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H^s wave front set for Schrödinger equations with sub-quadratic potential

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Abstract. The aim of this work is to study regularity of solutions of the initial value problem for the Schrödinger equations with sub-quadratic potential. More precisely, we determine the H^s wave front sets of solutions from the behavior at infinity of the initial data by using the characterization of the H^s wave front set via wave packet transform.

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§1. Introduction

In this paper, we study regularity of solutions of the initial value problem for the following Schrödinger equations with time dependent sub-quadratic potential,

$$\begin{cases} i\partial_t u(t,x) + \frac{1}{2}\Delta u(t,x) = V(t,x)u(t,x), & (t,x) \in \mathbb{R}\times\mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $i = \sqrt{-1}$, $u : \mathbb{R}\times\mathbb{R}^n \to \mathbb{C}$, $\partial_t = \frac{\partial}{\partial t}$, $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ and $V(t,x)$ is a real valued function.

A. Hassell and J. Wunsch [8] and S. Nakamura [17], [18] have studied the characterization of the wave front sets of solutions to Schrödinger equations in terms of initial data. In 2009, S. Nakamura [17] has determined the C^∞ wave front sets of solutions to Schrödinger equations with sub-linear potential. He has shown that $(x',\xi') \notin WF(e^{-itH_0}u_0)$ if and only if there exists $a(x,\xi) \in C_0^\infty(\mathbb{R}^{2n})$ such that $a(x',\xi') \neq 0$ and $\|a(x+tD_x, hD_x)u_0\| = O(h^\infty)$ as $h \to 0$, where $u_0 \in L^2(\mathbb{R}^n)$ and H_0 is the free Schrödinger operator; that is, the propagation of singularities for Schrödinger equations does not include effect of potential in the case of sub-linear potential. In 2009, S. Nakamura [18] has extended the result of [17] to sub-quadratic potential cases. In the case of sub-quadratic potential, the singularities of the solutions to Schrödinger equations propagate along the classical flow including effect of potential. In 2014, K. Horie and S. Nakamura [10] have treated the case that $\rho < 3/2$ by using Dollard type approximate solutions to the Hamilton-Jacobi equation and have also introduced an example of linear growth potential. S. Ito and one of the authors [13] have determined the C^∞ wave front sets of solutions to Schrödinger equations with time dependent sub-quadratic potential by using wave packet transform. In this study, we shall determine the H^s wave front sets of solutions to the Schrödinger equations (1.1) with time dependent sub-quadratic potential. We assume the following on $V(t,x)$.

Assumption A. *$V(t,x)$ is a real valued function in $C^\infty(\mathbb{R}\times\mathbb{R}^n)$ and there exists a real constant ρ satisfying $\rho<2$ such that for all multi-indices α*

$$|\partial_x^\alpha V(t,x)| \le C_\alpha\,(1+|x|)^{\rho-|\alpha|}$$

for some $C_\alpha>0$ and for all $(t,x)\in\mathbb{R}\times\mathbb{R}^n$.

In order to state our results precisely, we prepare several notations and give the definition of the H^s wave front set and wave packet transform which is defined by A. Córdoba and C. Fefferman [1]. We denote $\hat f(\xi)=\mathcal{F}[f](\xi)=(2\pi)^{-n/2}\int_{\mathbb{R}^n} f(x)e^{-ix\xi}dx$ by Fourier transform of f and $e^{it\Delta/2}$ by the evolution operator of the free Schrödinger equation. For $x=(x_1,\dots,x_n)\in\mathbb{R}^n$, we write $\langle x\rangle=(1+|x|^2)^{1/2}$ with $|x|^2=x_1^2+x_2^2+\cdots+x_n^2$. For $\xi_0\in\mathbb{R}^n\backslash\{0\}$, a conic neighborhood Γ of ξ_0 is a subset in $\mathbb{R}^n$ such that $\xi\in\Gamma$ and $\alpha>0$ imply $\alpha\xi\in\Gamma$. Let $x(\tau;t,x,\xi)$ and $\xi(\tau;t,x,\xi)$ be the solutions to

$$\begin{cases}\dot x(\tau)=\xi(\tau), & x(t)=x,\\ \dot\xi(\tau)=-\nabla_x V(\tau,x(\tau)), & \xi(t)=\xi,\end{cases} \tag{1.2}$$

where $\dot x(t)$ and $\dot\xi(t)$ stand for the derivatives of $x(t)$ and $\xi(t)$ respectively.

Definition 1.1 (Wave packet transform)**.** *Let $\varphi\in\mathcal{S}(\mathbb{R}^n)\backslash\{0\}$. For $f\in\mathcal{S}'(\mathbb{R}^n)$, the wave packet transform $W_\varphi f(x,\xi)$ of f with basic wave packet φ is defined as*

$$W_\varphi f(x,\xi)=\int_{\mathbb{R}^n}\overline{\varphi(y-x)}f(y)e^{-iy\xi}dy,\quad x,\xi\in\mathbb{R}^n.$$

Remark 1.2. *If f is in $L^2(\mathbb{R}^n)$,*

$$\|W_\varphi f\|_{L^2(\mathbb{R}^{2n})}=\|\varphi\|_{L^2(\mathbb{R}^n)}\,\|f\|_{L^2(\mathbb{R}^n)}$$

(Proof of Remark 1.2). Since $W_\varphi f(x,\xi) = \mathscr{F}_{y\to\xi}\left[\overline{\varphi(y-x)}f(y)\right]$, Plancherel's theorem shows

$$\|W_\varphi f\|_{L^2(\mathbb{R}^{2n})} = \left\|\overline{\varphi(y-x)}f(y)\right\|_{L^2(\mathbb{R}^n_x\times\mathbb{R}^n_y)} = \|\varphi\|_{L^2(\mathbb{R}^n)}\|f\|_{L^2(\mathbb{R}^n)}.$$

□

Definition 1.3 (H^s wave front set of f)**.** *For $f\in\mathcal{S}'(\mathbb{R}^n)$, $WF_{H^s}(f)$ a subset in $\mathbb{R}^n\times(\mathbb{R}^n\backslash\{0\})$ is said to be H^s wave front set if the following condition holds:*
$(x_0,\xi_0)\notin WF_{H^s}(f)$ if and only if there exist a function $\chi\in C_0^\infty(\mathbb{R}^n)$ with $\chi(x_0)\neq 0$ and a conic neighborhood Γ of ξ_0 in $\mathbb{R}^n\backslash\{0\}$ such that $\langle\xi\rangle^s\,|\widehat{\chi f}(\xi)|\in L^2(\Gamma)$.

The following theorem is our main result.

Theorem 1.4. *Let u be a solution of (1.1) in $C(\mathbb{R};L^2(\mathbb{R}^n))$ under Assumption A. For $s>0$, the following statements are equivalent.*

- *(i)* $(x_0,\xi_0)\notin WF_{H^s}[u(t)]$.
- *(ii) There exist a neighborhood K of x_0 and a neighborhood U of ξ_0 such that*

$$\int_1^\infty \lambda^{2s+n-1}\left\|W_{\varphi_\lambda^{(-t)}}u_0\left(x_\lambda(0),\xi_\lambda(0)\right)\right\|^2_{L^2(K\times U)}d\lambda<+\infty \tag{1.3}$$

for all $\varphi\in\mathcal{S}(\mathbb{R}^n)\backslash\{0\}$, where $x_\lambda(0)=x(0;t,x,\lambda\xi),\xi_\lambda(0)=\xi(0;t,x,\lambda\xi)$, $\varphi_\lambda^{(t)}(x)=e^{it\Delta/2}\varphi_\lambda(x)$ with $\varphi_\lambda(x)=\lambda^{nb/2}\varphi(\lambda^b x)$ and $b=\min\left\{\frac{2-\rho}{4},\frac14\right\}$.

