

## A note on the spectrum of a bounded operator on a complex interpolation space

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**Abstract.** Let  $(X_0, X_1)$  be a compatible couple of Banach spaces and  $T_j$  a bounded operator from  $X_j$  into itself ( $j = 0, 1$ ) satisfying  $T_0x = T_1x$  for all  $x \in X_0 \cap X_1$ . On an additional assumption concerning a boundedness of  $T_0|_{X_0 \cap X_1}$  ( $= T_1|_{X_0 \cap X_1}$ ), the next relations of spectra are proved:

$$\sigma(T_\theta) \subset \sigma(T_0) \cup \sigma(T_1) = \sigma(T_\Delta) \cup \sigma(T_\Sigma) \quad (\theta \in (0, 1)),$$

where  $T_\theta$ ,  $T_\Delta$  and  $T_\Sigma$  are the bounded operators induced by  $T_0$  and  $T_1$  on the complex interpolation space  $(X_0, X_1)_{[\theta]}$ , the intersection  $X_0 \cap X_1$  and the sum  $X_0 + X_1$ , respectively.

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### §1. Introduction and main result

This paper is concerned with the spectrum of a bounded operator on a complex interpolation space. The assumption of boundedness is not an essential restriction because considerations on the spectra of unbounded operators are reduced to those on the spectra of resolvents of the operators by the spectral mapping theorem. The definition of basic notions in the complex interpolation theory, which includes a compatible couple of Banach spaces, their intersection (resp. sum) and its norm  $\|\cdot\|_\Delta$  (resp.  $\|\cdot\|_\Sigma$ ) and a complex interpolation space, is described in Appendix of this paper.

Let  $(X_0, X_1)$  be a compatible couple of Banach spaces (for definition, see Definition A.1) and  $T_j$  a bounded operator from  $X_j$  into itself ( $j = 0, 1$ ), and

assume that  $T_0$  is consistent with  $T_1$ , i.e.,  $T_0x = T_1x$  for all  $x \in X_0 \cap X_1$ . Then, by the complex interpolation theory, for each  $\theta \in (0, 1)$ , a complex interpolation space  $(X_0, X_1)_{[\theta]}$  is defined (for definition, see Definition A.2), and there exists a unique bounded operator  $T_\theta$  from  $(X_0, X_1)_{[\theta]}$  into itself which is consistent with both  $T_0$  and  $T_1$ , i.e.,  $T_\theta x = T_0x = T_1x$  for all  $x \in X_0 \cap X_1$  (cf. [5, 4.1.2. Theorem and 4.2.2. Theorem (a)]). One of the typical examples of such a situation is given by an integrable function on  $\mathbb{R}^N$ . Indeed, such a function defines a convolution operator  $S_p$  on  $L^p(\mathbb{R}^N)$  for each  $p \in [1, \infty]$ , and each of these operators is bounded and consistent with one another. In this example,  $L^1(\mathbb{R}^N)$ ,  $L^\infty(\mathbb{R}^N)$ ,  $S_1$  and  $S_\infty$  are regarded as  $X_0$ ,  $X_1$ ,  $T_0$  and  $T_1$  above, respectively. Under this corresponding relationship, for each  $\theta \in (0, 1)$  and the exponent  $p$  satisfying  $1/p = 1 - \theta$ , the Lebesgue space  $L^p(\mathbb{R}^N)$  is isomorphic to the complex interpolation space  $(X_0, X_1)_{[\theta]}$  (cf. [5, 5.1.1. Theorem]), and  $S_p$  is nothing but the interpolation operator  $T_\theta$  on  $(X_0, X_1)_{[\theta]}$  whose existence and boundedness are guaranteed by applying Riesz–Thorin’s interpolation theorem [5, 1.1.1. Theorem] to  $T_0$  and  $T_1$ . On the spectrum of  $S_p$ , a remarkable result has been obtained, that is, K. Jörgens [8, Theorem 13.3] proved that the spectrum of  $S_p$  is independent of  $p \in [1, \infty]$ .

Another example is given by a Schrödinger semigroup or resolvents of its generator. B. Simon [14, THEOREM 1.1] and J. Voigt [17, 5.3. THEOREM and 5.8. PROPOSITION] proved that, for a class of potentials, the Schrödinger semigroup  $(T_p(t))_{t \geq 0}$  is a  $C_0$ -semigroup on  $L^p(\mathbb{R}^N)$  for each  $p \in [1, \infty)$  and each  $T_p(t)$  ( $t > 0$ ) is consistent with each  $T_q(t)$  for all  $p$  and  $q \in [1, \infty)$ . Based on this result, Simon [14, THEOREM 5.1] and R. Hempel–Voigt [7, Theorem] obtained the result that the generator  $-H_p$  of  $(T_p(t))_{t \geq 0}$ , i.e., a Schrödinger operator acting in  $L^p(\mathbb{R}^N)$ , has a spectrum independent of  $p \in [1, \infty)$ . As is stated in [7, 3.3 Proposition], the  $n$ th power  $(H_2 - z)^{-n}$  of a resolvent of  $H_2$  for a sufficiently large integer  $n$  is an integral operator, and its kernel exponentially decays away from the diagonal set. This fact implies that the bounded operator  $R_{p,n}$  on  $L^p(\mathbb{R}^N)$  induced from  $(H_2 - z)^{-n}$  coincides with the  $n$ th power of a resolvent of  $H_p$  and the integral operator whose kernel is the same as that of  $(H_2 - z)^{-n}$ . In addition, for each  $p \in [1, \infty)$ , the operator  $R_{p,n}$  is consistent with both  $R_{1,n}$  and the dual operator  $R_{\infty,n}$  of  $R_{1,n}$ , and hence  $R_{p,n}$  is nothing but the interpolation operator on  $L^p(\mathbb{R}^N)$  between  $R_{1,n}$  and  $R_{\infty,n}$ . Thus, this interpolation operator has a  $p$ -independent integral kernel.

