\mathrm{br}_P = \mathrm{br}_P^{\mathcal{O}G}$. For any $f \in \mathrm{Hom}_{\mathcal{O}G}(L, M)$, $k\overline{N}_G(P)$-homomorphism $\mathrm{Br}_P(f) \in \mathrm{Hom}_{k\overline{N}_G(P)}(\mathrm{Br}_P(L), \mathrm{Br}_P(M))$ is naturally determined. Hence, Br_P induces a functor

$$\mathrm{Br}_P : {}_{\mathcal{O}G}\mathbf{Mod} \to {}_{k\overline{N}_G(P)}\mathbf{Mod}.$$

We recall the definition of subpairs. A subpair of G is a pair (P, b_P) consisting of a p-subgroup P of G and a block b_P of $\mathcal{O}C_G(P)$. We call the subpair (P, b_P) a (G, b)-subpair if $\bar{b}_P \mathrm{br}_P(b) \neq 0$. For (G, b)-subpair (P, b_P), the block b_P is also a block of $\mathcal{O}H$ for a subgroup H such that $C_G(P) \leq H \leq N_G(P, b_P)$. The set of (G, b)-subpairs is a poset, and the group G acts on the set by conjugation.

We recall the definition of the Brauer functor with respect to (G, b)-subpair. Let (P, b_P) be a (G, b)-subpair, M an $\mathcal{O}Gb$-module. The Brauer construction of M with respect to the subpair (P, b_P) is the $k\overline{N}_G(P, b_P)\bar{b}_P$-module defined by $\mathrm{Br}_{(P,b_P)}(M) = \mathrm{Br}_P(b_P M)$, here we identify the block $\bar{b}_P$ of $kN_G(P, b_P)$ with an idempotent of $k\overline{N}_G(P, b_P)$. The $kN_G(P, b_P)$-epimorphism

$$\mathrm{br}_{(P,b_P)}^M : M^P \twoheadrightarrow \mathrm{Br}_{(P,b_P)}(M)$$

is defined by $m \mapsto \mathrm{br}_P^{b_P M}(b_P m)$. For any $f \in \mathrm{Hom}_{\mathcal{O}Gb}(L, M)$, we define

$$\mathrm{Br}_{(P,b_P)}(f) = \mathrm{Br}_P(b_P f b_P) \in \mathrm{Hom}_{k\overline{N}_G(P,b_P)\bar{b}_P}(\mathrm{Br}_{(P,b_P)}(L), \mathrm{Br}_{(P,b_P)}(M)).$$

So $\mathrm{Br}_{(P,b_P)}$ induces a functor

$$\mathrm{Br}_{(P,b_P)} : {}_{\mathcal{O}Gb}\mathbf{Mod} \to {}_{k\overline{N}_G(P,b_P)\bar{b}_P}\mathbf{Mod}.$$

We recall the definitions of Brauer categories and fusion systems. The Brauer category $\mathbf{Br}(G, b)$ is defined as follows: the objects of $\mathbf{Br}(G, b)$ are the (G, b)-subpairs, and for any two objects (P, b_P), (Q, b_Q), the morphism set $\mathrm{Hom}_{\mathbf{Br}(G,b)}((P, b_P), (Q, b_Q))$ is the set of all group homomorphisms $\phi : P \to Q$

such that there exists $g \in G$ satisfying ${}^g(P, b_P) \leq (Q, b_Q)$ and $\phi(x) = {}^g x$ for any $x \in P$. Let (P, b_P) be a (G, b)-subpair. The fusion system $\mathcal{F}_{(P,b_P)}(G, b)$ is defined as follows: the objects of $\mathcal{F}_{(P,b_P)}(G, b)$ are the subgroups of P, and for any two objects Q and R, the morphism set $\mathrm{Hom}_{\mathcal{F}_{(P,b_P)}(G,b)}(Q, R)$ is the set of all group homomorphisms $\phi : Q \to R$ such that there exists $g \in G$ satisfying ${}^g(Q, b_Q) \leq (R, b_R)$ for $(Q, b_Q), (R, b_R) \leq (P, b_P)$ and $\phi(x) = {}^g x$ for any $x \in Q$.

We review the definitions of vertex subpairs and source triples from [3]. Let M be an indecomposable $\mathcal{O}Gb$-module. A (G, b)-subpair (P, b_P) is called a vertex subpair of M if $M \mid b\mathcal{O}Gb_P \otimes_{\mathcal{O}P} V$ and $P \leq_G \mathrm{vtx}(M)$ for some indecomposable $\mathcal{O}P$-module V. For such V, it is called a source of M with respect to the vertex subpair (P, b_P). A triple (P, b_P, V) is called a source triple of M if V is a source of M with respect to the vertex subpair (P, b_P). If M has a source triple (P, b_P, V), then a vertex of M is P and a source of M is V from [3, Lemma 1]. We can consider the Green correspondence with respect to a source triple as follows.

Theorem 2.1 ([3, Lemma 1, Definition 2])**.** *Let (P, b_P) be a (G, b)-subpair. If M is an indecomposable $\mathcal{O}Gb$-module with source triple (P, b_P, V), then there exists a unique indecomposable $\mathcal{O}N_G(P, b_P)$-direct summand $f^b_{b_P}(M)$ of $b_P M$ with source triple (P, b_P, V). Then $f^b_{b_P}$ induces a one-to-one correspondence between the isomorphism classes of indecomposable $\mathcal{O}Gb$-modules with source triple (P, b_P, V) and the isomorphism classes of indecomposable $\mathcal{O}N_G(P, b_P)$-modules with source triple (P, b_P, V). The $f^b_{b_P}$ is called the Green correspondence with respect to (P, b_P).*

Brauer-friendly modules defined in the next section have fusion-stable endo-permutation moudules as sources. We recall the definition of fusion-stable endo-permutation modules. We call an $\mathcal{O}G$-module M an *endo-permutation $\mathcal{O}G$-module* if $\mathrm{End}_{\mathcal{O}}(M)$ is a permutation $\mathcal{O}G$-module.

Definition 2.2 ([13, Definition 9.9.1])**.** *Let (P, b_P) be a (G, b)-subpair, V an endo-permutation $\mathcal{O}P$-module, and set $\mathcal{F} = \mathcal{F}_{(P,b_P)}(G, b)$. We say that V is $\mathcal{F}$-stable if for any subgroup Q of P and any $\phi_{g^{-1}} \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$, the endo-permutation $\mathcal{O}Q$-module $Res^P_Q(V)$ and $Res_{\phi_{g^{-1}}}(V) = \mathrm{Res}^{{}^g P}_Q({}^g V)$ are compatible. We call the triple (P, b_P, V) a fusion-stable endo-permutation source triple if V is a $\mathcal{F}$-stable capped indecomposable endo-permutation $\mathcal{O}P$-module.*

In [7], E. C. Dade introduced slash constructions for endo-permutation modules over p-groups. For any endo-permutation $\mathcal{O}P$-module V and $Q \trianglelefteq P$, we denote by $V[Q]$ the slashed module of V with respect to Q.

§3. Brauer-friendly modules and slash functors

In this section, we review the definitions of Brauer-friendly modules and slash functors that were defined in [3].