- *(iii) There exist $\varphi\in\mathcal{S}(\mathbb{R}^n)\backslash\{0\}$, a neighborhood K of x_0 and a neighborhood U of ξ_0 such that (1.3) holds, where $\varphi_\lambda^{(t)}(x)$ is the same as in (ii).*

Theorem 1.5. *Let u be a solution of (1.1) in $C(\mathbb{R};L^2(\mathbb{R}^n))$ under Assumption A with $\rho<1$. For $s>0$, the following statements are equivalent.*

- *(i)* $(x_0,\xi_0)\notin WF_{H^s}[u(t)]$.
- *(ii) There exist a neighborhood K of x_0 and a neighborhood U of ξ_0 such that*

$$\int_1^\infty \lambda^{2s+n-1}\left\|W_{\varphi_\lambda^{(-t)}}u_0\left(x-\lambda\xi t,\lambda\xi\right)\right\|^2_{L^2(K\times U)}d\lambda<+\infty \tag{1.4}$$

for all $\varphi\in\mathcal{S}(\mathbb{R}^n)\backslash\{0\}$, where $\varphi_\lambda^{(t)}(x)=e^{it\Delta/2}\varphi_\lambda(x)$ with $\varphi_\lambda(x)=\lambda^{nb/2}\varphi(\lambda^b x)$ and $b=\min\left\{\frac{1-\rho}{2},\frac14\right\}$.

(iii) There exist $\varphi \in \mathcal{S}(\mathbb{R}^n)\backslash\{0\}$, a neighborhood K of x_0 and a neighborhood U of ξ_0 such that (1.4) *holds, where $\varphi_\lambda^{(t)}(x)$ is the same as in (ii).*

Remark 1.6. *If there exists a solution of* (1.1) *in $C(\mathbb{R}; H^r(\mathbb{R}^n))$ with $r \in \mathbb{R}$, Theorem 1.4 and Theorem 1.5 hold for $s > r$.*

Remark 1.7. *The assertion of Theorem 1.4 and of Theorem 1.5 is valid if $\partial_x^\alpha V(t,x)$ is continuous in $\mathbb{R} \times \mathbb{R}^n$ for $|\alpha| \le l$ with*

$$l > \max\left\{8s + 4n + 4\rho, \frac{8s + 4n + 4}{2 - \rho}, \frac{4s + 2n + 2}{1 - \rho}\right\},$$

which is however far from the best possible.

In 1970, L. Hörmander has introduced the wave front set (see [9]). The wave front set characterizes the singularities of generalized function f not only in which point f is singular but also in which direction f is singular at the point. He has shown that the wave front sets of the solutions to hyperbolic equation of principal type propagate along the associate Hamilton flow. R. Lascar [15] has treated firstly singularities of solutions microlocally for Schrödinger equations. He has studied propagation of singularities for Schrödinger equations by using quasi-homogeneous wave front set which he has introduced. C. Parenti and F. Segala [21] and T. Sakurai [22] have treated the singularities of solutions to Schrödinger equations in the same way. The singularities of the solutions to Schrödinger equations immediately go to the infinity if the initial data decays rapidly at infinity. W. Craig, T. Kappeler and W. Strauss [2] have treated this type of smoothing property microlocally. This type of microlocal smoothing property has been studied by several authors, including S. Doi [4], [5], S. Nakamura [16], T. Ōkaji [19], [20] and L. Robbiano and C. Zuily [23]. As stated in the beginning of this section, A. Hassell and J. Wunsch [8] have treated the problem in the framework of scattering metric and S. Nakamura [18] has treated the problem in semi-classical way. S. Ito and one of the authors [13] have treated the problem by way of wave packet transform.

The rest of the paper is organized as follows. In Section 2, we introduce characterization of wave front set via wave packet transfrom. In Section 3, we introduce transformed equation of (1.1) via wave packet transform according to [11] and [13], which is used for the proof of Therem 1.4. In Section 4, we prove Theorem 1.4. In Section 5, we prove Lemma 4.1, which is introduced in Section 4. In Section 6, we prove Theorem 1.5.

§2. Characterization of wave front set

In this section, we introduce the characterization of the H^s wave front set via wave packet transform due to K. Kato, M. Kobayashi and S. Ito [14], which plays an important role for the proof of Theorem 1.4 and Theorem 1.5.

Proposition 2.1 (K. Kato, M. Kobayashi and S. Ito [14] Theorem 1.2)**.** *Let* $s \in \mathbb{R}$, $0 < b < 1$, $(x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\})$, *and* $u \in \mathcal{S}'(\mathbb{R}^n)$. *The following statements are equivalent.*

(i) $(x_0, \xi_0) \notin WF_{H^s}(u)$.

(ii) There exist a neighborhood K *of* x_0 *and a neighborhood* U *of* ξ_0 *such that*

$$\int_1^\infty \lambda^{2s+n-1} \|W_{\varphi_\lambda} u(x, \lambda\xi)\|^2_{L^2(K\times U)}\, d\lambda < +\infty$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)\backslash\{0\}$, *where* $\varphi_\lambda(x) = \lambda^{nb/2}\varphi(\lambda^b x)$.

(iii) There exist $\varphi \in \mathcal{S}(\mathbb{R}^n)\backslash\{0\}$, *a neighborhood* K *of* x_0 *and a neighborhood* U *of* ξ_0 *such that*

$$\int_1^\infty \lambda^{2s+n-1} \|W_{\varphi_\lambda} u(x, \lambda\xi)\|^2_{L^2(K\times U)}\, d\lambda < +\infty,$$

where $\varphi_\lambda(x) = \lambda^{nb/2}\varphi(\lambda^b x)$.