These results naturally lead to the questions of whether the spectrum of the general interpolation operator  $T_\theta$  is independent of  $\theta$  or not, or what relationships exist among the spectrum of  $T_0$ ,  $T_1$  and  $T_\theta$ . To find directions for

generalization, it would not be meaningless to have an overview of some related results, including cases of unbounded operators, before we state the main result of this paper. W. Arendt [1, Corollary 4.3.] proved the  $p$ -independence of the spectrum of the generator of a  $C_0$ -semigroup on  $L^p$ , but not necessarily a Schrödinger semigroup, on the assumption of “upper Gaussian estimate”, which corresponds to domination by the Gauss semigroup. P. C. Kunstmann [9, THEOREM 1.1] and B. A. Barnes [2, THEOREM 4.8] obtained a result on the  $p$ -independence of the spectrum of an integral operator on  $L^p$  by improving Arendt’s method and using the theory of Banach algebras, respectively. S. Miyajima and the author replaced the Laplacian of Schrödinger semigroups or the Gauss semigroup with a fractional Laplacian  $-(-\Delta)^\alpha$  ( $0 < \alpha < 1$ ) and proved some results analogous to Hempel–Voigt’s or Arendt’s result by applying Kunstmann’s or Barnes’ result (cf. [11, Theorem 4.2], [10, Theorem 3.19 and Theorem 3.20] and [12, Theorem 3.9 and Theorem 4.2]).

So far we have overviewed the results on the  $p$ -independence of spectra, but in general, it does not hold (cf. [1, Section 3]). However, Barnes proved the following results which hold in such cases.

**Proposition 1.1.** *Let  $\Omega$  be a  $\sigma$ -finite measure space,  $p_0$  and  $p_1$  numbers satisfying  $1 \leq p_0 < p_1 \leq \infty$ , the Banach space  $X_0$  the Lebesgue space  $L^{p_0}(\Omega)$ , and the Banach space  $X_1$  the Lebesgue space  $L^{p_1}(\Omega)$  if  $p_1 \neq \infty$  or  $L_0^\infty(\Omega)$  if  $p_1 = \infty$ , where  $L_0^\infty(\Omega)$  is the closure of the subspace consisting of all integrable simple functions in  $L^\infty(\Omega)$ .*

(i) ([4, THEOREM 5.3]) *On this assumption, the inclusion relations*

$$\sigma(T_\theta) \subset \sigma(T_0) \cup \sigma(T_1) \cup \sigma(T_\Delta)$$

and

$$(1.1) \quad \sigma(T_\theta) \subset [\sigma(T_0) \cup \sigma(T_1)]^\wedge$$

hold for each  $\theta \in (0, 1)$ , where  $T_\Delta \in \mathcal{L}(X_0 \cap X_1)$  is defined by

$$(1.2) \quad T_\Delta x := T_0 x \quad (x \in X_0 \cap X_1)$$

and the right-hand side of (1.1) denotes the polynomial convex hull of  $\sigma(T_0) \cup \sigma(T_1)$  (for definition, see Remark 1.2 below). Note that, for all  $x \in X_0 \cap X_1$ , the image  $T_0 x$  belongs to  $X_0 \cap X_1$  because of the consistency of  $T_0$  and  $T_1$ . The boundedness of  $T_\Delta$  follows from that of  $T_0$  and  $T_1$  (cf. Definition A.1).

- (ii) ([4, THEOREM 4.3 (1)]) *In addition to the assumption of (i), assume that  $\Omega$  is either a finite or a special discrete measure space (for definition, see Remark 1.2 below). On these assumptions, the inclusion relation*

$$\sigma(T_\theta) \subset \sigma(T_0) \cup \sigma(T_1)$$

*holds for each  $\theta \in (0, 1)$ .*

*Remark 1.2.* Here, we state the definitions mentioned above.

- (i) ([15, p. 23]) For a compact subset  $K$  of  $\mathbb{C}$ , the polynomial convex hull  $\hat{K}$  is defined by

$$\hat{K} := \left\{ z \in \mathbb{C} \mid |f(z)| \leq \sup_{\zeta \in K} |f(\zeta)| \text{ for all polynomials } f \right\}.$$

- (ii) ([4, p. 367]) A measure space  $\Omega$  with positive measure  $\mu$  is said to be a special discrete measure space if  $\Omega = \{1, 2, 3, \dots\}$  and  $\mu$  satisfies  $\mu(\{k\}) < \infty$  for each  $k \in \Omega$  and  $\inf\{\mu(\{k\}) \mid k \in \Omega\} > 0$ .

In this paper, we generalize the Lebesgue spaces in Barnes' results stated just above Remark 1.2 to general complex interpolation spaces. By this generalization, Barnes' results (Proposition 1.1 (ii)) for the case of a finite and a special discrete measure space are derived from the main result of this paper. However, we do not assume that  $X_0 \cap X_1$  is dense in  $X_1$ , considering the case where  $X_0 = L^p$  ( $p \in [1, \infty)$ ),  $X_1 = L^\infty$  and bounded operators on  $X_0$  and  $X_1$  are given in advance. Hence,  $X_1$  is not necessarily  $L^\infty$  mentioned in Proposition 1.1. On the other hand, we assume that  $T_0$  is regarded as a bounded operator from  $X_0$  into  $X_1$ . This assumption seems to be reasonable in the view that most of the bounded operators treated in the articles above are  $L^p$ - $L^q$  bounded for appropriate exponents  $p$  and  $q$ . A precise statement is given by the following

**Theorem 1.3.** *Let  $(X_0, X_1)$  be a compatible couple (for its definition and the norm on  $X_0 \cap X_1$  or  $X_0 + X_1$ , see Definition A.1),  $T_j: X_j \rightarrow X_j$  a bounded operator ( $j = 0, 1$ ) and  $T_0$  consistent with  $T_1$ , i.e.,  $T_0x = T_1x$  is satisfied for all  $x \in X_0 \cap X_1$ . On this assumption, there exists, for each  $\theta \in (0, 1)$ , a unique bounded operator  $T_\theta$  from the complex interpolation space  $(X_0, X_1)_{[\theta]}$  (for definition, see Definition A.2) into itself, which is consistent with both  $T_0$  and  $T_1$ , i.e.,  $T_\theta x = T_0x = T_1x$  is satisfied for all  $x \in X_0 \cap X_1$ . In addition, assume that the following conditions (i) and (ii) hold.*

- (i)  $X_0 \cap X_1$  is dense in  $X_0$ .
- (ii) There exists a constant  $C > 0$  such that

$$\|T_0x\|_1 \leq C\|x\|_0$$

for all  $x \in X_0 \cap X_1$ .

