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Theta functions of binary quadratic forms with congruence conditions as ray class theta functions

Ryota Okano

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Abstract. We prove that theta functions of binary quadratic forms with congruence conditions coincide with theta functions attached to ray classes of quadratic fields. By Hecke's Theorem, we can construct Hecke eigenforms by forming linear combinations of them.

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§1. Introduction

In 1926, Hecke [3] constructed modular forms of weight one associated to both real and imaginary quadratic fields. These forms are theta functions of binary quadratic forms with congruence conditions. Kida-Namura [8] showed that these theta functions appear in inverse Mellin transforms of Hecke L-functions of certain ray class groups of quadratic fields. Conversely, in this paper, we prove that any theta functions with congruence conditions can be obtained as the inverse Mellin transforms of partial zeta functions of ray classes of certain quadratic fields. Using the action of Hecke operators, we express the weight one Hecke eigenforms in [10, Thorem 4.8.2 and Theorem 4.8.3] as a linear combinations of these theta functions.

Hecke [4] computed the action of Hecke operators on the weight one theta functions attached to ideal classes of orders of imaginary quadratic fields, and Kani [5] constructed Hecke eigenforms by forming linear combinations of them. We extend their results to the indefinite case and also ideal classes to ray classes.

We now define theta functions of binary quadratic forms with congruence conditions. While these theta functions are defined by Hecke [3], we follow

Kudla's definition in [9]. Let $A = k \begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix}$ be a symmetric matrix of size 2 over $\mathbb{Z}$ with $a, k > 0$ and $\gcd(a,b,c) = 1$ such that $b^2 - 4ac$ is a fundamental discriminant. We take a positive integer $N = k\,|b^2 - 4ac|$ and define

$$\mathscr{H}_A = \left\{ h \in \mathbb{Z}^2/N\mathbb{Z}^2 \mid Ah \equiv 0 \pmod{N} \right\}.$$

Here we consider $h \in \mathscr{H}_A$ as a column vector. For a column vector $m \in \mathbb{Z}^2$, we denote ${}^t mAm$ by $A[m]$. Let G be the subgroup of $\mathrm{SL}_2(\mathbb{R})$ in the identity component of the automorphism group of the quadratic form $Q(x,y) = \frac{1}{2} A\,[\boldsymbol{x}]$ consisting of elements preserving $h + N\mathbb{Z}^2$ with $\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$. If $-\det A < 0$, then G is a finite cyclic group, and if $-\det A > 0$, then G is an infinite cyclic group ([1, §8.3]).

Definition 1.1. Assumptions and notations being as above. Let $h \in \mathscr{H}_A$. If $-\det A < 0$, then we define

$$\theta_-(\tau; h, A, N) = \sum_{m \in h + N\mathbb{Z}^2/G} e\left(\frac{A[m]}{2N^2}\tau\right),$$

and if $-\det A > 0$, then we define

$$\theta_+(\tau; h, A, N) = \sum_{\substack{m \in h + N\mathbb{Z}^2/G \\ A[m] > 0}} \epsilon(m)\; e\left(\frac{A[m]}{2N^2}\tau\right),$$

where $\epsilon((x,y)) = \mathrm{sgn}\left(ax + \frac{b + \sqrt{b^2 - 4ac}}{2} y\right)$ is the sign function and $e(\tau) = \exp(2\pi i \tau)$ with τ in the complex upper half plane. Moreover, we set

$$\theta_\pm(\tau; h, A, N) = \begin{cases} \theta_-(\tau; h, A, N), & \text{if } -\det A < 0, \\ \theta_+(\tau; h, A, N), & \text{if } -\det A > 0. \end{cases}$$

Hecke [3] proved that these theta functions have the modular transformation law. We state it according to Shimura [12, Proposition2.1] and Kudla [9, Theorem].

Theorem 1.2. *For $\gamma = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \in \Gamma_0(N)$, we have*

$$\theta_\pm(\tau; h, A, N) \mid_1 \gamma = e\left(\frac{a'b'A[h]}{2N^2}\right) \psi_{-\det A}(d') \theta_\pm(\tau; a'h, A, N)$$

where $\psi_{-\det A}$ is the Kronecker character $\left(\frac{-\det A}{\cdot}\right)$.

We define

$$\mathscr{H}_A^1 = \left\{h \in \mathscr{H}_A \mid 2N^2 \text{ divides } A[h] \text{ and } \gcd(R_h) = 1\right\}$$

where $R_h = \left\{\frac{A[g]}{2N^2} \mid g \in h + N\mathbb{Z}^2\right\}$ is the set of representation numbers.

If $h \in \mathscr{H}_A^1$, then we see that $\theta_\pm(\tau; h, A, N)$ is a modular form on $\Gamma_1(N)$ by Theorem 1.2. In this paper, we deal with $\theta_\pm(\tau; h, A, N)$ for $h \in \mathscr{H}_A^1$.

This paper is organized as follows. In the following section, we define theta functions $\theta(\tau; \mathfrak{C})$ attached to the ray classes in quadratic fields (Definition 2.1) and we show a relation between $\theta_\pm(\tau; h, A, N)$ and $\theta(\tau; \mathfrak{C})$ (Theorem 2.2). In Section 3, we describe the action of Hecke operators on $\theta(\tau; \mathfrak{C})$ (Theorem 3.1) which implies the action of Hecke operators on $\theta_\pm(\tau; h, A, N)$. In Section 4, we express Hecke eigenforms as a linear combination of $\theta_\pm(\tau; h, A, N)$ using the result in Section 3 (Corollary 4.3).

§2. Theta functions attached to ray classes of quadratic fields

In this section, we define theta functions attached to the ray classes in quadratic fields, and we prove that they can be described in terms of $\theta_\pm(\tau; h, A, N)$ in Definition 1.1.