Remark 2.2. *For the* C^∞ *wave front set, G. B. Folland has firstly given the characterization in terms of wave packet transform with a positive symmetric Schwartz's function* φ *in [6]. T. Ōkaji [19] has given the characterization with the assumption that* φ *satisfies* $\int_{\mathbb{R}^n} x^\alpha \varphi(x)dx \neq 0$ *for some multi-index* α. *In [14], K. Kato, M. Kobayashi and S. Ito have removed any restriction on basic wave packet.*

Remark 2.3. *For the* H^s *wave front set, P. Gérard [7] has shown the equivalence between (i) and (iii) with* $\varphi(x) = e^{-x^2}$ *for* $b = \frac{1}{2}$ *(Proof is also in J. M. Delort [3]). In [19], T. Ōkaji has shown that (ii) implies (i) if* φ *satisfies* $\int_{\mathbb{R}^n} \varphi(x)dx \neq 0$ *for* $b = \frac{1}{2}$. *Conversely T. Ōkaji has shown that (i) with* $s = s_0$ *implies (ii) with* $s = s_0 - \varepsilon$ *for all* $\varepsilon > 0$ *in addition to the condition of* φ. *In [14], K. Kato, M. Kobayashi and S. Ito have shown Proposition 2.1 for* $0 < b < 1$ *without any restriction on basic wave packet.*

§3. Transformed equation via wave packet transform

In K. Kato and S. Ito [11] and [13], the initial value problem (1.1) is transformed to

$$\begin{cases} \left(i\partial_t + i\xi\cdot\nabla_x - i\nabla_x V(t,x)\cdot\nabla_\xi - \frac{1}{2}|\xi|^2 - \tilde{V}(t,x)\right) W_{\varphi^{(t)}} u(t,x,\xi) \\ \qquad\qquad = R_{\varphi^{(t)}} u(t,x,\xi), \\ W_{\varphi^{(0)}} u(0,x,\xi) = W_\varphi u_0(x,\xi), \end{cases} \tag{3.1}$$

where $\tilde{V}(t,x) = V(t,x) - \nabla_x V(t,x)\cdot x$,

$$\begin{aligned}
&R_{\varphi^{(t)}}u(t,x,\xi)\\
=&2\sum_{|\alpha|=2}\frac{1}{\alpha!}\int\overline{\varphi^{(t)}(y-x)}\\
&\qquad\times\left(\int_0^1\partial_x^\alpha V\left(t,x+\theta(y-x)\right)(1-\theta)d\theta\right)(y-x)^\alpha u(t,y)e^{-iy\xi}dy
\end{aligned}$$

and $\varphi^{(t)}(x) = e^{it\triangle/2}\varphi$ with $\varphi(x)\in\mathcal{S}(\mathbb{R}^n)\backslash\{0\}$. For reader's convenience, we briefly give a deduction of (3.1) from (1.1). The first term $i\partial_t u(t,\cdot)$ and the second term $\frac{1}{2}\Delta u(t,\cdot)$ of (1.1) are transformed to

$$\begin{aligned}
&W_{\varphi^{(t)}}[i\partial_t u](t,x,\xi) + W_{\varphi^{(t)}}\left[\frac{1}{2}\Delta u\right](t,x,\xi)\\
=&i\partial_t W_{\varphi^{(t)}}u(t,x,\xi) + W_{i\partial_t\varphi^{(t)}}u(t,x,\xi)\\
&\qquad\qquad+\frac{1}{2}\int\Delta_y\left\{\overline{\varphi^{(t)}(y-x)}e^{-iy\xi}\right\}u(t,y)dy\\
=&\left(i\partial_t + i\xi\cdot\nabla_x - \frac{1}{2}|\xi|^2\right)W_{\varphi^{(t)}}u(t,x,\xi) + W_{\left\{i\partial_t\varphi^{(t)}+\frac{1}{2}\Delta\varphi^{(t)}\right\}}u(t,x,\xi).
\end{aligned}\tag{3.2}$$

By Taylor's expansion of $V(t,y) = V\left(t,x+(y-x)\right)$ with respect to $y-x$, $V(t,\cdot)u(t,\cdot)$ is transformed to

$$\begin{aligned}
&W_{\varphi^{(t)}}[Vu](t,x,\xi)\\
=&\int\overline{\varphi^{(t)}(y-x)}\Big\{V(t,x)+\nabla_x V(t,x)\cdot(y-x)\\
&\qquad+2\sum_{|\alpha|=2}\frac{(y-x)^\alpha}{\alpha!}\int_0^1\partial_x V\left(t,x+\theta(y-x)\right)(1-\theta)d\theta\Big\}\\
&\qquad\qquad\times u(t,y)e^{-iy\xi}dy\\
=&\tilde{V}(t,x)W_{\varphi^{(t)}}u(t,x,\xi) + i\nabla_x V(t,x)\cdot\nabla_\xi W_{\varphi^{(t)}}u(t,x,\xi)\\
&\qquad+R_{\varphi^{(t)}}u(t,x,\xi).
\end{aligned}\tag{3.3}$$

Taking $\varphi^{(t)}(x)$ as the solution to the free Schrödinger equation

$$\begin{cases} i\partial_t\varphi(t,x)+\frac{1}{2}\Delta\varphi(t,x)=0,\\ \varphi(0,x)=\varphi(x)\in\mathcal{S}(\mathbb{R}^n)\backslash\{0\},\end{cases}$$

we have (3.1) from (3.2) and (3.3).

Solving (3.1) by the method of characteristics, we have the following integral equation

$$\begin{aligned} &W_{\varphi^{(t)}}u(t,x,\xi) \\ &= e^{-i\int_0^t\left\{\frac{1}{2}|\xi(\tau;t,x,\xi)|^2+\tilde{V}(\tau,x(\tau;t,x,\xi))\right\}d\tau}W_{\varphi}u_0\left(x(0;t,x,\xi),\xi(0;t,x,\xi)\right) \\ &\qquad - i\int_0^t e^{-i\int_\tau^t\left\{\frac{1}{2}|\xi(\tau_1;t,x,\xi)|^2+\tilde{V}(\tau_1,x(\tau_1;t,x,\xi))\right\}d\tau_1} \\ &\qquad\qquad \times R_{\varphi^{(t)}}u\left(\tau,x(\tau;t,x,\xi),\xi(\tau;t,x,\xi)\right)d\tau, \end{aligned} \tag{3.4}$$

where $x(\tau;t,x,\xi)$ and $\xi(\tau;t,x,\xi)$ are the solutions to (1.2).

§4. Proof of Theorem 1.4

In this section, we prove Theorem 1.4. Let t_0 be a fixed positive number. Since $\varphi_\lambda^{(t-t_0)}$ satisfies $(i\partial_t+\frac{1}{2}\Delta)\varphi=0$, we have

$$\begin{aligned} &W_{\varphi_\lambda^{(t-t_0)}}u\left(t,x(t;t_0,x,\lambda\xi),\xi(t;t_0,x,\lambda\xi)\right) \\ =&e^{-i\int_0^t\left\{\frac{1}{2}|\xi(\tau;t_0,x,\lambda\xi)|^2+\tilde{V}(\tau,x(\tau;t_0,x,\lambda\xi))\right\}d\tau} \\ &\qquad\qquad \times W_{\varphi_\lambda^{(-t_0)}}u_0\left(x(0;t_0,x,\lambda\xi),\xi(0;t_0,x,\lambda\xi)\right) \\ &- i\int_0^t e^{-i\int_\tau^t\left\{\frac{1}{2}|\xi(\tau_1;t_0,x,\lambda\xi)|^2+\tilde{V}(\tau_1,x(\tau_1;t_0,x,\lambda\xi))\right\}d\tau_1} \\ &\qquad\qquad \times R_{\varphi_\lambda^{(\tau-t_0)}}u\left(\tau,x(\tau;t_0,x,\lambda\xi),\xi(\tau;t_0,x,\lambda\xi)\right)d\tau. \end{aligned} \tag{4.1}$$