On these assumptions, the next relations of spectra

$$(1.3) \quad \sigma(T_\theta) \subset \sigma(T_0) \cup \sigma(T_1) = \sigma(T_\Delta) \cup \sigma(T_\Sigma)$$

hold for each  $\theta \in (0, 1)$ , where  $T_\Delta \in \mathcal{L}(X_0 \cap X_1)$  is defined by Eq. (1.2) and  $T_\Sigma \in \mathcal{L}(X_0 + X_1)$  denotes the operator whose image of  $x \in X_0 + X_1$  is defined by

$$T_\Sigma x := T_0u + T_1v \quad (x = u + v, u \in X_0, v \in X_1).$$

Note that this image is independent of the choice of  $u$  and  $v$ . The boundedness of  $T_\Sigma$  follows from that of  $T_0$  and  $T_1$ .

**Notation.** Let  $X$  and  $Y$  be Banach spaces. The space  $X'$  denotes the dual space of  $X$ . The set  $\mathcal{L}(X, Y)$  consists of all bounded operators from  $X$  into  $Y$ . We abbreviate  $\mathcal{L}(X, X)$  to  $\mathcal{L}(X)$ . For a  $T \in \mathcal{L}(X, Y)$ , the operator  $T'$  denotes the dual operator of  $T$ . Needless to say,  $T' \in \mathcal{L}(Y', X')$ . For a  $T \in \mathcal{L}(X)$ , the set  $\sigma(T)$  and  $\rho(T)$  denote the spectrum of  $T$  and the resolvent set of  $T$ , respectively. The relation “ $Y \hookrightarrow X$ ” means that  $Y$  is continuously embedded into  $X$ . In the case where  $Y \hookrightarrow X$ , the inclusion mapping from  $Y$  into  $X$  is written as  $\iota(Y, X)$ .

## §2. Proof of the main result

To prove Theorem 1.3, we generalize Barnes’ result (Proposition 1.1 (i)) to the case of complex interpolation spaces in the following

**Proposition 2.1** (cf. [4, THEOREM 5.3]). *Let  $(X_0, X_1)$  be a compatible couple,  $T_j$  a bounded operator from  $X_j$  into itself ( $j = 0, 1$ ) and  $T_0$  consistent with  $T_1$ , i.e.,  $T_0x = T_1x$  is satisfied for all  $x \in X_0 \cap X_1$ . On this assumption, the bounded operator  $T_\theta$  stated in Theorem 1.3 satisfies the inclusion relations of spectra*

$$(2.1) \quad \sigma(T_\theta) \subset \sigma(T_0) \cup \sigma(T_1) \cup \sigma(T_\Delta)$$

and

$$(2.2) \quad \sigma(T_\theta) \subset [\sigma(T_0) \cup \sigma(T_1)]^\wedge$$

for each  $\theta \in (0, 1)$ , where  $T_\Delta \in \mathcal{L}(X_0 \cap X_1)$  is defined by Eq. (1.2) and the right-hand side of (2.2) denotes the polynomial convex hull of  $\sigma(T_0) \cup \sigma(T_1)$  (for definition, see Remark 1.2 above).

*Proof.* To prove Eq. (2.1), i.e.,  $\rho(T_0) \cap \rho(T_1) \cap \rho(T_\Delta) \subset \rho(T_\theta)$ , suppose that  $\lambda \in \mathbb{C}$  belongs to  $\rho(T_0) \cap \rho(T_\Delta)$ . As is stated in [9, LEMMA 2.3], the resolvent  $(\lambda - T_0)^{-1}$  is consistent with  $(\lambda - T_\Delta)^{-1}$ . Indeed, since  $T_0$  is consistent with  $T_\Delta$ , the equality

$$(\lambda - T_0)(\lambda - T_\Delta)^{-1}x = x$$

holds for all  $x \in X_0 \cap X_1$ . Hence,  $(\lambda - T_0)^{-1}x = (\lambda - T_\Delta)^{-1}x$  for all  $x \in X_0 \cap X_1$ . In the same way, for each  $\lambda \in \rho(T_1) \cap \rho(T_\Delta)$ , the resolvent  $(\lambda - T_1)^{-1}$  is proved to be consistent with  $(\lambda - T_\Delta)^{-1}$ . Therefore, for each  $\lambda \in \rho(T_0) \cap \rho(T_1) \cap \rho(T_\Delta)$ , both  $(\lambda - T_0)^{-1}$  and  $(\lambda - T_1)^{-1}$  are consistent with  $(\lambda - T_\Delta)^{-1}$ . Hence,  $(\lambda - T_0)^{-1}$  is consistent with  $(\lambda - T_1)^{-1}$  for each  $\lambda \in \rho(T_0) \cap \rho(T_1) \cap \rho(T_\Delta)$ .