Let K be a quadratic field and $\mathfrak{f}$ a modulus of K. We denote by $Cl_K(\mathfrak{f})$ the ray class group modulo $\mathfrak{f}$ and, for an ideal $\mathfrak{a}$ prime to $\mathfrak{f}$, by $[\mathfrak{a}]$ the class of $\mathfrak{a}$ in $Cl_K(\mathfrak{f})$. If K is a real quadratic field, then we always assume that $\mathfrak{f}$ is divisible by the two infinite primes of K and we decompose $\mathfrak{f}$ into the finite part $\mathfrak{f}_0$ and the infinite part $\infty_1\infty_2$. Moreover, we take a totally positive integer ν of K with the property $\nu + 1 \in \mathfrak{f}_0$. If K is an imaginary quadratic field, then we understand $\mathfrak{f} = \mathfrak{f}_0$.

Definition 2.1. Let K be a quadratic field with the discriminant D, and $\mathfrak{f}$ a modulus of K. The theta function attached to the ray class $\mathfrak{C} \in Cl_K(\mathfrak{f})$ is the function on the upper half plane $\mathfrak{H}$ given by

$$\theta(\tau; \mathfrak{C}) = \sum_{\mathfrak{a} \in \mathfrak{C}} e(N_K(\mathfrak{a})\tau),$$

where the sum is over all integral ideals in the ray class $\mathfrak{C}$ and $N_K(\mathfrak{a})$ is the norm of $\mathfrak{a}$. Moreover, if $D < 0$, then we set

$$\theta_-(\tau; \mathfrak{C}) = \theta(\tau; \mathfrak{C}),$$

and if $D > 0$, then we set

$$\theta_+(\tau; \mathfrak{C}) = \theta(\tau; \mathfrak{C}) - \theta(\tau; \mathfrak{C}[(\nu)]).$$

Moreover, we set

$$\theta_\pm(\tau;\mathfrak{C}) = \begin{cases} \theta_-(\tau;\mathfrak{C}), & \text{if } D<0,\\ \theta_+(\tau;\mathfrak{C}), & \text{if } D>0. \end{cases}$$

The following theorem is the main theorem in this paper.

Theorem 2.2. *(i) For a theta function $\theta_\pm(\tau;\mathfrak{C})$ in Definition 2.1, there exist $A = k\begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix} \in \mathrm{M}_2(\mathbb{Z})$, $N = k|b^2-4ac|$, and $h \in \mathscr{H}_A^1$ such that*

$$\theta_\pm(\tau;h,A,N) = \theta_\pm(\tau;\mathfrak{C}).$$

(ii) For a theta function $\theta_\pm(\tau;h,A,N)$ in Definition 1.1 with $h\in\mathscr{H}_A^1$, there exist a quadratic field K, a modulus $\mathfrak{f}$ of K, and $\mathfrak{C}\in Cl_K(\mathfrak{f})$ such that

$$\theta_\pm(\tau;\mathfrak{C}) = \theta_\pm(\tau;h,A,N).$$

The following corollary follows immediately from Theorem 2.2.

Corollary 2.3. *We keep the notations in Theorem 2.2 (ii). We assume $-\det A > 0$. If there exists a totally positive unit u of K with $u+1 \in \mathfrak{f}_0$, then $\theta_+(\tau;h,A,N) = 0$.*

To prove Theorem 2.2 (i), we recast theta functions $\theta(\tau;\mathfrak{C})$ in different forms.

Proposition 2.4. *Let $K = \mathbb{Q}(\sqrt{D})$ be a quadratic field of discriminant D and $\mathfrak{f}$ a modulus of K. For a ray class $\mathfrak{C}\in Cl_K(\mathfrak{f})$, there exist a quadratic form $Q(x,y) = ax^2+bxy+cy^2$, and positive integers i,j,M,f satisfying $\gcd(i,j,M)=1$ and $\frac{MD}{f}\in\mathbb{Z}$ such that if $D<0$, then we have*

$$\theta_-(\tau;\mathfrak{C}) = \sum_{\left[\begin{smallmatrix} x\\ y\end{smallmatrix}\right] \in \left[\begin{smallmatrix} i\\ j\end{smallmatrix}\right] + M\mathbb{Z}^2/G'} e\left(\frac{1}{f}Q(x,y)\tau\right), \tag{2.1}$$

and if $D>0$, then we have

$$\theta_+(\tau;\mathfrak{C}) = \sum_{\substack{\left[\begin{smallmatrix} x\\ y\end{smallmatrix}\right] \in \left[\begin{smallmatrix} i\\ j\end{smallmatrix}\right] + M\mathbb{Z}^2/G' \\ Q(x,y)>0}} \operatorname{sgn}(ax+\omega y)\; e\left(\frac{1}{f}Q(x,y)\tau\right) \tag{2.2}$$

with $\omega = \frac{b+\sqrt{D}}{2}$. Here G' is the subgroup of $\mathrm{SL}_2(\mathbb{R})$ in the identity component of the automorphism group of Q consisting of elements preserving $(x_1,y_1)+\mathbb{Z}^2$. Moreover, we have

$$\frac{M}{f}\begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix}\begin{bmatrix} i \\ j \end{bmatrix} \in \mathbb{Z}^2. \tag{2.3}$$

Proof. Let $\mathscr{O}_K$ be the ring of integers of K. Let $\mathfrak{a}$ be an integral ideal of K in a ray class $\mathfrak{C}$ modulo $\mathfrak{f}$. For a subset X of K, we denote by X_+ the set of totally positive elements in X. Here we adopt the notational convention $X_+ = X$ if $D < 0$.