If we replace the initial condition in (3.1) with $W_{\varphi_\lambda}u(t_0,\cdot)$ for $t=t_0$, then the integral equation (3.4) corresponds to

$$\begin{aligned} &W_{\varphi_\lambda^{(t-t_0)}}u\left(t,x(t;t_0,x,\lambda\xi),\xi(t;t_0,x,\lambda\xi)\right) \\ =&e^{-i\int_{t_0}^t\left\{\frac{1}{2}|\xi(\tau;t_0,x,\lambda\xi)|^2+\tilde{V}(\tau,x(\tau;t_0,x,\lambda\xi))\right\}d\tau}W_{\varphi_\lambda}u(t_0,x,\lambda\xi) \\ &- i\int_{t_0}^t e^{-i\int_\tau^t\left\{\frac{1}{2}|\xi(\tau_1;t_0,x,\lambda\xi)|^2+\tilde{V}(\tau_1,x(\tau_1;t_0,x,\lambda\xi))\right\}d\tau_1} \\ &\qquad\qquad \times R_{\varphi_\lambda^{(\tau-t_0)}}u\left(\tau,x(\tau;t_0,x,\lambda\xi),\xi(\tau;t_0,x,\lambda\xi)\right)d\tau. \end{aligned} \tag{4.2}$$

In the following, we write $x_\tau=x(\tau;t_0,x,\lambda\xi)$, $\xi_\tau=\xi(\tau;t_0,x,\lambda\xi)$, and $t_\tau=\tau-t_0$ for brevity.

Now we prove Theorem 1.4. In Theorem 1.4, (ii) yields (iii) obviously. So we show that (i) implies (ii) and that (iii) implies (i). We denote

$P(r,\varphi)$:

$$\int_1^\infty \lambda^{2r+n-1}\sup_{t\in[0,t_0]}\left\|W_{\varphi_\lambda^{(t-t_0)}}u\left(t,x(t;t_0,x,\lambda\xi),\xi(t;t_0,x,\lambda\xi)\right)\right\|^2_{L^2(K\times U)}d\lambda<+\infty.$$

(Proof of Theorem 1.4). First we show that (i) implies (ii). Since Proposition 2.1(i) implies Proposition 2.1(ii), there exist a neighborhood K of x_0 and a neighborhood U of ξ_0 satisfying

$$\int_1^\infty \lambda^{2s+n-1} \left\| W_{\varphi_\lambda} u\,(t_0, x, \lambda\xi) \right\|^2_{L^2(K\times U)} d\lambda < +\infty \tag{4.3}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)\backslash\{0\}$. By induction with respect to r, we show that $P(r,\varphi)$ holds for all $\varphi \in \mathcal{S}(\mathbb{R}^n)\backslash\{0\}$ and for $r \le s$.

As basis step of induction, we show $P(-\varepsilon)$ for fixed $\varepsilon > 0$. Putting $\lambda\xi = \eta$, we have

$$\begin{aligned}
&\left\| W_{\varphi_\lambda^{(t-t_0)}} u\,(t, x(t;t_0,x,\lambda\xi), \xi(t;t_0,x,\lambda\xi)) \right\|^2_{L^2(K\times U)} \\
\le & \left\| W_{\varphi_\lambda^{(t-t_0)}} u\,(t, x(t;t_0,x,\lambda\xi), \xi(t;t_0,x,\lambda\xi)) \right\|^2_{L^2(\mathbb{R}^n_x\times\mathbb{R}^n_\xi)} \\
= & \lambda^{-n} \left\| W_{\varphi_\lambda^{(t-t_0)}} u\,(t, x(t;t_0,x,\eta), \xi(t;t_0,x,\eta)) \right\|^2_{L^2(\mathbb{R}^n_x\times\mathbb{R}^n_\eta)}.
\end{aligned} \tag{4.4}$$

By change of variables $X = x(t;t_0,x,\eta)$ and $\Xi = \xi(t;t_0,x,\eta)$, we have

$$\begin{aligned}
&\lambda^{-n} \left\| W_{\varphi_\lambda^{(t-t_0)}} u\,(t, x(t;t_0,x,\eta), \xi(t;t_0,x,\eta)) \right\|^2_{L^2(\mathbb{R}^n_x\times\mathbb{R}^n_\eta)} \\
= & \lambda^{-n} \iint_{\mathbb{R}^{2n}} \left| W_{\varphi_\lambda^{(t-t_0)}} u\,(t, X, \Xi) \right|^2 \left| \frac{\partial (X,\Xi)}{\partial (x,\eta)} \right|^{-1} dX d\Xi \\
= & \lambda^{-n} \left\| W_{\varphi_\lambda^{(t-t_0)}} u\,(t, X, \Xi) \right\|^2_{L^2(\mathbb{R}^{2n})}.
\end{aligned} \tag{4.5}$$

Here we use the fact that $\left|\frac{\partial(X,\Xi)}{\partial(x,\eta)}\right| = 1$, which is well known (see, for example Appendix A in K. Kato, M. Kobayashi and S. Ito [12] for the proof). As $u(t,x)$ is in $C(\mathbb{R}; L^2(\mathbb{R}^n))$, Remark 1.2 and the conservation of L^2 norm of solutions to the free Schrödinger equation show

$$\begin{aligned}
&\lambda^{-n} \left\| W_{\varphi_\lambda^{(t-t_0)}} u\,(t, X, \Xi) \right\|^2_{L^2(\mathbb{R}^{2n})} \\
= & \lambda^{-n} \left\| \varphi_\lambda^{(t-t_0)} \right\|^2_{L^2(\mathbb{R}^n)} \|u(t)\|^2_{L^2(\mathbb{R}^n)} \\
= & \lambda^{-n} \|\varphi\|^2_{L^2(\mathbb{R}^n)} \|u_0\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned} \tag{4.6}$$

Since (4.4), (4.5) and (4.6) show

$$\begin{aligned}
&\int_1^\infty \lambda^{-2\varepsilon+n-1} \sup_{t\in[0,t_0]} \left\| W_{\varphi_\lambda^{(t-t_0)}} u\,(t, x(t;t_0,x,\lambda\xi), \xi(t;t_0,x,\lambda\xi)) \right\|^2_{L^2(K\times U)} d\lambda \\
&\le \|\varphi\|^2_{L^2(\mathbb{R}^n)} \|u_0\|^2_{L^2(\mathbb{R}^n)} \int_1^\infty \lambda^{-1-2\varepsilon} d\lambda \\
&< +\infty,
\end{aligned} \tag{4.7}$$

we have $P(-\varepsilon,\varphi)$ for all $\varphi\in\mathcal{S}(\mathbb{R}^n)\backslash\{0\}$.

Assuming that $P(r,\varphi)$ holds for all $\varphi\in\mathcal{S}(\mathbb{R}^n)\backslash\{0\}$ with $r\in[-\varepsilon,s-2b]$, we show that $P(r+2b,\varphi)$ for all $\varphi\in\mathcal{S}(\mathbb{R}^n)\backslash\{0\}$. This assertion is showed by the following lemma, which is proven in Section 5.