Now, suppose  $\lambda \in \rho(T_0) \cap \rho(T_1) \cap \rho(T_\Delta)$  and  $\theta \in (0, 1)$ . To prove  $\lambda \in \rho(T_\theta)$ , we recall a fundamental theorem [5, 4.1.2. Theorem] in the complex interpolation theory and apply this theorem to  $(\lambda - T_0)^{-1}$  and  $(\lambda - T_1)^{-1}$ , that is, there exists a bounded operator  $R_\theta$  from the complex interpolation space  $(X_0, X_1)_{[\theta]}$  into itself which is consistent with both  $(\lambda - T_0)^{-1}$  and  $(\lambda - T_1)^{-1}$ . As is stated in [9, LEMMA 2.5], this  $\lambda$  belongs to  $\rho(T_\theta)$  and  $R_\theta = (\lambda - T_\theta)^{-1}$ . Indeed, since  $T_\theta$  (resp.  $R_\theta$ ) is consistent with  $T_\Delta$  (resp.  $(\lambda - T_\Delta)^{-1}$ ), the equalities

$$(\lambda - T_\theta)R_\theta x = R_\theta(\lambda - T_\theta)x = x$$

hold for all  $x \in X_0 \cap X_1$ . These equalities are valid for all  $x \in (X_0, X_1)_{[\theta]}$  because both  $(\lambda - T_\theta)R_\theta$  and  $R_\theta(\lambda - T_\theta)$  are continuous on  $(X_0, X_1)_{[\theta]}$  and  $X_0 \cap X_1$  is dense in  $(X_0, X_1)_{[\theta]}$  by [5, 4.2.2. Theorem]. Thus,  $\lambda$  belongs to  $\rho(T_\theta)$ , and we conclude  $\rho(T_0) \cap \rho(T_1) \cap \rho(T_\Delta) \subset \rho(T_\theta)$ , i.e.,  $\sigma(T_\theta) \subset \sigma(T_0) \cup \sigma(T_1) \cup \sigma(T_\Delta)$ .

To prove Eq. (2.2), we first show that the inclusion relation

$$A_\sigma(T_\Delta) \subset A_\sigma(T_0) \cup A_\sigma(T_1)$$

holds, where  $A_\sigma(T_\Delta)$ ,  $A_\sigma(T_0)$  and  $A_\sigma(T_1)$  denote the set of all approximate point spectra of  $T_\Delta$ ,  $T_0$  and  $T_1$ , respectively. This inclusion relation is proved in the same way as the proof of [4, THEOREM 5.1. (3)]. Indeed, suppose

that  $\lambda$  is an approximate point spectrum of  $T_\Delta$ , and  $\{x_n\}$  is a sequence of unit vectors in  $X_0 \cap X_1$  such that  $\|(\lambda - T_\Delta)x_n\|_\Delta \rightarrow 0$  as  $n \rightarrow \infty$ , where  $\|\cdot\|_\Delta$  denotes a norm of  $X_0 \cap X_1$  (for definition, see Definition A.1). From the definition of the norm  $\|\cdot\|_\Delta$ , the sequence  $\{x_n\}$  does not converge to zero in at least either  $X_0$  or  $X_1$  and the sequence  $\{(\lambda - T_j)x_n\}$  converges to zero in  $X_j$  for  $j = 0$  and  $1$ . Hence,  $\lambda$  is an approximate point spectrum of either  $T_0$  or  $T_1$ , i.e.,  $A_\sigma(T_\Delta) \subset A_\sigma(T_0) \cup A_\sigma(T_1)$ . Since the boundary  $\partial\sigma(T_\Delta)$  of  $\sigma(T_\Delta)$  is contained in  $A_\sigma(T_\Delta)$ , the inclusion relation

$$\partial\sigma(T_\Delta) \subset \sigma(T_0) \cup \sigma(T_1)$$

holds.

From this inclusion relation and the definition of the polynomial convex hull of a compact subset of  $\mathbb{C}$ , it is easy to prove the following inclusion relations and equality

$$\sigma(T_\Delta) \subset [\sigma(T_\Delta)]^\wedge = [\partial\sigma(T_\Delta)]^\wedge \subset [\sigma(T_0) \cup \sigma(T_1)]^\wedge$$

and

$$\sigma(T_0) \cup \sigma(T_1) \subset [\sigma(T_0) \cup \sigma(T_1)]^\wedge.$$

Thus,  $\sigma(T_0) \cup \sigma(T_1) \cup \sigma(T_\Delta) \subset [\sigma(T_0) \cup \sigma(T_1)]^\wedge$ . Combining this inclusion relation with Eq. (2.1) already proved, we have the inclusion relation  $\sigma(T_\theta) \subset [\sigma(T_0) \cup \sigma(T_1)]^\wedge$  for each  $\theta \in (0, 1)$ .  $\square$

The next lemma states, as a special case, relations between the spectrum of a bounded linear operator  $S$  on a Banach space  $Y$  and that of an extension of  $S$  on a Banach space  $X$  of which  $Y$  is a subspace. To replace  $X$  and  $Y$  with various Banach spaces is a key to the proof of Theorem 1.3. Although this lemma is surely well known, the author could not find an appropriate literature which includes all the assertions of this lemma. However, since it is easy to prove this lemma, we give some references instead of a complete proof.