Definition 3.1 ([3, Definition 6]). *Let (P_1, b_1, V_1) and (P_2, b_2, V_2) be fusion-stable endo-permutation source triples in (G, b). We say that (P_1, b_1, V_1) and (P_2, b_2, V_2) are compatible if the endo-permutation $\mathcal{O}Q$-modules $Res_{\phi_1}(V_1)$ and $Res_{\phi_2}(V_2)$ are compatible for any (G, b)-subpair (Q, b_Q) and any morphism $\phi_i \in \mathrm{Hom}_{\mathrm{Br}(G,b)}((Q, b_Q), (P_i, b_{P_i}))$ for $i \in \{1, 2\}$.*

Definition 3.2 ([3, Definition 8]). *Let M be an $\mathcal{O}Gb$-module which admits the decomposition $M = \bigoplus_{1 \le i \le n} M_i$ of M, where each M_i is indecomposable $\mathcal{O}Gb$-module with source triple (P_i, b_{P_i}, V_i). We say that $\mathcal{O}Gb$-module M is Brauer-friendly if (P_i, b_{P_i}, V_i) is a fusion-stable endo-permutation source triple for any $i \in \{1, \ldots, n\}$, and, (P_i, b_{P_i}, V_i) and (P_j, b_{P_j}, V_j) are compatible for every $i, j \in \{1, \ldots, n\}$.*

Definition 3.3 ([3, Definition 8]). *Let L and M be Brauer-friendly $\mathcal{O}Gb$-modules. We say that the L and M are compatible if $L \oplus M$ is a Brauer-friendly $\mathcal{O}Gb$-module.*

Definition 3.4 ([3, Definition 15]). *Let ${}_{\mathcal{O}Gb}\mathbf{M}$ be a subcategory of the category ${}_{\mathcal{O}Gb}\mathbf{Mod}$. We say that ${}_{\mathcal{O}Gb}\mathbf{M}$ is Brauer-friendly if any object of ${}_{\mathcal{O}Gb}\mathbf{M}$ is a Brauer-friendly $\mathcal{O}Gb$-module, and any two objects of ${}_{\mathcal{O}Gb}\mathbf{M}$ are compatible.*

Definition 3.5. *Let (P, b_P, V) be a fusion-stable endo-permutation source triples in (G, b). We say that a Brauer-friendly category is big enough with respect to (P, b_P, V) if any finite direct sum of indecomposable $\mathcal{O}Gb$-modules with source triple (P, b_P, V) belongs to the Brauer-friendly category. Let $\mathcal{S}$ be a set of compatible source triples of G. Also we define that big enough with respect to $\mathcal{S}$.*

Definition 3.6 ([3, Definition 14]). *Let G be a finite group, b a block of $\mathcal{O}G$, and ${}_{\mathcal{O}Gb}\mathbf{M}$ a subcategory of the category ${}_{\mathcal{O}Gb}\mathbf{Mod}$ of all $\mathcal{O}Gb$-modules. Let (P, b_P) be a (G, b)-subpair, and H a subgroup of G such that $PC_G(P) \le H \le N_G(P, b_P)$. We write $\overline{H} = H/P$. An additive functor $Sl : {}_{\mathcal{O}Gb}\mathbf{M} \to {}_{k\overline{H}\overline{b}_P}\mathbf{Mod}$ is called a (P, b_P)-slash functor if which is defined by the following data:*

- *for each $L, M \in {}_{\mathcal{O}Gb}\mathbf{M}$, there exists a map*

$$Sl^{L,M} : \mathrm{Hom}_{\mathcal{O}P}(L, M) \longrightarrow \mathrm{Hom}_k(Sl(L), Sl(M))$$

  satisfying the following conditions.

  - $Sl^{M,M}(1_{\mathrm{End}_{\mathcal{O}}(M)}) = 1_{\mathrm{End}_k(Sl(M))}$, *for any $M \in {}_{\mathcal{O}Gb}\mathbf{M}$;*

- *$Sl^{L,N}(g \circ f) = Sl^{M,N}(g) \circ Sl^{L,M}(f)$, for any $L, M, N \in {}_{\mathcal{O}Gb}\mathbf{M}$, and any $f \in \mathrm{Hom}_{\mathcal{O}P}(L, M)$, any $g \in \mathrm{Hom}_{\mathcal{O}P}(M, N)$;*
- *for any $L, M \in {}_{\mathcal{O}Gb}\mathbf{M}$, there exists a $k(C_G(P) \times C_G(P))\Delta H$-isomorphism*

$$f_{L,M} : \mathrm{Br}_{\Delta P}(\mathrm{Hom}_{\mathcal{O}}(b_P L, b_P M))) \xrightarrow{\sim} \mathrm{Hom}_k(Sl(L), Sl(M))$$

such that the following diagram is commutative.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{O}P}(L,M) & \xrightarrow{\quad Sl^{L,M} \quad} & \mathrm{Hom}_k(Sl(L), Sl(M)) \\ & {\scriptstyle \mathrm{br}^{\mathrm{Hom}_{\mathcal{O}}(L,M)}_{(\Delta P, b_P \otimes b_P)}} \searrow \qquad \nearrow {\scriptstyle f_{L,M}} & \\ & \mathrm{Br}_{\Delta P}(\mathrm{Hom}_{\mathcal{O}}(b_P L, b_P M))) & \end{array}$$

Biland has proven that there exists a slash functor for Brauer-friendly categories in [3].

Theorem 3.7 ([3, Theorem 18]). *Let b be a block of $\mathcal{O}G$ and ${}_{\mathcal{O}Gb}\mathbf{M}$ a Brauer-friendly category of $\mathcal{O}Gb$-modules. Let (P, b_P) be a (G, b)-subpair, H a subgroup of G such that $PC_G(P) \le H \le N_G(P, b_P)$, and we write $\overline{C}_G(P) = PC_G(P)/P$. Then the following statements hold.*

(i) There exists a (P, b_P)-slash functor $Sl_{(P,b_P)} : {}_{\mathcal{O}Gb}\mathbf{M} \to {}_{k\overline{H}\overline{b}_P}\mathbf{Mod}$.

(ii) If $Sl'_{(P,b_P)} : {}_{\mathcal{O}Gb}\mathbf{M} \to {}_{k\overline{H}\overline{b}_P}\mathbf{Mod}$ is another (P, b_P)-slash functor, then there exists a linear character $\chi : \overline{H}/\overline{C}_G(P) \to k^\times$ such that there exists an isomorphism of functors $\chi_* Sl_{(P,b_P)} \cong Sl'_{(P,b_P)}$.

Example 3.8. We denote by ${}_{\mathcal{O}Gb}\mathbf{Perm}$ the category of all p-permutation $\mathcal{O}Gb$-modules. Then ${}_{\mathcal{O}Gb}\mathbf{Perm}$ is a Brauer-friendly category, and the slash functor on ${}_{\mathcal{O}Gb}\mathbf{Perm}$ is the Brauer functor which is unique up to twisting by a linear character.

For Brauer-friendly modules, slash indecomposability can be defined as well as the Brauer indecomposability as follows (For Frobenius-friendly modules (*i.e.* endo-p-permutation modules), slash indecomposability was defined in [8, Definition 5.1]).

Definition 3.9. *Let ${}_{\mathcal{O}Gb}\mathbf{M}$ be a Brauer-friendly category of $\mathcal{O}Gb$-modules, $Sl_{(Q,b_Q)} : {}_{\mathcal{O}Gb}\mathbf{M} \to {}_{k\overline{N}_G(Q,b_Q)\bar{b}_Q}\mathbf{Mod}$ a (Q, b_Q)-slash functor for each (G, b)-subpair (Q, b_Q), and $M \in {}_{\mathcal{O}Gb}\mathbf{M}$. We say that M is slash indecomposable if for every (G, b)-subpair (Q, b_Q), $Res^{N_G(Q,b_Q)/Q}_{QC_G(Q)/Q}(Sl_{(Q,b_Q)}(M))$ is indecomposable or zero.*

Remark 3.10. The definition of the slash indecomposability is independent of the choice of Brauer-friendly categories and slash functors.

The following theorem is a generalization of [5, (3.2) THEOREM. (3)].