If we take an integral primitive ideal $\mathfrak{b}$ in the narrow ideal class of the ideal $\mathfrak{a}^{-1}\mathfrak{f}_0$, then there exists a totally positive number $z \in K$ satisfying $\mathfrak{b} = (z)\mathfrak{a}^{-1}\mathfrak{f}_0$. We can rewrite

$$\begin{aligned}\theta(\tau;\mathfrak{C}) &= \sum_{\substack{\mathfrak{a}\in\mathfrak{C}\\ \mathfrak{a}\subset\mathscr{O}_K}} q^{N_K(\mathfrak{a})}\\ &= \sum_{\alpha\in(1+\mathfrak{a}^{-1}\mathfrak{f}_0)_+/(\mathscr{O}_K^\times\cap(1+\mathfrak{f}_0))_+} q^{N_K(\alpha\mathfrak{a})}\\ &= \sum_{\beta\in(z+\mathfrak{b})_+/(\mathscr{O}_K^\times\cap(1+\mathfrak{f}_0))_+} q^{N_K(\mathfrak{b}^{-1}\mathfrak{f}_0)N_K(\beta)}\end{aligned}$$

with $q = \exp(2\pi i\tau)$. In the case of $D > 0$, if we take $\mathfrak{a}' = (\nu)\mathfrak{a}$, $z' = \nu z$, then we have $\mathfrak{b} = (z')\mathfrak{a}'^{-1}\mathfrak{f}_0$ and $\nu z + \mathfrak{b} = -z + (1+\nu)z + \mathfrak{b} = -z + \mathfrak{b}$.

By [1, Proposition 8.4.5], the integral primitive ideal $\mathfrak{b}$ can be written as

$$\mathfrak{b} = a\mathbb{Z} + \omega\mathbb{Z}, \quad \omega = \frac{b+\sqrt{D}}{2} \quad \text{with } D = b^2 - 4ac.$$

We may assume $a > 0$. Then, there exists a unique pair (x_1, y_1) of rational numbers such that

$$0 < x_1 \le 1,\ 0 \le y_1 < 1,\ ax_1 + \omega y_1 \in z + \mathfrak{b}.$$

Here supposing that $x_1 = 1$ and $y_1 = 0$ implies $z \in \mathfrak{b} = (z)\mathfrak{a}^{-1}\mathfrak{f}_0$, namely $1 \in \mathfrak{a}^{-1}\mathfrak{f}_0$. Therefore we have $\mathfrak{f}_0 \mid \mathfrak{a}$. This contradicts $\mathfrak{a} \in \mathfrak{C} \in Cl_K(\mathfrak{f})$ because $\mathfrak{f}_0 \neq \mathscr{O}_K$. Thus we must have $x_1 \neq 1$ or $y_1 \neq 0$.

We set

$$Q(x,y) = \frac{N_K(ax+\omega y)}{N_K(\mathfrak{b})} = ax^2 + bxy + cy^2$$

for $x, y \in \mathbb{Q}$. By [1, Proposition 8.3.3], we identify $(\mathscr{O}_K^\times \cap (1+\mathfrak{f}_0))_+$ with G' via the regular representation

$$\Phi : (\mathscr{O}_K^\times)_+ \longrightarrow \mathrm{SL}_2(\mathbb{R}) \quad ; \quad ax + \omega y \mapsto \begin{bmatrix} ax & -cy \\ ay & ax+by \end{bmatrix}.$$

Indeed, any $\epsilon \in (\mathscr{O}_K^\times)_+$ such that $\Phi(\epsilon) \in G'$ satisfy $\epsilon z + \mathfrak{b} = z + \mathfrak{b}$. By $\epsilon z = z + (\epsilon - 1)z$, this implies $\epsilon \in 1 + \mathfrak{f}_0$. Therefore, if $D < 0$, then we obtain

$$\theta_-(\tau,\mathfrak{C}) = \sum_{\left[\begin{smallmatrix} x\\ y\end{smallmatrix}\right]\in\left[\begin{smallmatrix} x_1\\ y_1\end{smallmatrix}\right]+\mathbb{Z}^2/G'} q^{N_K(\mathfrak{f}_0)Q(x,y)},$$

and if $D > 0$, then we obtain

$$\theta_+(\tau, \mathfrak{C}) = \sum_{\substack{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \left[\begin{smallmatrix} x_1 \\ y_1 \end{smallmatrix}\right] + \mathbb{Z}^2/G' \\ Q(x,y)>0}} \operatorname{sgn}(ax + \omega y)\ q^{N_K(\mathfrak{f}_0)Q(x,y)}.$$

Let M be the least common multiple of the denominators of x_1 and y_1. It follows that $i = Mx_1, j = My_1$ are non-negative integers. Multiplying M^2 to $N_K(\mathfrak{f}_0)Q(x + x_1, y + y_1)$ yields

$$\frac{1}{f}(a(Mx + i)^2 + b(Mx + i)(My + j) + c(My + j)^2)$$

with $f = \frac{M^2}{N(\mathfrak{f}_0)}$ and $x, y \in \mathbb{Z}$. This pair (i, j) determines a congruence condition, and Dirichlet's theorem on primes in arithmetic progressions [13, Chapter II, Theorem 3.4] shows that f is a positive integer. By the equation

$$\frac{1}{f}Q(Mx + i, My + j) = \frac{1}{f}\left(M^2Q(x, y) + M \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix} + Q(i, j)\right),$$

we see that f is a positive integer satisfying $f \mid M^2$ and $f \mid Q(i, j)$. The condition (2.3) follows from the fact that $N_K(\mathfrak{f}_0)Q(x+x_1, y+y_1)$ is an integer for all $x, y \in \mathbb{Z}$. Thus it remains to prove $\frac{MD}{f} \in \mathbb{Z}$. By (2.3), we get the equation

$$\begin{bmatrix} i \\ j \end{bmatrix} = -\frac{f}{MD}\begin{bmatrix} 2c & -b \\ -b & 2a \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} \qquad \text{with } X, Y \in \mathbb{Z}.$$

We set $\frac{\alpha}{\beta} = \frac{f}{MD}$ with $\gcd(\alpha, \beta) = 1$. By $\alpha \mid f \mid M^2$ and $\alpha \mid \gcd(i, j)$, we get $\alpha = 1$. This implies $\frac{MD}{f} \in \mathbb{Z}$. This completes the proof of Proposition 2.4. □

The following proposition shows that the right hand side of the equations (2.1) and (2.2) in Proposition 2.4 can be written as $\theta(\tau; h, A, N)$.