Lemma 4.1. *Assume A and let* $b=\min\left\{\frac{2-\rho}{4},\frac{1}{4}\right\}$. *For* $r\in\mathbb{R}$, $P(r,\varphi)$ *for all* $\varphi\in\mathcal{S}(\mathbb{R}^n)\backslash\{0\}$ *implies*

$$\int_1^\infty \lambda^{2(r+2b)+n-1}\left|\int_0^{t_0}\left\|R_{\varphi_\lambda^{(t_\tau)}}u(\tau,x_\tau,\xi_\tau)\right\|_{L^2(K\times U)}d\tau\right|^2 d\lambda<+\infty$$

for all $\varphi\in\mathcal{S}(\mathbb{R}^n)\backslash\{0\}$.

The equation (4.2) shows

$$\begin{aligned}&\sup_{t\in[0,t_0]}\left\|W_{\varphi_\lambda^{(t-t_0)}}u\left(t,x(t;t_0,x,\lambda\xi),\xi(t;t_0,x,\lambda\xi)\right)\right\|^2_{L^2(K\times U)}\\ &\le 2\left(\|W_{\varphi_\lambda}u(t_0,x,\lambda\xi)\|^2_{L^2(K\times U)}+\int_0^{t_0}\left\|R_{\varphi_\lambda^{(t_\tau)}}u(\tau,x_\tau,\xi_\tau)\right\|^2_{L^2(K\times U)}d\tau\right).\end{aligned}\tag{4.8}$$

Multiplying $\lambda^{2(r+2b)+n-1}$ to the both sides of (4.8) and integrating the both sides from 1 to infinity with respect to λ, we immediately have from (4.3) and Lemma 4.1 that $P(r+2b,\varphi)$ holds for all $\varphi\in\mathcal{S}(\mathbb{R}^n)\backslash\{0\}$.

Next, we show that (iii) implies (i). Let $\varphi_0\in\mathcal{S}(\mathbb{R}^n)\backslash\{0\}$, a neighborhood K of x_0 and a neighborhood U of ξ_0 satisfy

$$\int_1^\infty\lambda^{2s+n-1}\left\|W_{\varphi_{0,\lambda}^{(-t_0)}}u_0\left(x(0;t_0,x,\lambda\xi),\xi(0;t_0,x,\lambda\xi)\right)\right\|^2_{L^2(K\times U)}d\lambda<+\infty,\tag{4.9}$$

where $\varphi_{0,\lambda}^{(t)}(x)=e^{it\Delta/2}[(\varphi_0)_\lambda](x)$. It suffices to show that (iii) implies $P(s,\varphi_0)$.

Since $P(-\varepsilon,\varphi)$ is valid for all $\varphi\in\mathcal{S}(\mathbb{R}^n)\backslash\{0\}$, we show $P(s,\varphi_0)$ inductively. The equation (4.1) shows

$$\begin{aligned}&\sup_{t\in[0,t_0]}\left\|W_{\varphi_{0,\lambda}^{(t-t_0)}}u\left(t,x(t;t_0,x,\lambda\xi),\xi(t;t_0,x,\lambda\xi)\right)\right\|^2_{L^2(K\times U)}\\ &\le 2\left(\left\|W_{\varphi_{0,\lambda}^{(-t_0)}}u_0\left(x(0;t_0,x,\lambda\xi),\xi(0;t_0,x,\lambda\xi)\right)\right\|^2_{L^2(K\times U)}\right.\\ &\qquad\left.+\int_0^{t_0}\left\|R_{\varphi_\lambda^{(t_\tau)}}u(\tau,x_\tau,\xi_\tau)\right\|^2_{L^2(K\times U)}d\tau\right).\end{aligned}\tag{4.10}$$

Hence Lemma 4.1 and (4.9) yield $P(-\varepsilon+2b,\varphi_0)$, which and Proposition 2.1 yield $(x_0,\xi_0)\notin WF_{H^{(-\varepsilon+2b)}}(u(t_0,\cdot))$. From the argument for the proof that

(i) implies (ii), we have $P(-\varepsilon+2b,\varphi)$ for all $\varphi\in\mathcal{S}(\mathbb{R}^n)\backslash\{0\}$. Hence we have $P(-\varepsilon+4b,\varphi_0)$ from (4.9), (4.10) and Lemma 4.1. The same argument as above yields that $P(-\varepsilon+4b,\varphi)$ for all $\varphi\in\mathcal{S}(\mathbb{R}^n)\backslash\{0\}$. Repeating this argument, we have $P(s,\varphi_0)$, which and Proposition 2.1 yield (i). □

§5. Proof of Lemma 4.1

In this section, we prove Lemma 4.1. For an integer $L\geq 2$, Taylor's expansion of $V(\tau,y)$ shows that

$$\begin{aligned}
&R_{\varphi_\lambda^{(t_\tau)}}u(\tau,x_\tau,\xi_\tau)\\
&=\sum_{2\leq|\alpha|\leq L-1}\frac{\partial_x^\alpha V(\tau,x_\tau)}{\alpha!}\int(y-x_\tau)^\alpha\,\overline{\varphi_\lambda^{(t_\tau)}(y-x_\tau)}u(\tau,y)e^{-iy\xi_\tau}dy\\
&\quad+L\sum_{|\alpha|=L}\frac{1}{\alpha!}\int(y-x_\tau)^\alpha\,\overline{\varphi_\lambda^{(t_\tau)}(y-x_\tau)}\\
&\qquad\times\left(\int_0^1\partial_x^\alpha V\left(\tau,x_\tau+\theta(y-x_\tau)\right)(1-\theta)^{L-1}d\theta\right)u(\tau,y)e^{-iy\xi_\tau}dy.
\end{aligned}\tag{5.1}$$

We denote the first term of the right hand side of (5.1) by $R_{1,L}(\tau,x_\tau,\xi_\tau)$ and the second term of the right hand side of (5.1) by $R_{2,L}(\tau,x_\tau,\xi_\tau)$. For the proof, we prepare the following lemma.

Lemma 5.1. *Under Assumption A, there exists a positive constant λ_0 such that for all integers N there exist an integer L and a positive constant C_N satisfying*

$$|R_{2,L}(\tau,x_\tau,\xi_\tau)|\leq C_N\lambda^{-N}$$

for all $\lambda\geq\lambda_0$, $0\leq|t_\tau|\leq t_0$, $x\in K$, $\xi\in U$.

(Proof of Lemma 5.1). To prove Lemma 5.1, it suffices to show that there exists a positive constant λ_0 such that for all integers N there exist an integer L and a positive constant C_N satisfying

$$|I_\alpha(\tau,x_\tau,\xi_\tau,\lambda)|\leq C_N\lambda^{-N}$$

for $\lambda\geq\lambda_0$ and $|\alpha|=L$, where

$$I_\alpha(\tau,x_\tau,\xi_\tau,\lambda)=\int\overline{\varphi_\lambda^{(t_\tau)}(y-x_\tau)}\,V_\alpha(\tau,x_\tau,y)(y-x_\tau)^\alpha u(\tau,y)e^{-iy\xi_\tau}dy\tag{5.2}$$

and

$$V_\alpha(\tau,x_\tau,y)=\int_0^1\partial_x^\alpha V\left(\tau,x_\tau+\theta(y-x_\tau)\right)(1-\theta)^{L-1}d\theta.\tag{5.3}$$

We may assume that K and U are bounded sets and that $\inf\{|\xi| \mid \xi \in U\} > 0$. We fix α with $|\alpha| = L$. We show $|I_\alpha(\tau, x_\tau, \xi_\tau, \lambda)| \le C_N \lambda^{-N}$ in two cases that $0 \le |t_\tau| \le \lambda^{-2b}$ and $\lambda^{-2b} \le |t_\tau| \le t_0$.