Theorem 3.11 ([3, Theorem 23]). *Let b be a block of $\mathcal{O}G$, (P, b_P, V) a fusion-stable endo-permutation source triple, ${}_{\mathcal{O}Gb}\mathbf{M}$ a Brauer-friendly category of $\mathcal{O}Gb$-modules that is big enough with respect to (P, b_P, V), and $Sl_{(P,b_P)}$: ${}_{\mathcal{O}Gb}\mathbf{M} \to {}_{k[\overline{N}_G(P,b_P)]\overline{b}_P}\mathbf{Mod}$ a (P, b_P)-slash functor. Then $Sl_{(P,b_P)}$ induces a one-to-one correspondence between the isomorphism classes of indecomposable $\mathcal{O}Gb$-modules with source triple (P, b_P, V) and the isomorphism classes of projective indecomposable $k[\overline{N}_G(P, b_P)]\overline{b}_P$-modules.*

By this theorem, Brauer-friendly modules can be presented as follows.

Definition 3.12. *With the same notation as in Theorem 3.11, let $M \in {}_{\mathcal{O}Gb}\mathbf{M}$ be an indecomposable $\mathcal{O}Gb$-module with source triple (P, b_P, V). Then, by Theorem 3.11, there is up to isomorphism a unique simple $k[\overline{N}_G(P, b_P)]\overline{b}_P$-module S such that $Sl_{(P,b_P)}(M) \cong P(S)$. We denote the module M by $B(b,(P, b_P, V), Sl_{(P,b_P)}, S)$. In particular, if $S \cong k_{\overline{N}_G(P,b_P)\overline{b_P}}$, then we denote the module M by $BS(b,(P, b_P, V), Sl_{(P,b_P)})$. We call this module the Brauer-friendly Scott $\mathcal{O}Gb$-module with respect to (P, b_P, V).*

Remark 3.13.
(i) The above presentation of Brauer-friendly modules is a unique up to twisted by a linear character.
(ii) The Scott $\mathcal{O}G$-module $S(G, P)$ is presented by

$$S(G, P) = BS(b,(P, b_P, \mathcal{O}_P), Sl_{(P,b_P)})$$

for b is the principal block of $\mathcal{O}G$.

§4. Lemmas

In this section, we give lemmas for Brauer-friendly modules, Brauer-friendly Scott modules, and slash functors, which are analogies of lemmas for p-permutation modules, Scott modules, and Brauer functors respectively, which are used to prove the main theorem in [9].

NOTATION. Let M be a Brauer-friendly module and $\mathcal{S}_M$ be the set of source triples of any indecomposable summand of M. Hereinafter, we assume that M belongs to some Brauer-friendly categories that is big enough with respect to $\mathcal{S}_M$. Moreover, when we apply a slash functor to the Brauer-friendly module M, we assume that the domain of the slash functor is big enough with respect to $\mathcal{S}_M$.

Lemma 4.1. *Let (P, b_P) be a (G, b)-subpair, H a subgroup of G such that $PC_G(P) \leq H \leq N_G(P, b_P)$, M a Brauer-friendly $\mathcal{O}Gb$-module, and $Sl_{(P,b_P)}$ a (P, b_P)-slash functor. By [3, Lemma 10 (i)], we get a decomposition $b_P M = L \oplus L'$, where L is a Brauer-friendly $\mathcal{O}Hb_P$-module and L' is a direct sum of indecomposable $\mathcal{O}Hb_P$-modules with vertices that do not contain P. Then there exists an isomorphism of $k\overline{H}b_P$-modules*

$$\mathrm{Res}_{\overline{H}}^{\overline{N}_G(P,b_P)}(Sl_{(P,b_P)}(M)) \cong Sl'_{(P,b_P)}(L)$$

for some (P, b_P)-slash functor $Sl'_{(P,b_P)}$. In particular, if $H = N_G(P, b_P)$ and M has the source triple (P, b_P, V), then there exists an isomorphism of $k\overline{H}b_P$-modules

$$Sl_{(P,b_P)}(M) \cong Sl'_{(P,b_P)}(f^b_{b_P}(M)),$$

where $f^b_{b_P}$ is the Green correspondence with respect to (P, b_P).

Proof. Write $N_G = N_G(P, b_P)$. We have an isomorphism of $C_H(P)$-interior H-algebras

$$\begin{aligned}\mathrm{End}_k(\mathrm{Res}_H^{N_G}(Sl_{(P,b_P)}(M))) &\cong \mathbf{Res}_H^{N_G}(\mathrm{Br}_{\triangle P}(\mathrm{End}_{\mathcal{O}}(b_P M))) \\ &\cong \mathrm{Br}_{\triangle P}(\mathrm{End}_{\mathcal{O}}(b_P \mathrm{Res}_H^G(M))) \\ &\cong \mathrm{Br}_{\triangle P}(\mathrm{End}_{\mathcal{O}}(L)) \\ &\cong \mathrm{End}_k(Sl''_{(P,b_P)}(L)),\end{aligned}$$

where **Res** is a restriction to H as algebras and $Sl''_{(P,b_P)}$ is a (P, b_P)-slash functor. By [4, Lemma 3 (ii)], there exists a linear character $\chi : H/PC_H(P) \longrightarrow k^\times$ such that $b_P \mathrm{Res}_H^{N_G}(Sl_{(P,b_P)}(M)) \cong \chi_* Sl''_{(P,b_P)}(L)$. Hence, setting $Sl'_{(P,b_P)} = \chi_* Sl''_{(P,b_P)}$, we obtain

$$b_P \mathrm{Res}_H^{N_G}(Sl_{(P,b_P)}(M)) \cong Sl'_{(P,b_P)}(L).$$

The rest follows from $b_P M = f^b_{b_P}(M) \oplus Z$, where Z is a direct sum of indecomposable $\mathcal{O}N_G b_P$-modules with vertices that do not contain P. □

The following lemma is an analogy of [5, (3.2) THEOREM. (1)].

Lemma 4.2 ([2, Corollary 3.17])**.** *Let (Q, b_Q) be a (G, b)-subpair, $Sl_{(Q,b_Q)}$ a (Q, b_Q)-slash functor, and M an indecomposable Brauer-friendly $\mathcal{O}Gb$-module with source triple (P, b_P, V). Then the following conditions are equivalent.*

(i) $Sl_{(Q,b_Q)}(M) \neq 0$.

(ii) $(Q, b_Q) \leq_G (P, b_P)$.

We define the conjugation of slash functors by an element of a group.

Definition 4.3. *Let (P, b_P) be a (G, b)-subpair and $Sl_{(P,b_P)} : {}_{\mathcal{O}Gb}\mathbf{M} \to {}_{k\overline{N}_G(P,b_P)\bar{b}_P}\mathbf{Mod}$ a (P, b_P)-slash functor. For each $g \in G$, we denote by ${}^g(-)$ the conjugation functor by g, also we denote the functor ${}^g(-) \circ Sl_{(P,b_P)} : {}_{\mathcal{O}Gb}\mathbf{M} \to {}_{k\overline{N}_G({}^gP,{}^gb_P){}^g\bar{b}_P}\mathbf{Mod}$ by $g_\star Sl_{(P,b_P)}$. Then, by [3, Lemma 22 (ii)], the functor $g_\star Sl_{(P,b_P)}$ is a ${}^g(P, b_P)$-slash functor.*

Lemma 4.4. *Let (P, b_P) be a (G, b)-subpair. For each $g \in G$, we have an isomorphism of $\mathcal{O}G$-modules*

$$B(b,(P,b_P,V),Sl_{(P,b_P)},S) \cong B(b,({}^gP,{}^gb_P,{}^gV),g_\star Sl_{(P,b_P)},{}^gS).$$