Proposition 2.5. *Let $Q(x, y) = ax^2 + bxy + cy^2$ be a quadratic form and let i, j, M, f be positive integers. We set*

$$A_0 = \begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix}, h_0 = \begin{bmatrix} i \\ j \end{bmatrix}, k = \frac{M^2}{f}, v = \frac{M|\det A_0|}{f},$$
$$\textit{and } A = kA_0, h = vh_0, N = k|\det A_0| = Mv.$$

We assume $k, v \in \mathbb{Z}_{>0}$ and the condition (2.3)*. Then, if $-\det A < 0$, then we have*

$$\sum_{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \left[\begin{smallmatrix} i \\ j \end{smallmatrix}\right] + M\mathbb{Z}^2/G'} e\left(\frac{1}{f}Q(x, y)\tau\right) = \theta_-(\tau; h, A, N), \tag{2.4}$$

and if $-\det A > 0$*, then we have*

$$\sum_{\substack{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \left[\begin{smallmatrix} i \\ j \end{smallmatrix}\right] + M\mathbb{Z}^2/G' \\ Q(x,y)>0}} \operatorname{sgn}(ax+\omega y)\ e\left(\frac{1}{f}Q(x,y)\tau\right) = \theta_+(\tau; h, A, N). \tag{2.5}$$

Proof. By the assumption (2.3), $h \in \mathscr{H}_A$ follows from

$$Ah = N\frac{M}{f}A_0 h_0 \in N\mathbb{Z}^2.$$

Next, we show the equations (2.4) and (2.5). Let $g = h + N\begin{bmatrix} x \\ y \end{bmatrix} \in h + N\mathbb{Z}^2$. We compute

$$\begin{aligned} \frac{A[g]}{2N^2} &= \frac{M^2v^2}{fM^2v^2}\frac{1}{2}A_0\left[h_0 + M\begin{bmatrix} x \\ y \end{bmatrix}\right] \\ &= \frac{1}{f}\left(a(Mx+i)^2 + b(Mx+i)(My+j) + c(My+j)^2\right). \end{aligned}$$

This completes the proof. □

Now, we can prove Theorem 2.2 (i).

Proof of Theorem 2.2 (i). From Proposition 2.4 and Proposition 2.5, it follows that there exist A, N, h, m such that $\theta_\pm(\tau; \mathfrak{C}) = \theta_\pm(\tau; h, A, N)$. Dirichlet's theorem on primes in arithmetic progressions shows $h \in \mathscr{H}_A^1$. □

To prove Theorem 2.2 (ii) , we need the following two propositions. They are the converse of Proposition 2.4 and Proposition 2.5.

Proposition 2.6. *Let* $\theta_\pm(\tau; h, A, N)$ *be a theta function in Definition 1.1 with* $h \in \mathscr{H}_A^1$*. Then, there exist a quadratic form* $Q(x,y) = ax^2 + bxy + cy^2$ *and positive integers* i, j, M, f *such that if* $-\det A < 0$*, then we have*

$$\theta_-(\tau; h, A, N) = \sum_{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \left[\begin{smallmatrix} i \\ j \end{smallmatrix}\right] + M\mathbb{Z}^2/G'} e\left(\frac{1}{f}Q(x,y)\tau\right),$$

if $-\det A > 0$*, then we have*

$$\theta_+(\tau; h, A, N) = \sum_{\substack{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \left[\begin{smallmatrix} i \\ j \end{smallmatrix}\right] + M\mathbb{Z}^2/G' \\ Q(x,y)>0}} \operatorname{sgn}(ax+\omega y)\ e\left(\frac{1}{f}Q(x,y)\tau\right),$$

with $\gcd(i,j,M) = 1$ *and* $\gcd\left(\left\{\frac{1}{f}Q(Mx+i, My+j) \mid x, y \in \mathbb{Z}\right\}\right) = 1.$

Proof. We set $h = \begin{bmatrix} i_0 \\ j_0 \end{bmatrix} \in \mathscr{H}_A^1$, $v = \gcd(N, i_0, j_0)$, $M = N/v, i = i_0/v, j = j_0/v$ and obtain

$$\frac{A[g]}{2N^2} = \frac{k}{M^2} Q(Mx + i, My + j)$$

for $g = h + N \begin{bmatrix} x \\ y \end{bmatrix} \in h + N\mathbb{Z}^2$. Since $Q(Mx+i, My+j)$ is a positive integer for any $x, y \in \mathbb{Z}$, the denominator of $\frac{M^2}{k}$ is a divisor of $\gcd(R_h)$. By the assumption $h \in \mathscr{H}_A^1$, we see that $\frac{M^2}{k}$ is a positive integer. If we set $f = \frac{M^2}{k}$, then we get the positive integer f with $\gcd\left(\left\{ \frac{1}{f} Q(Mx + i, My + j) \mid x, y \in \mathbb{Z} \right\}\right) = 1$. This completes the proof. □