**Lemma 2.2.** *Let  $X$  and  $Y$  be Banach spaces and  $T$  a bounded operator from  $X$  into  $Y$ . Assume that  $Y$  is continuously embedded into  $X$ . On this assumption, the spectrum of  $T_X$  and that of  $T_Y$ , where  $T_X := \iota(Y, X)T \in \mathcal{L}(X)$  and  $T_Y := T\iota(Y, X) \in \mathcal{L}(Y)$ , have the next relations*

$$\sigma(T_X) \setminus \{0\} = \sigma(T_Y) \setminus \{0\}$$

and

$$\sigma(T_Y) \subset \sigma(T_X).$$

In addition, if  $Y$  is densely embedded into  $X$ , then these spectra are identical with each other, i.e.,

$$\sigma(T_X) = \sigma(T_Y).$$

*Proof.* The assertions of this lemma can be considered as a special case of spectral relationships of the operator products  $TS$  and  $ST$ , where  $T \in \mathcal{L}(X, Y)$  and  $S \in \mathcal{L}(Y, X)$  are bounded operators between Banach spaces  $X$  and  $Y$ . The equality  $\sigma(TS) \setminus \{0\} = \sigma(ST) \setminus \{0\}$  in the case of  $X = Y$  is proved in [16, Proposition 2.1]. This equality in the general case and the last equality of this lemma in the case where  $Y$  is a proper dense subspace of  $X$  are proved in [3, Lemma 5 (1)] and [3, Theorem 4 (2)], respectively. All the assertions of this lemma are obtained by replacing  $T$  and  $S$  in [13, Proposition 2] with  $T$  and  $\iota(Y, X)$  in this lemma, respectively.  $\square$

*Proof of Theorem 1.3.* On the assumptions (i) and (ii), the operator  $T_\Delta$  has a unique continuous extension  $T \in \mathcal{L}(X_0, X_0 \cap X_1)$ . By this definition,  $T$  is consistent with  $T_\Delta$  and  $T_0$ , and  $T_\Delta$  and  $T_0$  are written as  $T_\Delta = T\iota(X_0 \cap X_1, X_0)$  and  $T_0 = \iota(X_0 \cap X_1, X_0)T$ , respectively. On the assumption (i), Lemma 2.2 implies that the spectra of these operators are identical with each other:

$$(2.3) \quad \sigma(T_\Delta) = \sigma(T_0).$$

Hence, combining this equality with the inclusion relation (2.1) which has been proved, we have the next relation

$$\sigma(T_\theta) \subset \sigma(T_0) \cup \sigma(T_1)$$

for each  $\theta \in (0, 1)$ .

In addition, we define  $\tilde{T}: X_0 + X_1 \rightarrow X_1$  by

$$\tilde{T}x := T_\Sigma x \quad (x \in X_0 + X_1).$$

Note that the range of  $T_\Sigma$  is contained in  $X_1$  on the assumption (ii). This operator  $\tilde{T}$  is bounded, i.e.,  $\tilde{T} \in \mathcal{L}(X_0 + X_1, X_1)$ , because of the assumption (ii) and the boundedness of  $T_1$ . By using this operator,  $T_1$  and  $T_\Sigma$  are written as  $T_1 = \tilde{T}\iota(X_1, X_0 + X_1)$  and  $T_\Sigma = \iota(X_1, X_0 + X_1)\tilde{T}$ , respectively. By Lemma 2.2, the set  $\sigma(T_1) \setminus \{0\}$  is equal to  $\sigma(T_\Sigma) \setminus \{0\}$ . Hence, by using this equality and Eq. (2.3), we have

$$(\sigma(T_0) \cup \sigma(T_1)) \setminus \{0\} = (\sigma(T_\Delta) \cup \sigma(T_\Sigma)) \setminus \{0\}.$$



In the case where  $0 \in \sigma(T_0)$ , it means that  $0 \in \sigma(T_\Delta)$ , hence the equality  $\sigma(T_0) \cup \sigma(T_1) = \sigma(T_\Delta) \cup \sigma(T_\Sigma)$  holds. In the case where  $0 \notin \sigma(T_0)$ , the Banach space  $X_0 \cap X_1$  (resp.  $X_0 + X_1$ ) is isomorphic to  $X_0$  (resp.  $X_1$ ). Indeed, since  $T_0 = \iota(X_0 \cap X_1, X_0)T$  and  $T_0$  is bijective, so is  $\iota(X_0 \cap X_1, X_0)$ . By the open mapping theorem, the inverse of  $\iota(X_0 \cap X_1, X_0)$  is bounded. Thus,  $X_0 \cap X_1$  is isomorphic to  $X_0$  and hence  $X_0 \hookrightarrow X_1$ .

To prove that  $X_0 + X_1$  is isomorphic to  $X_1$ , suppose that  $x \in X_0 + X_1$  and  $x$  is written as  $x = u + v$  ( $u \in X_0, v \in X_1$ ). Since  $\|u\|_1 \leq \|\iota(X_0, X_1)\| \|u\|_0$ , the estimates

$$\|x\|_1 \leq \|u\|_1 + \|v\|_1 \leq \max\{\|\iota(X_0, X_1)\|, 1\}(\|u\|_0 + \|v\|_1)$$

hold. Taking the infimum with respect to such  $u$  and  $v$ , we have

$$\|x\|_1 \leq \max\{\|\iota(X_0, X_1)\|, 1\} \|x\|_\Sigma.$$

By this estimate and the trivial estimate  $\|x\|_\Sigma \leq \|x\|_1$ , the sum  $X_0 + X_1$  is isomorphic to  $X_1$ . Therefore,  $\sigma(T_0)$  (resp.  $\sigma(T_1)$ ) is equal to  $\sigma(T_\Delta)$  (resp.  $\sigma(T_\Sigma)$ ) and hence  $\sigma(T_0) \cup \sigma(T_1) = \sigma(T_\Delta) \cup \sigma(T_\Sigma)$ .  $\square$

### §3. Applications

We apply Theorem 1.3 to bounded operators on Lebesgue spaces to obtain a result of the spectra of integral operators. In what follows,  $L^p$  is the Lebesgue space for an exponent  $p \in [1, \infty]$  and a  $\sigma$ -finite measure space  $\Omega$  with the measure  $\mu$ , and  $\|\cdot\|_p$  denotes the norm of  $L^p$ . We identify the dual space  $(L^p)'$  with  $L^{p'}$  provided  $1 \leq p < \infty$ , where  $p'$  is the conjugate exponent of  $p$ . For a complex-valued function  $f$ , the function  $\bar{f}$  has the values which are the complex conjugate of those of  $f$ .