Proof. Set $X = B(b,({}^gP,{}^gb_P,{}^gV),g_\star Sl_{(P,b_P)},{}^gS)$. Then X also has the source triple (P, b_P, V) and we have ${}^g(Sl_{(P,b_P)}(X)) = g_\star Sl_{(P,b_P)}(X) = {}^gP(S)$. Thus $Sl_{(P,b_P)}(X) = P(S)$. Hence we obtain

$$B(b,(P,b_P,V),Sl_{(P,b_P)},S) \cong B(b,({}^gP,{}^gb_P,{}^gV),g_\star Sl_{(P,b_P)},{}^gS).$$

□

Lemma 4.5. *Let (P, b_P) be a (G, b)-subpair and $f^b_{b_P}$ the Green correspondence with respect to (P, b_P). Then there exists a (P, b_P)-slash functor $Sl'_{(P,b_P)}$ such that there exists an isomorphism of $\mathcal{O}N_G(P, b_P)b_P$-modules*

$$f^b_{b_P}(B(b,(P,b_P,V),Sl_{(P,b_P)},S)) \cong B(b_P,(P,b_P,V),Sl'_{(P,b_P)},S).$$

In particular, we have an isomorphism

$$f^b_{b_P}(BS(b,(P,b_P,V),Sl_{(P,b_P)})) \cong BS(b_P,(P,b_P,V),Sl'_{(P,b_P)}).$$

Proof. Set $M = B(b,(P,b_P,V),Sl_{(P,b_P)},S)$. Then, by Lemma 4.1, there exists a (P, b_P)-slash functor $Sl'_{(P,b_P)}$ such that there exists an isomorphism of $k\overline{N}_G(P, b_P)\bar{b}_P$-modules

$$Sl'_{(P,b_P)}(f^b_{b_P}(M)) \cong Sl_{(P,b_P)}(M) \cong P(S).$$

□

The following lemma is an analogy of [14, Chapter 4, Theorem 8.6 (ii)] for Brauer-friendly modules.

Lemma 4.6. *Let P be a p-subgroup of G, H a subgroup of G such that $PC_G(P) \leq H$, b' a block of $\mathcal{O}H$, and (P, b_P) a (G, b)-subpair. We assume that (P, b_P) is an (H, b')-subpair, and (P, b_P, V) is a fusion-stable endo-permutation* *source triple. Then there exist* $t \in N_G(P, b_P)$*, a* (P, b_P)*-slash functor* $Sl''_{(P,b_P)}$*, and a simple* $k[\overline{N}_H(P, b_P)]\bar{b}_P$*-module* S' *such that*

$$B(b', (P, b_P, {}^tV), Sl''_{(P,b_P)}, S') \mid \mathrm{Res}^G_H(B(b, (P, b_P, V), Sl_{(P,b_P)}, S)).$$

In particular, we have

$$BS(b', (P, b_P, {}^tV), Sl''_{(P,b_P)}) \mid \mathrm{Res}^G_H(BS(b, (P, b_P, V), Sl_{(P,b_P)})).$$

To prove Lemma 4.6, we need the following lemma.

Lemma 4.7 (Burry [14, Chapter 4, Theorem 4.8 (i)])**.** *Let* H *be a subgroup of* G *containing* $PC_G(P)$*,* b' *a block of* $\mathcal{O}H$*, and* (P, b_P) *a* (G, b)*-subpair. We assume that* (P, b_P) *is an* (H, b')*-subpair. Let* $f^b_{b_P}$ *and* $f^{b'}_{b_P}$ *be the Green correspondences with respect to* (P, b_P)*. Then, for any indecomposable* $\mathcal{O}Gb$*-module* V *with vertex subpair* (P, b_P) *and any indecomposable* $\mathcal{O}Hb'$*-module* W *with vertex subpair* (P, b_P)*, the following conditions are equivalent.*

- *(i)* $W \mid \mathrm{Res}^G_H(V)$.
- *(ii)* $f^{b'}_{b_P}(W) \mid \mathrm{Res}^{N_G(P,b_P)}_{N_H(P,b_P)}(f^b_{b_P}(V))$.

Proof. (Proof of Lemma 4.6) We prove Lemma 4.6 in a similar way as the proof of [14, Chapter 4, Theorem 8.6 (ii)]. Set $N_G = N_G(P, b_P)$ and $N_H = N_H(P, b_P)$. By Lemma 4.7, it is sufficient to show the following:

$$f^{b'}_{b_P}(B(b', (P, b_P, {}^tV), Sl''_{(P,b_P)}, S')) \mid \mathrm{Res}^{N_G}_{N_H}(f^b_{b_P}(B(b, (P, b_P, V), Sl_{(P,b_P)}, S))).$$

Also, by Lemma 4.5, this statement is equivalent to the following:

$$B(b_P, (P, b_P, {}^tV), Sl'''_{(P,b_P)}, S') \mid \mathrm{Res}^{N_G}_{N_H}(B(b_P, (P, b_P, V), Sl'_{(P,b_P)}, S)).$$

Set $B_G = B(b, (P, b_P, V), Sl_{(P,b_P)}, S)$. It is equivalent to show that there exist $t \in N_G$, a (P, b_P)-slash functor $Sl'''_{(P,b_P)}$, a simple $k\overline{N}_H\bar{b}_P$-module S', and an indecomposable direct summand X of $b_P\mathrm{Res}^{N_G}_{N_H}(f^b_{b_P}(B_G))$ such that X has a source triple $(P, b_P, {}^tV)$ and $Sl'_{(P,b_P)}(X) \cong P(S')$. By [3, Lemma 10 (i)], we get a decomposition $b_P\mathrm{Res}^{N_G}_{N_H}(f^b_{b_P}(B_G)) = L \oplus L'$, where L is a Brauer-friendly $\mathcal{O}N_Hb_P$-module and L' is a direct sum of indecomposable $\mathcal{O}N_Hb_P$-modules with vertices that do not contain P. Since $f^b_{b_P}(B_G) = B(b_P, (P, b_P, V), Sl'_{(P,b_P)}, S)$, we obtain $f^b_{b_P}(B_G) \mid \mathrm{Ind}^{N_G}_P(V)$. The Mackey formula gives the relation

$$L \mid \mathrm{Res}^{N_G}_{N_H}(f^b_{b_P}(B_G)) \mid \bigoplus_{t \in N_H \backslash N_G / P} \mathrm{Ind}^{N_H}_P({}^tV).$$

Let $L = \oplus_{i\in I} L_i$ be a decomposition of L as a direct sum of indecomposable $\mathcal{O}N_H b_P$-modules. Then each L_i has the vertex subpair (P, b_P). Hence for each $i \in I$, there exists $t_i \in N_G$ such that $\mathrm{s}(L_i) = {}^{t_i}V$. By Lemma 4.1, there exists a (P, b_P)-slash functor $Sl'''_{(P,b_P)}$ such that

$$\mathrm{Res}^{N_G}_{N_H}(Sl'_{(P,b_P)}(f^b_{b_P}(B_G))) \cong Sl'''_{(P,b_P)}(L).$$

There exists a simple $k\overline{N}_H\bar{b}_P$-module S'_i such that $Sl'''_{(P,b_P)}(L_i) \cong P(S'_i)$, by the above argument and Theorem 3.11. This shows

$$L_i = B(b_P, (P, b_P, {}^tV), Sl'''_{(P,b_P)}, S').$$

In particular, if $S = k_{\overline{N}_G b_P}$, then $P(k_{\overline{N}_H b_P}) \mid \mathrm{Res}^{N_G}_{N_H}(Sl'_{(P,b_P)}(f^b_{b_P}(B_G)))$. Thus there exists $i \in I$ such that $Sl'''_{(P,b_P)}(L_i) \cong P(k_{\overline{N}_H b_P})$. This shows $L_i = BS(b_P, (P, b_P, {}^tV), Sl'''_{(P,b_P)})$. □

Lemma 4.8 (Burry-Carlson, Puig)**.** *Let (P, b_P) be a (G, b)-subpair, $H := N_G(P, b_P)$, $f^b_{b_P}$ the Green correspondence with respect to (P, b_P), V an indecomposable $\mathcal{O}Gb$-module, and W an indecomposable summand of $b_P\mathrm{Res}^G_H(V)$. Then the following condition (i) implies (ii) and $f^b_{b_P}(V) = W$.*

(i) W has a vartex subpair (P, b_P).