Proposition 2.7. *We assume that* $Q(x, y) = ax^2 + bxy + cy^2$ *is a quadratic form with* $a > 0$*. Let* i, j, M *be positive integers with* $\gcd(i, j, M) = 1$*, and let* f *is the positive number such that* $\gcd\left(\left\{ \frac{1}{f} Q(Mx + i, My + j) \mid x, y \in \mathbb{Z} \right\}\right) = 1$*. We set*

$$D = b^2 - 4ac,\ \ K = \mathbb{Q}(\sqrt{D}),\ \ z_0 = a\frac{i}{M} + \frac{b + \sqrt{D}}{2}\frac{j}{M},$$
$$\mathfrak{b} = a\mathbb{Z} + \frac{b + \sqrt{D}}{2}\mathbb{Z},\ \ \text{and } (z) = \frac{\mathfrak{n}}{\mathfrak{m}}$$

where $z \in z_0 + \mathfrak{b}$ *is a totally positive number and* $\mathfrak{m}, \mathfrak{n}$ *are coprime integral ideals. We define* $\mathfrak{f}_0 = \frac{\mathfrak{b}\mathfrak{m}}{\gcd(\mathfrak{b}, \mathfrak{n})}$ *and put* $\mathfrak{a} = (z)\mathfrak{b}^{-1}\mathfrak{f}_0$*. Then, there exists* $\mathfrak{C} \in Cl_K(\mathfrak{f})$ *with* $\mathfrak{a} \in \mathfrak{C}$ *such that*

$$\sum_{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \left[\begin{smallmatrix} i \\ j \end{smallmatrix}\right] + M\mathbb{Z}^2/G'} e\left(\frac{1}{f} Q(x, y)\tau\right) = \theta_-(\tau; \mathfrak{C}) \qquad \text{if} -\det A < 0,$$

and

$$\sum_{\substack{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \left[\begin{smallmatrix} i \\ j \end{smallmatrix}\right] + M\mathbb{Z}^2/G' \\ Q(x,y) > 0}} \operatorname{sgn}(ax + \omega y)\ e\left(\frac{1}{f} Q(x, y)\tau\right) = \theta_+(\tau; \mathfrak{C}) \qquad \text{if} -\det A > 0.$$

Proof. Since the integral ideal $\mathfrak{a} = \frac{\mathfrak{n}}{\gcd(\mathfrak{b}, \mathfrak{n})}$ is prime to $\mathfrak{f}_0$, there exists $\mathfrak{C} \in Cl_K(\mathfrak{f})$ containing $\mathfrak{a} = (z)\mathfrak{b}^{-1}\mathfrak{f}_0$. By following the proof of Proposition 2.4, we obtain the proposition. □

Now, we can prove Theorem 2.2 (ii).

Proof of Theorem 2.2 (ii). By Proposition 2.6 and Proposition 2.7, we see that there exists a ray class $\mathfrak{C} \in Cl_K(\mathfrak{f})$ such that $\theta_\pm(\tau; h, A, N) = \theta_\pm(\tau; \mathfrak{C})$. □

The following example is a definite case of Theorem 2.2.

Example 2.8. Let $K = \mathbb{Q}(\sqrt{-23})$ and $\mathfrak{f} = (1 + \sqrt{-23})$. The ray class group $Cl_K(\mathfrak{f}) \simeq \mathbb{Z}/6\mathbb{Z}$ has a generator $\mathfrak{C}_- = [(15\omega + 20, 40\omega - 15)]$ with $\omega = \frac{1+\sqrt{-23}}{2}$. Using Proposition 2.4 and Proposition 2.5, we have the following equations:

$$\theta_-(\tau; [\mathscr{O}_K]) = \theta_-\left(\tau; \begin{bmatrix} 0 \\ 46 \end{bmatrix}, A_1, N\right), \ \theta_-(\tau; \mathfrak{C}_-) = \theta_-\left(\tau; \begin{bmatrix} 230 \\ 184 \end{bmatrix}, A_3, N\right),$$
$$\theta_-(\tau; \mathfrak{C}_-^2) = \theta_-\left(\tau; \begin{bmatrix} 138 \\ 506 \end{bmatrix}, A_2, N\right), \ \theta_-(\tau; \mathfrak{C}_-^3) = \theta_-\left(\tau; \begin{bmatrix} 0 \\ 230 \end{bmatrix}, A_1, N\right),$$
$$\theta_-(\tau; \mathfrak{C}_-^4) = \theta_-\left(\tau; \begin{bmatrix} 46 \\ 368 \end{bmatrix}, A_3, N\right), \ \theta_-(\tau; \mathfrak{C}_-^5) = \theta_-\left(\tau; \begin{bmatrix} 138 \\ 322 \end{bmatrix}, A_2, N\right),$$

with $N = 552, A_1 = 24\begin{bmatrix} 2 & 1 \\ 1 & 12 \end{bmatrix}, A_2 = 24\begin{bmatrix} 4 & 1 \\ 1 & 6 \end{bmatrix}, A_3 = 24\begin{bmatrix} 4 & -1 \\ -1 & 6 \end{bmatrix}$.

The following example is an indefinite case of Theorem 2.2.

Example 2.9. Let $K = \mathbb{Q}(\sqrt{10})$, and $\mathfrak{f}_0 = (3, 1 + \sqrt{10})$. The ray class group $Cl_K(\mathfrak{f}) \simeq \mathbb{Z}/4\mathbb{Z}$ has a generator $\mathfrak{C}_+ = [(2, \sqrt{10})]$. Using Proposition 2.4 and Proposition 2.5, we have the following equations:

$$\theta_+(\tau; [\mathscr{O}_K]) = \theta_+\left(\tau; \begin{bmatrix} 0 \\ 80 \end{bmatrix}, A_1, N\right), \ \theta_+(\tau; \mathfrak{C}_+) = \theta_+\left(\tau; \begin{bmatrix} 40 \\ 80 \end{bmatrix}, A_2, N\right).$$

with $N = 120, A_1 = 3\begin{bmatrix} 4 & 4 \\ 4 & -6 \end{bmatrix}, A_2 = 3\begin{bmatrix} 2 & 6 \\ 6 & -2 \end{bmatrix}$.