In the case that $0 \le |t_\tau| \le \lambda^{-2b}$, the fact that $xe^{it\Delta/2} = e^{it\Delta/2}(x - it\partial_x)$ shows

$$
\begin{aligned}
&(y-x_\tau)^\alpha \varphi_\lambda^{(t_\tau)}(y-x_\tau)\\
=&(y-x_\tau)^\alpha e^{it_\tau\Delta/2}\left[\varphi_\lambda(y-x_\tau)\right]\\
=&e^{it_\tau\Delta/2}\left[(y-x_\tau-it_\tau\partial_y)^\alpha\varphi_\lambda\right](y-x_\tau)\\
=&\sum_{\substack{\beta+\gamma=\alpha\\ \beta'\le\beta,\gamma'\le\gamma}} C_{\beta,\beta',\gamma,\gamma'}\ t_\tau^{|\beta|}\lambda^{b(|\beta|-|\gamma|)}\varphi^{(t_\tau)}_{\beta',\gamma',\lambda}(y-x_\tau),
\end{aligned}
\tag{5.4}
$$

where $\varphi_{\beta,\gamma}(x) = x^\gamma\partial_x^\beta\varphi(x)$ and $\varphi^{(t)}_{\beta,\gamma,\lambda}(x) = e^{it\Delta/2}\left[(\varphi_{\beta,\gamma})_\lambda\right](x)$. The equality (5.4), Schwarz's inequality, the conservation of L^2 norm of solutions to the free Schrödinger equation and of (1.1) show

$$
\begin{aligned}
&|I_\alpha(\tau,x_\tau,\xi_\tau,\lambda)|\\
\le&C\sum_{\substack{\beta+\gamma=\alpha\\ \beta'\le\beta,\gamma'\le\gamma}} |t_\tau|^{|\beta|}\lambda^{b(|\beta|-|\gamma|)}\int\left|\overline{\varphi^{(t_\tau)}_{\beta',\gamma',\lambda}(y-x_\tau)}\ V_\alpha(\tau,x_\tau,y)u(\tau,y)\right|dy\\
\le&C\lambda^{-bL}\sum_{\substack{\beta+\gamma=\alpha\\ \beta'\le\beta,\gamma'\le\gamma}}\int\left|\overline{\varphi^{(t_\tau)}_{\beta',\gamma',\lambda}(y-x_\tau)}u(\tau,y)\right|dy\\
\le&C\sum_{\substack{\beta+\gamma=\alpha\\ \beta'\le\beta,\gamma'\le\gamma}}\left\|\varphi_{\beta',\gamma'}\right\|_{L^2}\|u_0\|_{L^2}\lambda^{-bL}.
\end{aligned}
\tag{5.5}
$$

In the case that $\lambda^{-2b} \le |t_\tau| \le t_0$, we divide (5.2) into two terms

$$
\begin{aligned}
&|I_\alpha(\tau,x_\tau,\xi_\tau,\lambda)|\\
\le&\int_{|y-x_\tau|\le\lambda^{1-\delta}|t_\tau|}\left|\overline{\varphi_\lambda^{(t_\tau)}(y-x_\tau)}\ V_\alpha(\tau,x_\tau,y)(y-x_\tau)^\alpha u(\tau,y)\right|dy\\
&\qquad+C\int_{|y-x_\tau|\ge\lambda^{1-\delta}|t_\tau|}\left|\overline{\varphi_\lambda^{(t_\tau)}(y-x_\tau)}(y-x_\tau)^\alpha u(\tau,y)\right|dy
\end{aligned}
\tag{5.6}
$$

with $0 < \delta < b$. Under the case that $\lambda^{-2b} \le |t_\tau| \le t_0$, there exists a positive constant λ_0 such that

$$
|x_\tau| \ge \frac{d_U}{2}|t_\tau|\lambda \tag{5.7}
$$

for all $\lambda \geq \lambda_0$, $x \in K$ and $\xi \in U$, where $d_U = \inf\{|\xi| \mid \xi \in U\}$ (see Appendix A in K. Kato and S. Ito [13] for the proof of (5.7)). On $|y - x_\tau| \leq \lambda^{1-\delta}|t_\tau|$, the inequality (5.7) and Assumption A yield

$$\begin{aligned}|V_\alpha(\tau, x_\tau, y)(y - x_\tau)^\alpha| \leq& C\left(1 + |x_\tau + \theta(y - x_\tau)|\right)^{\rho-L}|y - x_\tau|^L \\ \leq& C\left(1 + |x_\tau| - |y - x_\tau|\right)^{\rho-L}\left(\lambda^{1-\delta}|t_\tau|\right)^L \\ \leq& C\left(1 + \frac{d_V}{2}|t_\tau|\lambda - \lambda^{1-\delta}|t_\tau|\right)^{\rho-L}(\lambda|t_\tau|)^L\lambda^{-\delta L} \\ \leq& C(\lambda|t_\tau|)^\rho\lambda^{-\delta L}.\end{aligned}$$

Hence Schwarz's inequality, the conservation of L^2 norm of solutions to the free Schrödinger equation and of (1.1) show that

$$\begin{aligned}&|(\text{The first term of the right hand side of (5.6)})| \\ &\leq C(\lambda|t_\tau|)^\rho\lambda^{-\delta L}\int_{\mathbb{R}^n}\left|\overline{\varphi_\lambda^{(t_\tau)}(y - x_\tau)}u(\tau, y)\right|dy \\ &\leq C\|\varphi\|_{L^2}\|u_0\|_{L^2}\lambda^{\rho-\delta L}.\end{aligned} \tag{5.8}$$

Since $xe^{it\Delta/2} = e^{it\Delta/2}(x - it\nabla_x)$, we have for any integer M

$$\begin{aligned}(1 + |x|^2)^M\varphi_\lambda^{(t)}(x) =& e^{it\Delta/2}\left[(1 + |x - it\nabla_x|^2)^M\varphi_\lambda(x)\right] \\ =& e^{it\Delta/2}\left[\sum_{|\beta+\gamma|\leq 2M} C_{\beta,\gamma}(\lambda^b t)^{|\beta|}\lambda^{-b|\gamma|}(\varphi_{\beta,\gamma})_\lambda\right] \\ =& \sum_{|\beta+\gamma|\leq 2M} C_{\beta,\gamma}(\lambda^b t)^{|\beta|}\lambda^{-b|\gamma|}\varphi_{\beta,\gamma,\lambda}^{(t)}(x).\end{aligned} \tag{5.9}$$