(ii) V has a vartex subpair (P, b_P).

The following lemma is a generalization of H. Kawai [10, Theorem 1.7] for Brauer-friendly modules. We prove the lemma with the similar argument as [10, Theorem 1.7].

Lemma 4.9. *Let (P, b_P) be a (G, b)-subpair, $(Q, b_Q) \leq_G (P, b_P)$, and set $H = N_G(Q, b_Q)$ and $B_G = B(b, (P, b_P, V), Sl_{(P,b_P)}, S)$. If $R = {}^gP \cap H$ is a maximal element of $\{{}^iP \cap H \mid i \in G, (Q, b_Q) \leq {}^i(P, b_P)\}$, then there exist an (R, b_R)-slash functor $Sl_{(R,b_R)}$, a $z \in G$, and a simple $k[\overline{N}_H(R, b_R)]\bar{b}_R$-module S' such that*

$$B(b_Q, (R, b_R, \mathrm{Cap}(\mathrm{Res}^{{}^zP}_R({}^zV))), Sl_{(R,b_R)}, S') \mid \mathrm{Res}^G_H(B_G),$$

where b_R is the unique block satisfying $(R, b_R) \leq {}^g(P, b_P)$.

Proof. We prove this by induction on $|P|/|R|$.

If $|P|/|R| = 1$, *i.e.* ${}^gP = R$, then ${}^g(P, b_P)$ is a (G, b)-subpair. By $(Q, b_Q) \leq (R, b_R)$, $(R, b_R) = {}^g(P, b_P)$ is an (H, b_Q)-subpair. Hence, by Lemma 4.4 and Lemma 4.6, there exist an (R, b_R)-slash functor $Sl_{(R,b_R)}$ and $z \in N_G(R, b_R)$ such that

$$B(b_Q, (R, b_R, {}^zV), Sl_{(R,b_R)}, S') \mid \ \mathrm{Res}^G_H(B(b, ({}^gP, {}^gb_P, V), g_\star Sl_{(P,b_P)}, {}^gS),$$

and

$$\mathrm{Res}^G_H(B(b,({}^gP,{}^gb_P,V),g_\star Sl_{(P,b_P)},{}^gS)\cong \mathrm{Res}^G_H(B_G).$$

In this case, the statement follows.

Now suppose that $|P|/|R|\gneq 1$, *i.e.* $R\lneq_G P$. We set $H_1=N_G(R,b_R)$ and $\Omega=\{{}^iP\cap H_1|i\in G,(R,b_R)\leq {}^i(P,b_P)\}$. From $(R,b_R)\leq {}^g(P,b_P)$, we see $\Omega\neq\emptyset$. Let R_1 be a maximal element of Ω. Then H_1 and (R_1,b_{R_1}) satisfy the condition of the lemma. Therefore, by induction hypothesis, there exist an (R_1,b_{R_1})-slash functor $Sl_{(R_1,b_{R_1})}$, an $x\in G$, and a simple $k[\overline{N}_{H_1}(R_1,b_{R_1})]\overline{b}_{R_1}$-module S_{R_1} such that

$$B(b_R,(R_1,b_{R_1},\mathrm{Cap}(\mathrm{Res}^{{}^xP}_{R_1}({}^xV))),Sl_{(R_1,b_{R_1})},S_{R_1})\mid \mathrm{Res}^G_{H_1}(B_G).$$

Set $N=B(b_R,(R_1,b_{R_1},\mathrm{Cap}(\mathrm{Res}^{{}^xP}_{R_1}({}^xV))),Sl_{(R_1,b_{R_1})},S_{R_1})$, $T=N_H(R,b_R)$. By [3, Lemma 10 (i)], we get a decomposition $b_R\mathrm{Res}^{H_1}_T(N)=L\oplus L'$, where L is a Brauer-friendly $\mathcal{O}Tb_R$-module and L' is a direct sum of indecomposable $\mathcal{O}Tb_R$-modules with vertices that do not contain R. Let $L=\oplus_{i\in I}L_i$ be a decomposition of L as a direct sum of indecomposable $\mathcal{O}Tb_R$-modules. Then, for any $i\in I$, there exist a vertex of L_i which contains R. Here, the Mackey formula gives the relation

$$\begin{aligned}\bigoplus_{i\in I}L_i\ |\mathrm{Res}^{H_1}_T(\mathrm{Ind}^{H_1}_{R_1}(\mathrm{Cap}(\mathrm{Res}^{{}^xP}_{R_1}({}^xV))))\\ \cong \bigoplus_{h\in T\backslash H_1/R_1}\mathrm{Ind}^T_{{}^hR_1\cap T}(\mathrm{Res}^{{}^hR_1}_{{}^hR_1\cap T}({}^h(\mathrm{Cap}(\mathrm{Res}^{{}^xP}_{R_1}({}^xV)))))\\ \cong \bigoplus_{h\in T\backslash H_1/R_1}\mathrm{Ind}^T_R(\mathrm{Res}^{{}^hR_1}_R({}^h(\mathrm{Cap}(\mathrm{Res}^{{}^xP}_{R_1}({}^xV))))),\end{aligned}$$

where $R={}^hR_1\cap T$, for any $h\in H_1$. Hence, for any $i\in I$, we have $\mathrm{vtx}(L_i)=R$. Therefore, for any $i\in I$, we can take a vertex subpair of L_i as (R,b_R). We may assume that

$$L_i\mid \mathrm{Ind}^H_R(\mathrm{Res}^{{}^{h_i}R_1}_R({}^{h_i}(\mathrm{Cap}(\mathrm{Res}^{{}^xP}_{R_1}({}^xV))))),$$

for some $h_i\in H_1$. Let $\mathrm{Res}^{{}^{h_i}R_1}_R({}^{h_i}(\mathrm{Cap}(\mathrm{Res}^{{}^xP}_{R_1}({}^xV))))=\bigoplus_{j\in J}Z_j$ be a decomposition as a direct sum of indecomposable $\mathcal{O}R$-modules. Then, there exists $j\in J$ such that $\mathrm{s}(L_i)=Z_j$. Since we can take a vertex of Z_j as R, therefore we have $Z_j\cong \mathrm{Cap}(\mathrm{Res}^{{}^{h_i}R_1}_R({}^{h_i}(\mathrm{Cap}(\mathrm{Res}^{{}^xP}_{R_1}({}^xV)))))$. Moreover, we see that

$$\mathrm{Cap}(\mathrm{Res}^{{}^{h_i}R_1}_R({}^{h_i}(\mathrm{Cap}(\mathrm{Res}^{{}^xP}_{R_1}({}^xV)))))=\mathrm{Cap}(\mathrm{Res}^{{}^{h_ix}P}_R({}^{h_ix}V)).$$