§3. The action of Hecke operators

In this section, we describe the action of Hecke operators on theta functions attached to ray classes. Let p be a prime. Define the U-operator U_p and the V-operator V_p by

$$U_p\left(\sum_{n\geq 0}^{\infty} a_n q^n\right) = \sum_{n\geq 0}^{\infty} a_{pn} q^n, \quad V_p\left(\sum_{n\geq 0}^{\infty} a_n q^n\right) = \sum_{n\geq 0}^{\infty} a_n q^{pn}.$$

The Hecke operator T_p acts on a modular form $f \in M_1(\Gamma_1(N))$ by

$$T_p f = U_p f + (V_p f) \mid_1 \gamma_p \text{ with } \gamma_p = \begin{bmatrix} * & * \\ * & p \end{bmatrix} \in \Gamma_0(N).$$

The following theorem extends Hecke's result [4] to the ray class group case. Our result also includes the indefinite case.

Theorem 3.1. *Let p be a prime with $p \nmid N$. We have*

$$T_p\theta_\pm(\tau;\mathfrak{C}) = \begin{cases} \theta_\pm(\tau;\mathfrak{C}[\mathfrak{p}^{-1}]) + \theta_\pm(\tau;\mathfrak{C}[\mathfrak{p}'^{-1}]), & \text{if } \left(\frac{D}{p}\right) = 1 \text{ and } (p) = \mathfrak{p}\mathfrak{p}', \\ 0, & \text{if } \left(\frac{D}{p}\right) = -1. \end{cases}$$

To prove Theorem 3.1, we need the following lemma.

Lemma 3.2. *Let a be a positive integer with $\gcd(a,N) = 1$. If*

$$\theta_\pm(\tau;\mathfrak{C}) = \theta_\pm(\tau;h,A,N),$$

then

$$\theta_\pm(\tau;\mathfrak{C}[(a)]) = \theta_\pm(\tau;ah,A,N).$$

Proof. We compute

$$\theta_-(\tau;\mathfrak{C}[(a)]) = \sum_{\mathfrak{a}\in\mathfrak{C}} e(a^2N_K(\mathfrak{a})\tau) = \sum_{g\equiv h \bmod N} e\left(a^2\frac{A[g]}{2N^2}\tau\right) = \theta_-(\tau;ah,A,N).$$

In the same way, we can show the case of $\theta_+(\tau;\mathfrak{C}[(a)])$. □

Proof of Theorem 3.1. By Theorem 2.2, Theorem 1.2 and Lemma 3.2, we have

$$(V_p\theta_-(\tau,\mathfrak{C})) \mid_1 \gamma_p = \left(\frac{D}{p}\right)\theta_-(p\tau,\mathfrak{C}[(p)]^{-1}).$$

If $\left(\frac{D}{p}\right) = 1$ and $(p) = \mathfrak{p}\mathfrak{p}'$, then we have

$$\sum_{\substack{\mathfrak{a}\in\mathfrak{C} \\ N(\mathfrak{a})\equiv 0 \bmod p}} q^{N_K(\mathfrak{a})} = \sum_{\substack{\mathfrak{a}\in\mathfrak{C} \\ \mathfrak{a}\equiv 0 \bmod \mathfrak{p}}} q^{N_K(\mathfrak{a})} + \sum_{\substack{\mathfrak{a}\in\mathfrak{C} \\ \mathfrak{a}\equiv 0 \bmod \mathfrak{p}'}} q^{N_K(\mathfrak{a})} - \sum_{\substack{\mathfrak{a}\in\mathfrak{C} \\ \mathfrak{a}\equiv 0 \bmod \mathfrak{p}\mathfrak{p}'}} q^{N_K(\mathfrak{a})}.$$

This shows

$$\begin{aligned} T_p\theta_-(\tau;\mathfrak{C}) &= \sum_{\substack{\mathfrak{a}\in\mathfrak{C} \\ N_K(\mathfrak{a})\equiv 0 \bmod p}} q^{p^{-1}N_K(\mathfrak{a})} + \sum_{\mathfrak{a}\in\mathfrak{C}[(p)]^{-1}} q^{pN_K(\mathfrak{a})} \\ &= \sum_{\mathfrak{a}\in\mathfrak{C}[\mathfrak{p}]^{-1}} q^{N_K(\mathfrak{a})} + \sum_{\mathfrak{a}\in\mathfrak{C}[\mathfrak{p}']^{-1}} q^{N_K(\mathfrak{a})} \\ &= \theta_-(\tau;\mathfrak{C}[\mathfrak{p}^{-1}]) + \theta_-(\tau;\mathfrak{C}[\mathfrak{p}'^{-1}]). \end{aligned}$$

If $\left(\frac{D}{p}\right) = -1$, then we have

$$\sum_{\substack{\mathfrak{a}\in\mathfrak{C} \\ N(\mathfrak{a})\equiv 0 \bmod p}} q^{N_K(\mathfrak{a})} = \sum_{\substack{\mathfrak{a}\in\mathfrak{C} \\ \mathfrak{a}\equiv 0 \bmod (p)}} q^{N_K(\mathfrak{a})}.$$

This implies

$$T_p\theta_-(\tau;\mathfrak{C}) = \sum_{\substack{\mathfrak{a}\in\mathfrak{C} \\ N_K(\mathfrak{a})\equiv 0 \bmod p}} q^{p^{-1}N_K(\mathfrak{a})} - \sum_{\mathfrak{a}\in\mathfrak{C}[(p)]^{-1}} q^{pN_K(\mathfrak{a})} = 0.$$

In the same way, we can show the case of $\theta_+(\tau;\mathfrak{C})$. □

Using Theorem 2.2 and Theorem 3.1, we see the action of Hecke operators on $\theta_\pm(\tau;h,A,N)$. The following example is a definite case of Theorem 3.1. An indefinite case will be given in Example 4.6.