**Corollary 3.1.** *Let  $p$  and  $q$  be numbers satisfying  $1 \leq p < q \leq \infty$ , and  $T_p$  (resp.  $T_q$ ) a bounded operator from  $L^p$  (resp.  $L^q$ ) into itself which is consistent with  $T_q$  (resp.  $T_p$ ). Then, for each  $r \in (p, q)$ , there exists a unique bounded operator  $T_r \in \mathcal{L}(L^r)$  consistent with both  $T_p$  and  $T_q$ . In addition, assume that  $T_p$  is  $L^p$ - $L^q$  bounded, i.e., there exists a constant  $C > 0$  such that*

$$\|T_p f\|_q \leq C \|f\|_p$$

for all  $f \in L^p \cap L^q$ . Then, the following assertions hold:

- (i) For each  $r \in (p, q)$ , the operator  $T_r$  satisfies the following relations of spectra

$$\sigma(T_r) \subset \sigma(T_p) \cup \sigma(T_q) = \sigma(T_\Delta) \cup \sigma(T_\Sigma),$$

where  $T_\Delta$  and  $T_\Sigma$  are defined in the same way in Theorem 1.3 by replacing  $T_0, T_1, X_0$  and  $X_1$  with  $T_p, T_q, L^p$  and  $L^q$ , respectively.

- (ii) In the particular case where  $1 \leq p \leq 2$  and  $q = p'$ , on the additional assumption that  $(T_p)^* = T_{p'}$ , where  $(T_p)^*$  is defined by

$$(3.1) \quad (T_p)^* f := \overline{(T_p)' f} \quad (f \in L^{p'})$$

and hence  $(T_p)^*$  is isomorphic to  $(T_p)'$ , the relations of spectra

$$\sigma(T_2) \subset \sigma(T_p) = \sigma(T_{p'}) = \sigma(T_\Delta) = \sigma(T_\Sigma)$$

hold.

*Proof.* Suppose  $1 \leq p < q \leq \infty$  and let  $\theta$  be the number satisfying  $1/r = (1 - \theta)/p + \theta/q$ . Then,  $L^r$  is the complex interpolation space  $(L^p, L^q)_{[\theta]}$  by [5, 5.1.1 Theorem]. In addition, assume that  $T_p$  is  $L^p$ - $L^q$  bounded. Then, the conditions (i) and (ii) in Theorem 1.3 are satisfied. Now, we regard  $L^p, L^q, T_p$  and  $T_q$  as  $X_0, X_1, T_0$  and  $T_1$  in Theorem 1.3, respectively, and apply Eq. (1.3) in Theorem 1.3 to them. Since  $T_r$  is nothing but  $T_\theta$  in Theorem 1.3, we have

$$\sigma(T_r) \subset \sigma(T_p) \cup \sigma(T_q) = \sigma(T_\Delta) \cup \sigma(T_\Sigma).$$

In the particular case where  $1 \leq p \leq 2, q = p'$  and  $(T_p)^* = T_{p'}$ , combining this inclusion relation with the equalities  $\sigma(T_p) = \sigma((T_p)^*) = \sigma(T_{p'})$ , we have  $\sigma(T_2) \subset \sigma(T_p) = \sigma(T_{p'})$ . Since  $L^p \cap L^{p'}$  is dense in  $L^p$ , Lemma 2.2 implies that  $\sigma(T_\Delta)$  is equal to  $\sigma(T_p)$ . Since  $L^{p'}$  is continuously embedded into  $L^p + L^{p'}$ , Lemma 2.2 implies that  $\sigma(T_{p'}) \setminus \{0\} = \sigma(T_\Sigma) \setminus \{0\}$  and  $\sigma(T_{p'}) \subset \sigma(T_\Sigma)$ . In the case where  $0 \in \sigma(T_p)$ , it means that  $0 \in \sigma(T_{p'})$ , hence  $\sigma(T_{p'}) = \sigma(T_\Sigma)$ . In the case where  $0 \notin \sigma(T_p)$ , for the same reason stated in the last paragraph of the proof of Theorem 1.3, we conclude that  $\sigma(T_{p'}) = \sigma(T_\Sigma)$ . Thus,  $\sigma(T_p) = \sigma(T_{p'}) = \sigma(T_\Delta) = \sigma(T_\Sigma)$ .  $\square$

**Corollary 3.2.** *Let  $K: \Omega \times \Omega \rightarrow \mathbb{C}$  be a measurable function and assume that  $K$  satisfies the next three estimates:*

$$\begin{aligned} \operatorname{ess.\,sup}_{x \in \Omega} \int_{\Omega} |K(x, y)| \, d\mu(y) &< \infty, \\ \operatorname{ess.\,sup}_{y \in \Omega} \int_{\Omega} |K(x, y)| \, d\mu(x) &< \infty, \\ \operatorname{ess.\,sup}_{(x, y) \in \Omega \times \Omega} |K(x, y)| &< \infty. \end{aligned}$$

*On these assumptions, for each  $p \in [1, \infty]$ , the integral operator  $T_p$  with kernel  $K$  is defined by*

$$(T_p f)(x) := \int_{\Omega} K(x, y) f(y) \, d\mu(y) \quad (f \in L^p, \text{ a.e. } x \in \Omega).$$

*Then,  $T_p$  is a bounded operator from  $L^p$  into itself, and  $T_p$  is consistent with  $T_q$  for all  $p, q \in [1, \infty]$ . These operators have the relations of spectra*

$$\sigma(T_r) \subset \sigma(T_p) \cup \sigma(T_q) = \sigma(T_{\Delta}) \cup \sigma(T_{\Sigma})$$

*provided  $1 \leq p < r < q \leq \infty$ , where  $T_{\Delta}$  and  $T_{\Sigma}$  are the same operators appearing in Corollary 3.1. In addition, if  $K$  is a Hermitian kernel, i.e.,  $K(x, y) = \overline{K(y, x)}$  for a.e.  $(x, y) \in \Omega \times \Omega$ , the relations of spectra*

$$\sigma(T_2) \subset \sigma(T_p) = \sigma(T_{p'}) = \sigma(T_{\Delta}) = \sigma(T_{\Sigma})$$