From the above, for any $i\in I$, there exist an (R,b_R)-slash functor $Sl_{(R,b_R)}$ and a simple $k[\overline{N}_T(R,b_R)]\overline{b}_R$-module S'_i such that

$$L_i=B(b_R,(R,b_R,\mathrm{Cap}(\mathrm{Res}^{{}^{hx}P}_R({}^{hx}V))),Sl_{(R,b_R)},S').$$

We choose $i \in I$ and set $h = h_i$, $S' = S'_i$, $z = hx \in G$. Then we have

$$B(b_R, (R, b_R, \mathrm{Cap}(\mathrm{Res}^{{}^zP}_R({}^zV))), Sl_{(R,b_R)}, S') \mid \mathrm{Res}^H_T(\mathrm{Res}^G_H(B_G)).$$

Therefore there exists a direct summand U of $\mathrm{Res}^G_H(B_G)$ such that

$$B(b_R, (R, b_R, \mathrm{Cap}(\mathrm{Res}^{{}^zP}_R({}^zV))), Sl_{(R,b_R)}, S') \mid \mathrm{Res}^H_T(U).$$

By Lemma 4.8 and [3, Theorem 4], the module U has a vertex subpair (R, b_R) and lies in the block b_R of $\mathcal{O}H$ and

$$f^{b_Q}_{b_R}(U) = B(b_R, (R, b_R, \mathrm{Cap}(\mathrm{Res}^{{}^zP}_R({}^zV))), Sl_{(R,b_R)}, S').$$

Hence, by Lemma 4.5, we have

$$U \cong B(b_Q, (R, b_R, \mathrm{Cap}(\mathrm{Res}^{{}^zP}_R({}^zV))), Sl_{(R,b_R)}, S').$$

From the above, it follows that

$$B(b_Q, (R, b_R, \mathrm{Cap}(\mathrm{Res}^{{}^zP}_R({}^zV))), Sl_{(R,b_R)}, S') \mid \mathrm{Res}^G_H(B_G).$$

□

The following lemma is a generalization of J. Thévenaz [15, Exercises (27.4)] for Brauer-friendly modules.

Lemma 4.10. *Let (P, b_P) be a (G, b)-subpair and $Q \leq_G P$, and set $M = B(b, (P, b_P, V), Sl_{(P,b_P)}, S)$ and $H = N_G(Q, b_Q)$. By [3, Lemma 10 (i)], we get a decomposition $b_Q Res^G_H(M) = L \oplus L'$, where L is a Brauer-friendly $\mathcal{O}Hb_Q$-module and L' is a direct sum of indecomposable $\mathcal{O}Hb_Q$-modules with vertices that do not contain Q. Let $L = \oplus_{i \in I} L_i$ be a decomposition of L as a direct sum of indecomposable $\mathcal{O}Hb_Q$-modules and we set $Z_i = \mathrm{vtx}(L_i)$. Then, for each $1 \leq i \leq n$ and any (Q, b_Q)-slash functor $Sl_{(Q,b_Q)}$, there exist a $g_i \in G$ and a simple $k[\overline{N}_H(Z_i, b_{Z_i})]\overline{b}_{Z_i}$-module S_i such that*

$$Sl_{(Q,b_Q)}(L_i) \cong B(b_Q, (Z_i, b_{Z_i}, \mathrm{Cap}(\mathrm{Res}^{{}^{g_i}P}_{Z_i}({}^{g_i}V))[Q]), Sl_{(Z_i,b_{Z_i})}, S_i) \oplus (\bigoplus_j X_{i,j}),$$

where $X_{i,j}$ is indecomposable Brauer-friendly kHb_Q-module with source triple $(\mathrm{vtx}(X_{i,j}), b_{\mathrm{vtx}(X_{i,j})}, \mathrm{s}(X_{i,j}))$ such that

$$(Q, b_Q) \leq (\mathrm{vtx}(X_{i,j}), b_{\mathrm{vtx}(X_{i,j})}) \leq (Z_i, b_{Z_i})$$

and

$$\mathrm{s}(X_{i,j}) \mid \mathrm{Res}^{Z_i}_{\mathrm{vtx}(X_{i,j})}(\mathrm{Cap}(\mathrm{Res}^{{}^{g_i}P}_{Z_i}({}^{g_i}V)))[Q].$$

Therefore, we have

$$\begin{aligned} &Sl_{(Q,b_Q)}(M) \cong Sl_{(Q,b_Q)}(L) \\ &\cong \bigoplus_{1 \leq i \leq n} \big(B(b_Q, (Z_i, b_{Z_i}, \mathrm{Cap}(\mathrm{Res}^{{}^{g_i}P}_{Z_i}({}^{g_i}V))[Q]), Sl_{(Z_i,b_{Z_i})}, S_i) \oplus (\bigoplus_j X_{i,j})\big).\end{aligned}$$

Remark 4.11. If $Sl_{(Q,b_Q)}(L_i)$ is indecomposable, then we have

$$Sl_{(Q,b_Q)}(L_i) \cong B(b_Q, (Z_i, b_{Z_i}, \mathrm{Cap}(\mathrm{Res}^{g_i P}_{Z_i}({}^{g_i}V))[Q]), Sl_{(Z_i,b_{Z_i})}, S_i).$$

Proof. By Lemma 4.1, we have

$$Sl_{(Q,b_Q)}(M) \cong Sl_{(Q,b_Q)}(L) \cong \bigoplus_{1\le i\le n} Sl_{(Q,b_Q)}(L_i).$$

First, we determine the structure of each L_i. By [3, Theorem 4], we see that there exists $g_i \in G$ such that $(Q, b_Q) \trianglelefteq (Z_i, b_{Z_i}) \le {}^{g_i}(P, b_P)$ and $\mathrm{s}(L_i) = \mathrm{Cap}(\mathrm{Res}^{g_i P}_{Z_i}({}^{g_i}V))$. Therefore, there exist a $g_i \in G$, a (Z_i, b_{Z_i})-slash functor $Sl_{(Hb_Q,Z_i,b_{Z_i})}$, and a simple $k[\overline{N}_H(Z_i, b_{Z_i})]\overline{b}_{Z_i}$-module S_i such that

$$L_i \cong B(b_Q, (Z_i, b_{Z_i}, \mathrm{Cap}(\mathrm{Res}^{g_i P}_{Z_i}({}^{g_i}V))), Sl_{(Z_i,b_{Z_i})}, S_i).$$

Next, we determine the structure of $Sl_{(Q,b_Q)}(L_i)$. Since we have $(Q, b_Q) \trianglelefteq (Z_i, b_{Z_i})$ by [2, Lemma 3.16 (i)], we see

$$P(S_i) \cong Sl_{(Z_i,b_{Z_i})}(L_i) \cong Sl_{(Z_i,b_{Z_i})} \circ Sl_{(Q,b_Q)}(L_i).$$

Thus, there exists the unique direct summand X_i of $Sl_{(Q,b_Q)}(L_i)$ such that $Sl_{(Z_i,b_{Z_i})}(X_i) \cong P(S_i)$. From [4, Lemma 3 (iii)] and Lemma 4.2, we see $\mathrm{vtx}(X_i) = \mathrm{vtx}(L_i)$ and $\mathrm{s}(X) = \mathrm{Cap}(\mathrm{Res}^{g_i P}_{Z_i}({}^{g_i}V))[Q]$. Hence, we get

$$X_i = B(b_Q, (Z_i, b_{Z_i}, \mathrm{Cap}(\mathrm{Res}^{g_i P}_{Z_i}({}^{g_i}V))[Q]), Sl_{(Z_i,b_{Z_i})}, S_i).$$