Example 3.3. We keep the notation in Example 2.8. Let $\mathfrak{p}_1 = (13, 2\omega+8)$ and $\mathfrak{p}_2 = (13, 2\omega+3)$. Then, we have $[\mathfrak{p}_1^{-1}] = \mathfrak{C}_-^5$, $[\mathfrak{p}_2^{-1}] = \mathfrak{C}_-$ and $(13) = \mathfrak{p}_1\mathfrak{p}_2$. By Theorem 3.1 and Example 2.8, we obtain

$$T_{13}\theta_-(\tau;[\mathscr{O}_K]) = \theta_-(\tau;\mathfrak{C}_-^5) + \theta_-(\tau;\mathfrak{C}_-).$$

This implies

$$T_{13}\left(\sum_{\substack{x\equiv 0 \bmod 12 \\ y\equiv 1 \bmod 12}} q^{\frac{1}{6}(x^2+xy+6y^2)}\right) = \sum_{\substack{x\equiv 3 \bmod 12 \\ y\equiv 7 \bmod 12}} q^{\frac{1}{6}(2x^2+xy+3y^2)} + \sum_{\substack{x\equiv 5 \bmod 12 \\ y\equiv 4 \bmod 12}} q^{\frac{1}{6}(2x^2-xy+3y^2)}.$$

§4. Hecke eigenforms as a linear combination of $\theta_\pm(\tau;h,A,N)$

In this section, we express Hecke eigenforms in [10, Thorem 4.8.2 and Theorem 4.8.3] as linear combinations of the theta function $\theta(\tau;h,A,N)$. First, we define theta functions attached to Hecke characters by linear combinations of $\theta(\tau;\mathfrak{C})$.

Let $K = \mathbb{Q}(\sqrt{D})$ be a quadratic field with the discriminant D, $\mathfrak{f}$ a modulus of K, and $\mathfrak{f}_0$ the finite part of $\mathfrak{f}$. We denote a' by the conjugate of a. In this paper, we assume that a Hecke character χ modulo $\mathfrak{f}_0$ is a class character in the sense of [10, §3.3] and satisfies

$$\chi((a)) = \begin{cases} 1, & D<0, \\ \operatorname{sgn}(a) \text{ or } \operatorname{sgn}(a'), & D>0, \end{cases} \tag{4.1}$$

for $a \equiv 1 \bmod \mathfrak{f}_0$. Since χ is a class character, we identify χ with a character on the ray class group $Cl_K(\mathfrak{f})$. The theta function attached to the Hecke character χ is the function on the upper half plane $\mathfrak{H}$ defined by

$$\Theta(\tau;\mathfrak{f},\chi) = \sum_{\mathfrak{C}\in Cl_K(\mathfrak{f})} \chi(\mathfrak{C})\theta(\tau;\mathfrak{C}). \tag{4.2}$$

Let $\mathbb{T}(N)$ be the algebra generated by all the Hecke operators T_n with $\gcd(N,n)=1$. Recall that a Hecke character χ decomposes $\chi=\chi_f\chi_\infty$ in the sense of [10, §3.3].

Theorem 4.1 ([10, Thorem 4.8.2 and Theorem 4.8.3])**.** *If χ is a Hecke character modulo $\mathfrak{f}_0$ satisfying* (4.1)*, then $\Theta(\tau;\mathfrak{f},\chi)=\sum_{n=0}^{\infty}a_nq^n\in M_1(N,\chi_f\psi_D)$ is a $\mathbb{T}(N)$-eigenform with*

$$a_p=\begin{cases}\chi([\mathfrak{p}])+\chi([\mathfrak{p}']), & \left(\frac{D}{p}\right)=1 \text{ and } (p)=\mathfrak{p}\mathfrak{p}',\\ 0, & \left(\frac{D}{p}\right)=-1,\end{cases}\tag{4.3}$$

where p is a prime with $p\nmid N$ and ψ_D is a Kronecker character $\left(\frac{D}{\cdot}\right)$. Moreover, if χ is primitive, then $\Theta(\tau;\mathfrak{f},\chi)$ is a newform.

Remark 4.2. We can check $\Theta(\tau;\mathfrak{f},\chi)\in M_1(N,\chi_f\psi_D)$ by Theorem 1.2 and Theorem 2.2. By Theorem 3.1, we see that $\Theta(\tau;\mathfrak{f},\chi)$ is a Hecke eigenform and that a_p satisfies the equation (4.3).

Theorem 2.2 enables us to express the newform $\Theta(\tau;\mathfrak{f},\chi)$ in (4.2) as a linear combination of $\theta_\pm(\tau;h,A,N)$ in Definition 1.1.

Corollary 4.3. *We have*

$$\Theta(\tau;\mathfrak{f},\chi)=u_D\sum_{\mathfrak{C}\in Cl_K(\mathfrak{f})}\chi(\mathfrak{C})\theta_\pm(\tau;h,A,N),$$

where h,A,N are determined by $\mathfrak{C}$ by Theorem 2.2 (i) and $u_D=1$ or $\frac{1}{2}$ according as $D<0$ or $D>0$.

If the ray class field modulo $\mathfrak{f}$ is a ring class field, then it is known that a newform in $S_1(|D|,\psi_D)$ is expressed as a linear combination of binary theta functions *without* congruence conditions, but if not, we need theta functions with congruence conditions.

Remark 4.4. Kani [6] showed that the space generated by the binary theta function attached to the positive definite forms without congruence conditions as a subspace of $M_1(|D|,\psi_D)$ is precisely the subspace of modular forms with CM by ψ_D. However, if χ_f is not a Kronecker character, then $\Theta(\tau;\mathfrak{f},\chi)$ in (4.2) is not contained in the space constructed by Kani, because Theorem 3.1 shows that $\Theta(\tau;\mathfrak{f},\chi)\in M_1(N,\chi_f\psi_D)$ has CM by ψ_D. In particular, if $D>0$, there exist theta functions $\Theta(\tau;\mathfrak{f},\chi)$ which are not contained in Kani's space. For example, see [7, Example 6.12].

The following example is a definite case of Theorem 4.1.