Hence we have

$$\begin{aligned}&|(\text{The second term of the right hand side of (5.6)})| \\ =& C\int_{|y-x_\tau|\geq\lambda^{1-\delta}|t_\tau|}\left(1 + |y - x_\tau|^2\right)^{-M}\left(1 + |y - x_\tau|^2\right)^M \\ &\qquad\times\left|\overline{\varphi_\lambda^{(t_\tau)}(y - x_\tau)}(y - x_\tau)^\alpha u(\tau, y)\right|dy\end{aligned}$$

$$\begin{aligned}\leq& C\int_{|y-x_\tau|\geq\lambda^{1-\delta}|t_\tau|}\left(1 + (\lambda^{1-\delta}|t_\tau|)^2\right)^{-M}\sum_{L\leq|\beta+\gamma|\leq 2M+L}(\lambda^b|t_\tau|)^{|\beta|}\lambda^{-b|\gamma|} \\ &\qquad\times\left|\overline{\varphi_{\beta,\gamma,\lambda}^{(t_\tau)}(y - x_\tau)}u(\tau, y)\right|dy\end{aligned} \tag{5.10}$$

$$\le C \sum_{L\le|\beta+\gamma|\le 2M+L} \int_{\mathbb{R}^n} \left(1+\lambda^{2(1-\delta-2b)}\right)^{-M} \lambda^{b(2M+L)} \times \left|\overline{\varphi^{(t_\tau)}_{\beta,\gamma,\lambda}(y-x_\tau)}u(\tau,y)\right| dy$$

$$\le C \sum_{L\le|\beta+\gamma|\le 2M+L} \lambda^{-2M(1-\delta-3b)}\lambda^{bL}\|\varphi_{\beta,\gamma}\|_{L^2}\|u_0\|_{L^2}.$$

Since $1-\delta-3b > 1-4b \ge 0$, we have Lemma 5.1 from (5.5), (5.8) and (5.10) if we take L and M sufficiently large. □

(Proof of Lemma 4.1). By Lemma 5.1, we have

$$\int_1^\infty \lambda^{2(r+2b)+n-1} \left|\int_0^{t_0} \|R_{2,L}(\tau,x_\tau,\xi_\tau)\|_{L^2(K\times U)}\right|^2 d\lambda < +\infty \tag{5.11}$$

for bounded set U with $\inf\{|\xi| \mid \xi\in U\} > 0$ if we take L sufficiently large. Hence we only have to show that

$$\int_1^\infty \lambda^{2(r+2b)+n-1} \left|\int_0^{t_0} \|R_{1,L}(\tau,x_\tau,\xi_\tau)\|_{L^2(K\times U)}\, d\tau\right|^2 d\lambda < +\infty.$$

By (5.4) and the Assumption A, we have

$$\begin{aligned}
&|R_{1,L}(\tau,x_\tau,\xi_\tau)| \\
&\le C \sum_{2\le|\alpha|\le L-1} \sum_{\substack{\beta+\gamma=\alpha\\ \beta'\le\beta,\gamma'\le\gamma}} |\partial_x^\alpha V(\tau,x_\tau)||t_\tau|^{|\beta|}\lambda^{b(|\beta|-|\gamma|)} \left|W_{\varphi^{(t_\tau)}_{\beta',\gamma',\lambda}} u(\tau,x_\tau,\xi_\tau)\right| \\
&\le C \sum_{2\le|\alpha|\le L-1} \sum_{\substack{\beta+\gamma=\alpha\\ \beta'\le\beta,\gamma'\le\gamma}} (1+|x_\tau|)^{\rho-|\alpha|}|t_\tau|^{|\beta|}\lambda^{b(|\beta|-|\gamma|)} \left|W_{\varphi^{(t_\tau)}_{\beta',\gamma',\lambda}} u(\tau,x_\tau,\xi_\tau)\right|.
\end{aligned}$$

In the case that $0 \le |t_\tau| \le \lambda^{-2b}$,

$$\begin{aligned}
|R_{1,L}(\tau,x_\tau,\xi_\tau)| &\le C \sum_{2\le|\alpha|\le L-1} \sum_{\substack{\beta+\gamma=\alpha\\ \beta'\le\beta,\gamma'\le\gamma}} \lambda^{-2b|\beta|}\lambda^{b(|\beta|-|\gamma|)} \left|W_{\varphi^{(t_\tau)}_{\beta',\gamma',\lambda}} u(\tau,x_\tau,\xi_\tau)\right| \\
&\le C \sum_{2\le|\alpha|\le L-1} \sum_{\substack{\beta+\gamma=\alpha\\ \beta'\le\beta,\gamma'\le\gamma}} \lambda^{-2b} \left|W_{\varphi^{(t_\tau)}_{\beta',\gamma',\lambda}} u(\tau,x_\tau,\xi_\tau)\right|.
\end{aligned}$$

The inequality (5.7) yields

$$
\begin{aligned}
&|R_{1,L}(\tau, x_\tau, \xi_\tau)| \\
\le& C \sum_{2\le|\alpha|\le L-1} \sum_{\substack{\beta+\gamma=\alpha \\ \beta'\le\beta, \gamma'\le\gamma}} \left(1+\frac{d_U}{2}|t_\tau|\lambda\right)^{\rho-|\alpha|} (|t_\tau|\lambda^b+\lambda^{-b})^{|\alpha|} \left|W_{\varphi^{(t_\tau)}_{\beta',\gamma',\lambda}} u(\tau, x_\tau, \xi_\tau)\right| \\
\le& C \sum_{2\le|\alpha|\le L-1} \sum_{\substack{\beta+\gamma=\alpha \\ \beta'\le\beta, \gamma'\le\gamma}} |t_0|^\rho \lambda^{\rho-(1-b)2} \left|W_{\varphi^{(t_\tau)}_{\beta',\gamma',\lambda}} u(\tau, x_\tau, \xi_\tau)\right|
\end{aligned}
$$

for all $\lambda \ge \lambda_0$, $\lambda^{-2b} \le |t_\tau| \le t_0$, $x \in K$ and $\xi \in U$, where $d_U = \inf\{|\xi| \mid \xi \in U\}$. Since $b = \min\left\{\frac{2-\rho}{4}, \frac{1}{4}\right\}$, we have

$$
|R_{1,L}(\tau, x_\tau, \xi_\tau)| \le C \sum_{2\le|\alpha|\le L-1} \sum_{\substack{\beta+\gamma=\alpha \\ \beta'\le\beta, \gamma'\le\gamma}} \lambda^{-2b} \left|W_{\varphi^{(t_\tau)}_{\beta',\gamma',\lambda}} u(\tau, x_\tau, \xi_\tau)\right| \tag{5.12}
$$

for all $\lambda \ge \lambda_0$, $0 \le |t_\tau| \le t_0$, $x \in K$ and $\xi \in U$. The inequality (5.12) and the assumption that $P(r, \varphi)$ holds for all $\varphi \in \mathcal{S}(\mathbb{R}^n)\backslash\{0\}$ with $r \in \mathbb{R}$ show that

$$
\begin{aligned}
&\int_{\lambda_0}^\infty \lambda^{2(r+2b)+n-1} \left|\int_0^{t_0} \|R_{1,L}\|_{L^2(K\times U)} d\tau\right|^2 d\lambda \\
\le& C \sum_{2\le|\alpha|\le L-1} \sum_{\substack{\beta+\gamma=\alpha \\ \beta'\le\beta, \gamma'\le\gamma}} \\
&\quad \int_1^\infty \lambda^{2r+n+4b-1} \left|\lambda^{-2b} \int_0^{t_0} \left\|W_{\varphi^{(t_\tau)}_{\beta',\gamma',\lambda}} u(\tau, x_\tau, \xi_\tau)\right\|_{L^2(K\times U)} d\tau\right|^2 d\lambda \\
\le& C \sum_{2\le|\alpha|\le L-1} \sum_{\substack{\beta+\gamma=\alpha \\ \beta'\le\beta, \gamma'\le\gamma}} \\
&\quad \int_1^\infty \lambda^{2r+n-1} \sup_{\tau\in[0,t_0]} \left\|W_{\varphi^{(t_\tau)}_{\beta',\gamma',\lambda}} u(\tau, x_\tau, \xi_\tau)\right\|^2_{L^2(K\times U)} d\lambda \\
<& +\infty.
\end{aligned} \tag{5.13}
$$