Let $Sl_{(Q,b_Q)}(L_i) = X_i \oplus (\bigoplus_j X_{i,j})$ be a decomposition of $Sl_{(Q,b_Q)}(L_i)$ as a direct sum of indecomposable $\mathcal{O}Hb_Q$-modules. By [4, Lemma 3 (iii)], we have $(Q, b_Q) \le (\mathrm{vtx}(X_{i,j}), b_{\mathrm{vtx}(X_{i,j})}) \le (Z_i, b_{Z_i})$ and

$$\mathrm{s}(X_{i,j}) \mid \mathrm{Res}^{Z_i}_{\mathrm{vtx}(X_{i,j})}(\mathrm{Cap}(\mathrm{Res}^{g_i P}_{Z_i}({}^{g_i}V)))[Q].$$

From the above, we have

$$\begin{aligned} &Sl_{(Q,b_Q)}(M) \cong Sl_{(Q,b_Q)}(L) \\ &\cong \bigoplus_{1\le i\le n} \big(B(b_Q, (Z_i, b_{Z_i}, \mathrm{Cap}(\mathrm{Res}^{g_i P}_{Z_i}({}^{g_i}V))[Q]), Sl_{(Z_i,b_{Z_i})}, S_i) \oplus (\bigoplus_j X_{i,j})\big). \end{aligned}$$

□

The following lemma is the subpair version of [9, Lemma 3.1]. It can be proved in a similar way as the proof of [9, Lemma 3.1].

Lemma 4.12. *Let (P, b_P) be a (G, b)-subpair and Q a fully $\mathcal{F}_{(P,b_P)}(G, b)$-normalized subgroup of G. Assume that $(Q, b_Q) \leq (P, b_P)$. Then, $N_P(Q)$ is a maximal element of*

$$\{{}^gP \cap N_G(Q, b_Q) \mid g \in G, (Q, b_Q) \leq {}^g(P, b_P)\}.$$

The following lemma is the subpair version of [9, Lemma 3.2].

Lemma 4.13. *Let (P, b_P) be a (G, b)-subpair and set $\mathcal{F} = \mathcal{F}_{(P,b_P)}(G, b)$. If Q is a fully $\mathcal{F}$-automized and $\mathcal{F}$-receptive subgroup of P, then we have $N_{{}^gP}(Q) \leq_{N_G(Q,b_Q)} N_P(Q)$, for any $g \in G$ such that $(Q, b_Q) \leq ({}^gP, {}^gb_P)$.*

Proof. Assume that $(Q, b_Q) \leq ({}^gP, {}^gb_P)$ for some $g \in G$. Then ${}^{g^{-1}}Q$ and Q are $\mathcal{F}$-conjugate. Therefore, by [1, I, Lemma 2.6 (c)], there exists $\varphi_x \in \mathrm{Hom}_{\mathcal{F}}(N_P({}^{g^{-1}}Q), N_P(Q))$ such that $\varphi_x|_{{}^{g^{-1}}Q} \in \mathrm{Iso}_{\mathcal{F}}({}^{g^{-1}}Q, Q)$. Thus $xg^{-1} \in N_G(Q, b_Q)$ and

$$N_{{}^gP}(Q) = {}^gN_P({}^{g^{-1}}Q) =_{N_G(Q,b_Q)} {}^{(xg^{-1})g}N_P({}^{g^{-1}}Q) = {}^xN_P({}^{g^{-1}}Q) \leq N_P(Q).$$

□

§5. Main theorem

NOTATION. Let (P, b_P) be a (G, b)-subpair, set $\mathcal{F} = \mathcal{F}_{(P,b_P)}(G, b)$, Q be a fully $\mathcal{F}$-normalized subgroup of P, and $M = B(b, (P, b_P, V), Sl_{(P,b_P)}, S)$ a Brauer-friendly $\mathcal{O}Gb$-module. Then, from Lemma 4.12, the subgroup $N_P(Q)$ is a maximal element of

$$\{{}^gP \cap N_G(Q, b_Q) \mid g \in G, (Q, b_Q) \leq {}^g(P, b_P)\}.$$

Therefore, by Lemma 4.9, there exist an $n \in G$, an $(N_P(Q), b_{N_P(Q)})$-slash functor $Sl_{(N_P(Q),b_{N_P(Q)})}$, and a simple $k[\overline{N}_{N_G(Q,b_Q)}(N_P(Q), b_{N_P(Q)})]\overline{b}_{N_P(Q)}$-module S_Q such that

$$B(b_Q, (N_P(Q), b_{N_P(Q)}, W_Q), Sl_{(N_P(Q),b_{N_P(Q)})}, S_Q) \mid \mathrm{Res}^G_{N_G(Q,b_Q)}(M),$$

where $W_Q = \mathrm{Cap}(\mathrm{Res}^{{}^nP}_{N_P(Q)}({}^nV))$. Also, by Lemma 4.10, for any (Q, b_Q)-slash functor $Sl_{(Q,b_Q)}$, we have

$$B(b_Q, (N_P(Q), b_{N_P(Q)}, V_Q), Sl_{(N_P(Q),b_{N_P(Q)})}, S_Q) \mid Sl_{(Q,b_Q)}(M),$$

where $V_Q = W_Q[Q]$. In this section, we set

$$B_Q = B(b_Q, (N_P(Q), b_{N_P(Q)}, V_Q), Sl_{(N_P(Q),b_{N_P(Q)})}, S_Q).$$

The following theorem is a generalization of [9, Theorem 1.3].

Theorem 5.1. *Let G be a finite group, b a block of $\mathcal{O}G$, and (P, b_P) a (G, b)-subpair. We set $M = B(b, (P, b_P, V), Sl_{(P,b_P)}, S)$, $\mathcal{F} = \mathcal{F}_{(P,b_P)}(G, b)$, $N_Q = N_G(Q, b_Q)$, and $H_Q = N_P(Q)$ for $Q \leq P$. Suppose that $\mathcal{F}$ is saturated and $\mathrm{Res}^{N_P}_{PC_G(P)}(S)$ is a simple $\mathcal{O}PC_G(P)$-module. The following conditions are equivalent.*

(i) M is slash indecomposable.

(ii) $\mathrm{Res}^{N_Q}_{QC_G(Q)}(B_Q)$ is indecomposable for each fully $\mathcal{F}$-normalized subgroup Q of P.

If these conditions are satisfied, then for each fully $\mathcal{F}$-normalized subgroup Q of P and any (Q, b_Q)-slash functor $Sl_{(Q,b_Q)}$, we have

$$Sl_{(Q,b_Q)}(M) \cong B_Q.$$

Proof. If (i) holds, *i.e.* $\mathrm{Res}^{N_Q}_{QC_G(Q)}(Sl_{(Q,b_Q)}(M))$ is indecomposable, for each fully $\mathcal{F}$-normalized subgroup Q of P and any (Q, b_Q)-slash functor $Sl_{(Q,b_Q)}$, then by the definition of B_Q, we have

$$\mathrm{Res}^{N_Q}_{QC_G(Q)}(B_Q) \cong \mathrm{Res}^{N_Q}_{QC_G(Q)}(Sl_{(Q,b_Q)}(M)).$$

Hence, $\mathrm{Res}^{N_Q}_{QC_G(Q)}(B_Q)$ is indecomposable. This shows (ii). Moreover, since $\mathrm{Res}^{N_Q}_{QC_G(Q)}(Sl_{(Q,b_Q)}(M))$ is indecomposable, the module $Sl_{(Q,b_Q)}(M)$ is also indecomposable. Therefore, we get

$$Sl_{(Q,b_Q)}(M) \cong B_Q.$$

Conversely, suppose that (ii) holds. It is sufficient to prove that for each $Q \leq P$, $\mathrm{Res}^{N_Q}_{QC_G(Q)}(Sl_{(Q,b_Q)}(M))$ is indecomposable. We prove this by induction on $|P : Q|$.