Example 4.5. We keep the notation in Example 2.8. Let χ_- be a character on the ray class group $Cl_K(\mathfrak{f})$. Using Theorem 4.1 and Example 2.8, we see that

$$\begin{aligned}
\Theta(\tau;\mathfrak{f},\chi_-) &= \sum_{n=0}^{5} \chi_-(\mathfrak{C}_-^n)\theta(\tau;\mathfrak{C}_-^n) \\
&= \left(\sum_{\substack{x\equiv 0 \bmod 12 \\ y\equiv 1 \bmod 12}} + \chi_-(\mathfrak{C}_-^3) \sum_{\substack{x\equiv 0 \bmod 12 \\ y\equiv 5 \bmod 12}} \right) q^{\frac{1}{6}(x^2+xy+6y^2)} \\
&\quad + \left(\chi_-(\mathfrak{C}_-) \sum_{\substack{x\equiv 3 \bmod 12 \\ y\equiv 7 \bmod 12}} + \chi_-(\mathfrak{C}_-^4) \sum_{\substack{x\equiv 3 \bmod 12 \\ y\equiv 11 \bmod 12}} \right) q^{\frac{1}{6}(2x^2+xy+3y^2)} \\
&\quad + \left(\chi_-(\mathfrak{C}_-^2) \sum_{\substack{x\equiv 1 \bmod 12 \\ y\equiv 8 \bmod 12}} + \chi_-(\mathfrak{C})^5 \sum_{\substack{x\equiv 5 \bmod 12 \\ y\equiv 4 \bmod 12}} \right) q^{\frac{1}{6}(2x^2-xy+3y^2)}
\end{aligned}$$

is a $\mathbb{T}(552)$-eigenform in $M_1(552, \chi_-\psi_{-23})$. Moreover, if χ_- is not the trivial character, then $\Theta(\tau;\mathfrak{f},\chi_-) \in S_1(552, \chi_-\psi_{-23})$ is a newform.

The following example is an indefinite case of Theorem 4.1.

Example 4.6. We keep the notation in Example 2.9. Let χ_+ be a character on the ray class group $Cl_K(\mathfrak{f})$. Using Theorem 4.1 and Example 2.9, we see that

$$\begin{aligned}
\Theta(\tau;\mathfrak{f},\chi_+) &= \sum_{n=0}^{4} \chi_+(\mathfrak{C}_+^n)\theta(\tau;\mathfrak{C}_+^n) \\
&= \sum_{\substack{x\equiv 0 \bmod 3 \\ y\equiv 2 \bmod 3 \\ 2x^2+4xy-3y^2>0 \\ \bmod \langle X_1\rangle}} \operatorname{sgn}\left(2x+(2+\sqrt{10})y\right) q^{\frac{1}{3}(2x^2+4xy-3y^2)} \\
&\quad + \chi_+(\mathfrak{C}_+) \sum_{\substack{x\equiv 1 \bmod 3 \\ y\equiv 2 \bmod 3 \\ x^2+6xy-y^2>0 \\ \bmod \langle X_2\rangle}} \operatorname{sgn}\left(x+(3+\sqrt{10})y\right) q^{\frac{1}{3}(x^2+6xy-y^2)}
\end{aligned}$$

is a newform in $S_1(120, \chi_{+f}\psi_{40})$ where $X_1 = \begin{bmatrix} 31 & 18 \\ 12 & 7 \end{bmatrix}, X_2 = \begin{bmatrix} 1 & 6 \\ 6 & 37 \end{bmatrix}$. In the last equation, the first sum means that $\begin{bmatrix} x \\ y \end{bmatrix}$ runs over $\begin{bmatrix} 0 \\ 2 \end{bmatrix} + 3\mathbb{Z}^2/\langle X_1\rangle$ and $2x^2+4xy-3y^2>0$. The same applies to the second sum.

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References

[1] J. Buchmann and U. Vollmer. *Binary quadratic forms*, volume 20 of *Algorithms and Computation in Mathematics*. Springer, Berlin, 2007.

[2] F. Diamond and J. Shurman. *A first course in modular forms*, volume 228 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005.

[3] E. Hecke. Zur Theorie der elliptischen Modulfunktionen. *Math. Ann.*, 97(1):210–242, 1927.

[4] E. Hecke. Analytische Arithmetik der positiven quadratischen Formen. *Danske Vid. Selsk. Math.-Fys. Medd.*, 17(12):134, 1940.

[5] E. Kani. The space of binary theta series. *Ann. Sci. Math. Québec*, 36(2):501–534 (2013), 2012.

[6] E. Kani. Binary theta series and modular forms with complex multiplication. *Int. J. Number Theory*, 10(4):1025–1042, 2014.

[7] M. Kida and G. Koda. Isoclinism classes of Galois groups of number fields. *Acta Arith.*, 191(2):115–149, 2019.

[8] M. Kida and N. Namura. On Artin L-functions of certain central extensions. *J. Number Theory*, 173:147–169, 2017.

[9] S. S. Kudla. Holomorphic Siegel modular forms associated to $\mathrm{SO}(n, 1)$. *Math. Ann.*, 256(4):517–534, 1981.

[10] T. Miyake. *Modular forms*. Springer-Verlag, Berlin, 1989. Translated from the Japanese by Yoshitaka Maeda.

[11] K. Ono. *The web of modularity: arithmetic of the coefficients of modular forms and q-series*, volume 102 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2004.

[12] G. Shimura. On modular forms of half integral weight. *Ann. of Math. (2)*, 97:440–481, 1973.

[13] J. H. Silverman. *Advanced topics in the arithmetic of elliptic curves*, volume 151 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994.

[14] S. Yamamoto. On Kronecker limit formulas for real quadratic fields. *J. Number Theory*, 128(2):426–450, 2008.

Ryota Okano
Department of Mathematics, Tokyo University of Science
Kagurazaka 1-3, Shinjuku, Tokyo 162-0827, Japan
E-mail: 1118701@ed.tus.ac.jp