*hold for each  $p \in [1, 2]$ .*

*Proof.* Suppose  $1 \leq p < r < q \leq \infty$ . By the estimates for  $K$ , the operator  $T_p$  is  $L^p$ - $L^q$  bounded. Since  $T_r$  is unique as a bounded operator which is consistent with both  $T_p$  and  $T_q$ , the spectrum  $\sigma(T_r)$  has the same relation stated in Corollary 3.1 (i):

$$\sigma(T_r) \subset \sigma(T_p) \cup \sigma(T_q) = \sigma(T_{\Delta}) \cup \sigma(T_{\Sigma}).$$

In addition, if  $K$  is a Hermitian kernel, the operator  $T_{p'}$  coincides with  $(T_p)^*$  defined by Eq. (3.1). By Corollary 3.1 (ii), the relations of spectra

$$\sigma(T_2) \subset \sigma(T_p) = \sigma(T_{p'}) = \sigma(T_{\Delta}) = \sigma(T_{\Sigma})$$

hold for each  $p \in [1, 2]$ . □

**Example 3.3.** Let  $N$  be a natural number,  $\alpha$  a real number satisfying  $1 < \alpha < 2$ ,  $(-\Delta)^{\alpha/2}$  the fractional Laplacian of order  $\alpha/2$ , where  $\Delta$  is the usual Laplacian in  $L^2(\mathbb{R}^N)$  with domain  $H^2(\mathbb{R}^N)$ , and  $b: \mathbb{R}^N \rightarrow \mathbb{R}^N$  a measurable function belonging to the Kato class  $\mathcal{K}_N^{\alpha-1}$ , i.e.,  $b$  satisfies

$$\lim_{\varepsilon \downarrow 0} \operatorname{ess.\,sup}_{x \in \mathbb{R}^N} \int_{|y-x| < \varepsilon} |b(y)| |y-x|^{\alpha-1-N} dy = 0.$$

K. Bogdan and T. Jakubowski [6, Theorem 1] proved that there exists a continuous transition density  $K: (0, \infty) \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$  such that

$$(3.2) \quad \lim_{t \downarrow 0} \int_{\mathbb{R}^N} \frac{(T(t)f)(x) - f(x)}{t} g(x) dx \\ = \int_{\mathbb{R}^N} \{ -((-\Delta)^{\alpha/2} f)(x) + b(x) \cdot (\nabla f)(x) \} g(x) dx,$$

where  $f, g \in C_c^\infty(\mathbb{R}^N)$  and

$$(T(t)f)(x) := \int_{\mathbb{R}^N} K(t, x, y) f(y) dy \quad (t > 0, \text{ a.e. } x \in \mathbb{R}^N).$$

Eq. (3.2) and the estimate stated in [6, Lemma 3] for  $K$  mean that, for each  $p \in [1, \infty]$ , the transition density  $K$  defines a semigroup on  $L^p(\mathbb{R}^N)$  with generator  $-(-\Delta)^{\alpha/2} + b \cdot \nabla$  in a weak sense. By [6, Lemma 3], it is proved that, for each  $t > 0$  and  $p \in [1, \infty]$ , an integral operator  $T_p(t)$  with the kernel  $K(t, \cdot, \cdot)$  exists as a bounded operator from  $L^p(\mathbb{R}^N)$  into itself, and each  $T_p(t)$  is  $L^p$ - $L^q$  bounded provided  $1 \leq p < q \leq \infty$ . Needless to say, for each  $t > 0$  and  $p, q \in [1, \infty]$ , the operator  $T_p(t)$  is consistent with  $T_q(t)$ . Hence, by Corollary 3.1 (i), the relations of spectra

$$\sigma(T_r(t)) \subset \sigma(T_p(t)) \cup \sigma(T_q(t)) = \sigma(T_\Delta(t)) \cup \sigma(T_\Sigma(t))$$

hold for each  $t > 0$  and  $p, q, r$  satisfying  $1 \leq p < r < q \leq \infty$ , where  $T_\Delta(t) \in \mathcal{L}(L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N))$  and  $T_\Sigma(t) \in \mathcal{L}(L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N))$  are defined by

$$T_\Delta(t)f := T_p(t)f \quad (f \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N))$$

and

$$T_\Sigma(t)f := T_p(t)u + T_q(t)v \quad (f = u + v, u \in L^p(\mathbb{R}^N), v \in L^q(\mathbb{R}^N)).$$

In the particular case where  $1 \leq p \leq 2$ ,  $q = p'$  and  $b$  is a constant function, the operator  $(T_p(t))^*$  coincides with  $T_{p'}(t)$  for each  $t > 0$ , and hence Corollary 3.1 (ii) implies that the relations of spectra

$$\sigma(T_2(t)) \subset \sigma(T_p(t)) = \sigma(T_{p'}(t)) = \sigma(T_\Delta(t)) = \sigma(T_\Sigma(t))$$

hold for each  $t > 0$  and  $p \in [1, 2]$ .

*Remark 3.4.* The author proved that in some special cases,  $(T_p(t))_{t \geq 0}$  is a  $C_0$ -semigroup on  $L^p(\mathbb{R}^N)$  generated by the operator sum  $A_p$  of the fractional Laplacian  $-(-\Delta)^{\alpha/2}$  and the advection operator  $b \cdot \nabla$ , and the spectrum of  $A_p$  is independent of  $p \in [1, \infty)$ . For this result, another paper is in preparation.

Finally, we verify that Barnes' result (Proposition 1.1 (ii)) is derived from Theorem 1.3.