If $|P : Q| = 1$, then this case is similar to the proof of [11, Lemma 4.3 (ii)], by the assumption of the theorem.

Now consider the case that $|P : Q| \gneq 1$. For some $g \in G$, ${}^gQ \leq P$ and ${}^gQ$ is fully $\mathcal{F}$-normalized. We see that for any $({}^gQ, b_{{}^gQ})$-slash functor $Sl_{({}^gQ,b_{{}^gQ})}$,

$${}^g(\mathrm{Res}^{N_Q}_{QC_G(Q)}(Sl_{(Q,b_Q)}(M))) \cong \mathrm{Res}^{N_{{}^gQ}}_{{}^gQC_G({}^gQ)}(Sl_{({}^gQ,b_{{}^gQ})}(M)).$$

Therefore, it is sufficient to prove that $\mathrm{Res}^{N_{{}^gQ}}_{{}^gQC_G({}^gQ)}(Sl_{({}^gQ,b_{{}^gQ})}(M))$ is indecomposable. Hence, without loss of generality, we may assume that Q is fully $\mathcal{F}$-normalized.

We set $N_1 = B_Q$. Let $Sl_{(Q,b_Q)}(M) = \bigoplus_{1\le i\le r} N_i$ be a decomposition of $Sl_{(Q,b_Q)}(M)$ as a direct sum of indecomposable kN_Qb_Q-modules. Then, by Lemma 4.10 and its proof, for N_i, there exist $L_j \mid \mathrm{Res}^G_{N_Q}(M)$ and $g_i \in G$ such that

$$(Q, b_Q) \le (R, b_R) \le (\mathrm{vtx}(L_j), b_{\mathrm{vtx}(L_j)}) \le {}^{g_i}(P, b_P).$$

where $R = \mathrm{vtx}(N_i)$. By Lemma 4.2, $Sl_{(R,b_R)}(N_i) \neq 0$. Since Q is fully $\mathcal{F}$-normalized, Q is fully $\mathcal{F}$-automized and $\mathcal{F}$-receptive, and hence $N_{^{g_i}P}(Q) \le_{N_Q} H_Q$, from Lemma 4.13. Thus

$$R \le {}^{g_i}P \cap N_Q = N_{^{g_i}P}(Q) \le_{N_Q} H_Q$$

and $Sl_{(R,b_R)}(N_1) \neq 0$. Now we have

$$Sl_{(R,b_R)}(N_1) \oplus Sl_{(R,b_R)}(N_i) \mid Sl_{(R,b_R)}(Sl_{(Q,b_Q)}(M)) \cong \mathrm{Res}^{N_R}_{N_R\cap N_Q}(Sl_{(R,b_R)}(M)).$$

Thus $\mathrm{Res}^{N_R}_{N_R\cap N_Q}(Sl_{(R,b_R)}(M))$ is decomposable and $\mathrm{Res}^{N_R}_{RC_G(R)}(Sl_{(R,b_R)}(M))$ is decomposable, by $RC_G(R) \le N_R \cap N_Q$. If $Q = R$, then we see $P = Q$ from [4, Lemma 5] and Lemma 4.8. This is a contradiction. Hence $Q \lneqq R$ holds and we have that $|P : Q| \gneqq |P : R|$. By the induction hypothesis, the module $\mathrm{Res}^{N_R}_{N_R\cap N_Q}(Sl_{(R,b_R)}(M))$ is indecomposable. Hence $r = 1$, and we have that

$$Sl_{(Q,b_Q)}(M) \cong N_1 = B_Q.$$

Hence, $\mathrm{Res}^{N_Q}_{QC_G(Q)}(N_1)$ is indecomposable, and $\mathrm{Res}^{N_Q}_{QC_G(Q)}(Sl_{(Q,b_Q)}(M))$ is also indecomposable, by our hypothesis, . □

The following lemma can be proved in a similar way as [9, Lemma 4.3].

Lemma 5.2. *Let (P, b_P) be a (G, b)-subpair, $\mathcal{F} := \mathcal{F}_{(P,b_P)}(G, b)$, and Q a fully $\mathcal{F}$-automized subgroup of P. If there exists $N_P(Q) \le H_Q \le N_G(Q, b_Q)$ such that $|N_G(Q, b_Q) : H_Q| = p^a$ $(a \ge 0)$, then $N_G(Q, b_Q) = C_G(Q)H_Q$.*

The following proposition is a special analogy of [9, Theorem 1.4].

Proposition 5.3. *Let (P, b_P) be a (G, b)-subpair and Q a fully $\mathcal{F}_{(P,b_P)}(G, b)$-normalized subgroup of P. Suppose that $\mathcal{F} = \mathcal{F}_{(P,b_P)}(G, b)$ is saturated. Moreover, we assume that the following two conditions:*

(i) $|N_G(Q, b_Q) : N_P(Q)| = p^a$ $(a \ge 0)$.

(ii) $\mathrm{Res}^{N_P(Q)}_{QC_G(Q)\cap N_P(Q)}(V_Q)$ *is indecomposable.*

Then $\mathrm{Res}^{N_G(Q,b_Q)}_{QC_G(Q)}(B_Q)$ *is indecomposable.*

Proof. We set $N_G = N_G(Q, b_Q)$. Since $\mathcal{F}$ is saturated, Q is a fully $\mathcal{F}$-automized subgroup of P. From the Mackey formula, Lemma 5.2, and the condition (i), we have

$$\mathrm{Res}^{N_G}_{QC_G(Q)}(\mathrm{Ind}^{N_G}_{N_P(Q)}(V_Q)) \cong \mathrm{Ind}^{QC_G(Q)}_{QC_G(Q)\cap N_P(Q)}(\mathrm{Res}^{N_P(Q)}_{QC_G(Q)\cap N_P(Q)}(V_Q)).$$

Hence, $\mathrm{Res}^{N_G}_{QC_G(Q)}(\mathrm{Ind}^{N_G}_{N_P(Q)}(V_Q))$ is indecomposable, by the condition (ii) and Green's indecomposability theorem, so

$$\mathrm{Res}^{N_G}_{QC_G(Q)}(B_Q) \cong \mathrm{Res}^{N_G}_{QC_G(Q)}(\mathrm{Ind}^{N_G}_{N_P(Q)}(V_Q))$$

is indecomposable. □

The following corollary is a consequence of Theorem 5.1 and Proposition 5.3.

Corollary 5.4. *Let (P, b_P) be a (G, b)-subpair, $B(b, (P, b_P, V), Sl_{(P,b_P)}, S)$ a Brauer-friendly $\mathcal{O}Gb$-module, and suppose that $\mathcal{F}_{(P,b_P)}(G, b)$ is saturated. If for every fully $\mathcal{F}_{(P,b_P)}(G, b)$-normalized subgroup Q of P, the subgroup $N_P(Q)$ and the module V_Q satisfy the conditions of Proposition 5.3, then the module $B(b, (P, b_P, V), Sl_{(P,b_P)}, S)$ is slash indecomposable.*

The following example is a generalization of [16, Lemma 2.2] to Brauer-friendly modules.

Example 5.5. Let G be a p-group, $(P, 1_{C_G(P)})$ a $(G, 1_G)$-subpair, and suppose that $\mathcal{F} = \mathcal{F}_P(G)$ is saturated. Set $M = BS(1_G, (P, b_P, V), Sl_{(P,1_{C_G(P)})})$. Moreover, we assume that $\mathrm{Res}^{N_P(Q)}_{QC_G(Q)\cap N_P(Q)}(V_Q)$ is indecomposable, for any fully $\mathcal{F}$-normalized subgroup Q of P. From Corollary 5.4, M is slash indecomposable.

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