By (5.11) and (5.13), we have

$$
\int_1^\infty \lambda^{2(r+2b)+n-1} \left|\int_0^{t_0} \left\|R_{\varphi^{(t_\tau)}_\lambda} u(\tau, x_\tau, \xi_\tau)\right\|_{L^2(K\times U)} d\tau\right|^2 d\lambda < +\infty. \tag{5.14}
$$

□

§6. Proof of Theorem 1.5

In this section, we prove Theorem 1.5.

(Proof of Theorem 1.5). Since

$$
\begin{aligned}
&W_{\varphi^{(t)}}\left[Vu\right](t,x,\xi) \\
&= \int \overline{\varphi^{(t)}(y-x)} \\
&\qquad \times \left\{ V(t,x) + \sum_{|\alpha|=1} (y-x)^\alpha \int_0^1 \partial_x^\alpha V\left(t, x+\theta(y-x)\right) d\theta \right\} \\
&\qquad\qquad \times u(t,y) e^{-iy\xi} dy \\
&= V(t,x) W_{\varphi^{(t)}} u(t,x,\xi) + \tilde{R}_{\varphi^{(t)}} u(t,x,\xi),
\end{aligned} \tag{6.1}
$$

where

$$
\tilde{R}_{\varphi^{(t)}} u(t,x,\xi) = \sum_{|\alpha|=1} \int \overline{\varphi^{(t)}(y-x)} (y-x)^\alpha \left(\int_0^1 \partial_x^\alpha V\left(t, x+\theta(y-x)\right) d\theta \right) \times u(t,y) e^{-iy\xi} dy,
$$

the initial value problem (1.1) is transformed to

$$
\begin{cases}
\left(i\partial_t + i\xi \cdot \nabla_x - \frac{1}{2}|\xi|^2 - V(t,x) \right) W_{\varphi^{(t)}} u(t,x,\xi) = \tilde{R}_{\varphi^{(t)}} u(t,x,\xi), \\
W_{\varphi^{(0)}} u(0,x,\xi) = W_\varphi u_0(x,\xi),
\end{cases} \tag{6.2}
$$

where $\varphi^{(t)}(x) = e^{it\Delta/2}\varphi(x)$ for all $\varphi \in \mathcal{S}(\mathbb{R}^n)\backslash\{0\}$. By the same calculations as in Section 3, we have

$$
\begin{aligned}
&W_{\varphi_\lambda^{(t-t_0)}} u\left(t, x+\lambda\xi(t-t_0), \lambda\xi\right) \\
&\quad = e^{-i\int_0^t \left\{ \frac{1}{2}\lambda^2|\xi|^2 + V(\tau, x+t_\tau\lambda\xi) \right\} d\tau} W_{\varphi_\lambda^{(-t_0)}} u_0\left(x - \lambda\xi t_0, \lambda\xi\right) \\
&\qquad - i \int_0^t e^{-i\int_\tau^t \left\{ \frac{1}{2}\lambda^2|\xi|^2 + V(\tau_1, x+(\tau_1-t_0)\lambda\xi) \right\} d\tau_1} \tilde{R}_{\varphi_\lambda^{(\tau-t_0)}} u\left(\tau, x+\lambda\xi t_\tau, \lambda\xi\right) d\tau,
\end{aligned}
$$

where $t_\tau = \tau - t_0$, $x + \lambda\xi(t-t_0)$ and $\lambda\xi$ are the solutions to

$$
\begin{cases}
\dot{x}(\tau) = \xi(\tau), & x(t_0) = x, \\
\dot{\xi}(\tau) = 0, & \xi(t_0) = \lambda\xi,
\end{cases}
$$

for $\lambda \geq 1$. For an integer $L \geq 1$, Taylor's expansion of $V(\tau, y)$ shows that

$$
\begin{aligned}
&\tilde{R}_{\varphi_\lambda^{(t_\tau)}} u(\tau, x+\lambda\xi t_\tau, \lambda\xi) \\
&= \sum_{1\leq|\alpha|\leq L-1} \frac{\partial_x^\alpha V(\tau, x+\lambda\xi t_\tau)}{\alpha!} \int (y-x-\lambda\xi t_\tau)^\alpha \\
&\qquad\qquad \times \overline{\varphi_\lambda^{(t_\tau)}(y-x-\lambda\xi t_\tau)} u(\tau,y) e^{-iy\lambda\xi} dy \\
&\quad + L \sum_{|\alpha|=L} \frac{1}{\alpha!} \int (y-x-\lambda\xi t_\tau)^\alpha \, \overline{\varphi_\lambda^{(t_\tau)}(y-x-\lambda\xi t_\tau)} \\
&\qquad \times \left(\int_0^1 \partial_x^\alpha V\left(\tau, x+\lambda\xi t_\tau + \theta(y-x-\lambda\xi t_\tau)\right) (1-\theta)^{L-1} d\theta \right) \\
&\qquad\qquad \times u(\tau,y) e^{-iy\lambda\xi} dy.
\end{aligned} \tag{6.3}
$$

We denote the first term of the right hand side of (6.3) by $\tilde{R}_{1,L}(\tau, x+\lambda\xi t_\tau, \lambda\xi)$ and the second term of the right hand side of (6.3) by $\tilde{R}_{2,L}(\tau, x+\lambda\xi t_\tau, \lambda\xi)$. The same argument as in the proof of Lemma 4.1 shows the assertion of Lemma 5.1 is valid for $\tilde{R}u$. That is, $P(r,\varphi)$ for all $\varphi \in \mathcal{S}(\mathbb{R}^n)\backslash\{0\}$ implies

$$
\int_1^\infty \lambda^{2(r+b)+n-1} \left| \int_0^{t_0} \left\| \tilde{R}_{\varphi_\lambda^{(t_\tau)}} u(\tau, x+\lambda\xi t_\tau, \lambda\xi) \right\|_{L^2(K\times U)} d\tau \right|^2 d\lambda < +\infty. \tag{6.4}
$$

In fact, the same argument shows

$$
\int_1^\infty \lambda^{2(r+b)+n-1} \left| \int_0^{t_0} \left\| \tilde{R}_{1,L} u(\tau, x+\lambda\xi t_\tau, \lambda\xi) \right\|_{L^2(K\times U)} d\tau \right|^2 d\lambda < +\infty,
$$

since $\rho < 1$. Exactly the same proof as for $R_{2,L}$ is valid for $\tilde{R}_{2,L}$. By using (6.4), we can show Theorem 1.5 in the same procedure as in Section 4. □

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