*Proof of Proposition 1.1 (ii).* Suppose  $1 \leq p_0 < p_1 \leq \infty$ . In the case where  $\Omega$  is a finite measure space,  $L^{p_1}$  is continuously embedded into  $L^{p_0}$ . Hence,  $L^{p_0} \cap L^{p_1}$  is isomorphic to  $L^{p_1}$ , and  $T_1$  is  $L^{p_1}$ - $L^{p_0}$  bounded. By regarding  $L^{p_1}$ ,  $L^{p_0}$ ,  $T_1$  and  $T_0$  as  $X_0$ ,  $X_1$ ,  $T_0$  and  $T_1$  in Theorem 1.3, respectively, we can apply Theorem 1.3 to them. Since  $T_\theta$  induced from  $L^{p_1}$ ,  $L^{p_0}$ ,  $T_1$  and  $T_0$  is equal to  $T_{1-\theta}$  in the assertion of Theorem 1.3 by the equality  $(L^{p_1}, L^{p_0})_{[\theta]} = (L^{p_0}, L^{p_1})_{[1-\theta]}$  proved in [5, 4.2.1. Theorem], we have  $\sigma(T_{1-\theta}) \subset \sigma(T_1) \cup \sigma(T_0)$  for each  $\theta \in (0, 1)$ . Thus,  $\sigma(T_\theta) \subset \sigma(T_0) \cup \sigma(T_1)$  for each  $\theta \in (0, 1)$ .

Next, suppose that  $\Omega$  is a special discrete measure space. In the case where numbers  $p_0$  and  $p_1$  satisfy  $1 \leq p_0 < p_1 < \infty$ , the Lebesgue space  $L^{p_0}$  is continuously embedded into  $L^{p_1}$ . As was already stated,  $(L^{p_0}, L^{p_1})_{[\theta]} = L^{p_\theta}$  for each  $\theta \in (0, 1)$ , where  $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$ . In the case where numbers  $p_0$  and  $p_1$  satisfy  $1 \leq p_0 < p_1 = \infty$ , the Lebesgue space  $L^{p_0}$  is continuously embedded into  $L_0^\infty$ . By [5, 4.2.2. Theorem (b)], the complex interpolation space  $(L^{p_0}, L_0^\infty)_{[\theta]}$  is isomorphic to  $(L^{p_0}, L^\infty)_{[\theta]}$ , i.e.,  $L^{p_\theta}$  for each  $\theta \in (0, 1)$ , where  $1/p_\theta = (1-\theta)/p_0$ . Thus, in both cases, by a similar argument used above, we have the asserted inclusion relation of spectra.  $\square$

### §A. Definitions related to complex interpolation spaces

Here, we state definitions related to complex interpolation spaces.

**Definition A.1.** Let  $X_j$  be a Banach space with norm  $\|\cdot\|_j$  ( $j = 0, 1$ ) and assume that there exists a Hausdorff topological vector space  $Z$  such that both  $X_0$  and  $X_1$  are continuously embedded into  $Z$ . We say that such a couple of  $X_0$  and  $X_1$  is compatible. For a compatible couple of  $X_0$  and  $X_1$ , we define the norms on  $X_0 \cap X_1$  and  $X_0 + X_1$  by

$$\begin{aligned} \|x\|_\Delta &:= \|x\|_0 + \|x\|_1 \quad (x \in X_0 \cap X_1), \\ \|x\|_\Sigma &:= \inf\{\|u\|_0 + \|v\|_1 \mid u \in X_0, v \in X_1, u + v = x\} \quad (x \in X_0 + X_1), \end{aligned}$$

respectively. Both  $(X_0 \cap X_1, \|\cdot\|_\Delta)$  and  $(X_0 + X_1, \|\cdot\|_\Sigma)$  are Banach spaces.

The complex interpolation spaces in this paper are defined as follows.

**Definition A.2** (cf. [5, 4.1.2. Theorem]). Let  $(X_0, X_1)$  be a compatible couple. The vector space  $\mathcal{F}$  consists of all functions  $f$  from  $S := \{z \in \mathbb{C} \mid 0 \leq \operatorname{Re} z \leq 1\}$  into  $X_0 + X_1$  which satisfy the following conditions (i) through (iv). In what follows, the number  $i$  denotes the imaginary unit:  $i = \sqrt{-1}$ .

- (i)  $f$  is bounded and continuous on  $S$ .
- (ii)  $f$  is analytic on the interior  $S^\circ = \{z \in \mathbb{C} \mid 0 < \operatorname{Re} z < 1\}$ .
- (iii) The functions  $t \mapsto f(j+it)$  from  $\mathbb{R}$  into  $X_j$  are continuous on  $\mathbb{R}$  ( $j = 0, 1$ ).
- (iv)  $\lim_{|t| \rightarrow \infty} f(j+it) = 0$  in  $X_j$  ( $j = 0, 1$ ).

We provide  $\mathcal{F}$  with the norm

$$\|f\|_{\mathcal{F}} := \max \left\{ \sup_{t \in \mathbb{R}} \|f(it)\|_0, \sup_{t \in \mathbb{R}} \|f(1+it)\|_1 \right\},$$

where  $\|\cdot\|_j$  is the norm of  $X_j$  ( $j = 0, 1$ ). This normed space is a Banach space. For each  $\theta \in (0, 1)$ , we define the following normed space  $(X_0, X_1)_{[\theta]}$ :

$$\begin{aligned} (X_0, X_1)_{[\theta]} &:= \{f(\theta) \in X_0 + X_1 \mid f \in \mathcal{F}\}, \\ \|x\|_{[\theta]} &:= \inf\{\|f\|_{\mathcal{F}} \mid x = f(\theta), f \in \mathcal{F}\} \quad (x \in (X_0, X_1)_{[\theta]}). \end{aligned}$$

This normed space is a Banach space. We say that this Banach space is a complex interpolation space of index  $\theta$  generated from the couple  $(X_0, X_1)$ .

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