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Some inequalities for weighted and integral means of operator convex functions

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Abstract. Let f be an operator convex function on I and A, $B \in \mathcal{SA}_I(H)$, the convex set of selfadjoint operators with spectra in I. If $A \neq B$ and f, as an operator function, is Gâteaux differentiable on

$$[A, B] := \{(1-t)A + tB \mid t \in [0,1]\},$$

while $p : [0,1] \to [0,\infty)$ is Lebesgue integrable satisfying the condition

$$0 \le \int_0^\tau p(s)\,ds \le \int_0^1 p(s)\,ds \text{ for all } \tau \in [0,1]$$

and symmetric, namely $p(1-t) = p(t)$ for all $t \in [0,1]$, then

$$\begin{aligned}
&-\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right] \\
&\le \frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 p(\tau) f((1-\tau)A + \tau B)\,d\tau - \int_0^1 f((1-\tau)A + \tau B)\,d\tau \\
&\le \frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right].
\end{aligned}$$

Some particular examples of interest are also given.

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§1. Introduction

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A + \lambda B) \le (\ge)\,(1-\lambda) f(A) + \lambda f(B) \tag{1.1}$$

in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A \leq B$ with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$ imply $f(A) \leq f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [9] and the references therein.

As examples of such functions, we note that $f(t) = t^r$ is operator monotone on $[0,\infty)$ if and only if $0 \leq r \leq 1$. The function $f(t) = t^r$ is operator convex on $(0,\infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0,\infty)$ if $0 \leq r \leq 1$. The logarithmic function $f(t) = \ln t$ is operator monotone and operator concave on $(0,\infty)$. The entropy function $f(t) = -t \ln t$ is operator concave on $(0,\infty)$. The exponential function $f(t) = e^t$ is neither operator convex nor operator monotone.

In [6] we obtained among others the following Hermite-Hadamard type inequalities for operator convex functions $f : I \to \mathbb{R}$

$$f\left(\frac{A+B}{2}\right) \leq \int_0^1 f((1-s)A + sB)\,ds \leq \frac{f(A)+f(B)}{2}, \tag{1.2}$$

where A, B are selfadjoint operators with spectra included in I.

For two distinct operators A, $B \in \mathcal{SA}_I(H)$ we consider the segment of selfadjoint operators

$$[A,B] := \{(1-t)A + tB \mid t \in [0,1]\}.$$

We observe that A, $B \in [A,B]$ and $[A,B] \subset \mathcal{SA}_I(H)$.

A continuous function $g : \mathcal{SA}_I(H) \to \mathcal{B}(H)$ is said to be *Gâteaux differentiable* in $A \in \mathcal{SA}_I(H)$ along the direction $B \in \mathcal{B}(H)$ if the following limit exists in the strong topology of $\mathcal{B}(H)$

$$\nabla g_A(B) := \lim_{s\to 0} \frac{g(A+sB) - g(A)}{s} \in \mathcal{B}(H). \tag{1.3}$$

If the limit (1.3) exists for all $B \in \mathcal{B}(H)$, then we say that f is *Gâteaux differentiable* in A and we can write $g \in \mathcal{G}(A)$. If this is true for any A in a subset $\mathcal{S}$ from $\mathcal{SA}_I(H)$ we write that $g \in \mathcal{G}(\mathcal{S})$.

In the recent paper [8], we obtained the following operator *Féjer's type inequalities:*

Theorem 1. Let f be an operator convex function on I and A, $B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$ and $p : [0,1] \to [0,\infty)$ is Lebesgue integrable and symmetric, namely $p(1-t) = p(t)$ for all $t \in [0,1]$, then

$$0 \le \int_0^1 p(t) f((1-t)A + tB)\, dt - \left(\int_0^1 p(t)\, dt\right) f\left(\frac{A+B}{2}\right) \le \frac{1}{2}\left(\int_0^1 \left|t - \frac{1}{2}\right| p(t)\, dt\right) [\nabla f_B(B-A) - \nabla f_A(B-A)]. \tag{1.4}$$

In particular, for $p \equiv 1$ we get

$$0 \le \int_0^1 f((1-t)A + tB)\, dt - f\left(\frac{A+B}{2}\right) \le \frac{1}{8} [\nabla f_B(B-A) - \nabla f_A(B-A)]. \tag{1.5}$$

We also have:

Theorem 2. Let f be an operator convex function on I and $A, B \in \mathcal{SA}_I(H)$, with $A \ne B$. If $f \in \mathcal{G}([A,B])$ and $p : [0,1] \to [0,\infty)$ is Lebesgue integrable and symmetric, namely $p(1-t) = p(t)$ for all $t \in [0,1]$, then

$$0 \le \left(\int_0^1 p(t)\, dt\right) \frac{f(A) + f(B)}{2} - \int_0^1 p(t) f((1-t)A + tB)\, dt \le \frac{1}{2}\int_0^1 \left(\frac{1}{2} - \left|t - \frac{1}{2}\right|\right) p(t)\, dt\, [\nabla f_B(B-A) - \nabla f_A(B-A)]. \tag{1.6}$$

In particular, for $p \equiv 1$ we get

$$0 \le \frac{f(A) + f(B)}{2} - \int_0^1 f((1-t)A + tB)\, dt \le \frac{1}{8} [\nabla f_B(B-A) - \nabla f_A(B-A)]. \tag{1.7}$$

For recent inequalities for operator convex functions see [1]-[7] and [10]-[20].

Motivated by the above results, we establish in this paper some upper and lower bounds in the operator order for the difference

$$\int_0^1 p(\tau) f((1-\tau)A + \tau B)\, d\tau - \int_0^1 p(\tau)\, d\tau \int_0^1 f((1-\tau)A + \tau B)\, d\tau$$

in the case when the operator convex function f is Gâteaux differentiable as a function of selfadjoint operators. Two particular examples of interest for $f(x) = -\ln x$ and $f(x) = x^{-1}$ are also given.

§2. Some preliminary facts

Let f be an operator convex function on I. For A, $B \in \mathcal{SA}_I(H)$, the class of all selfadjoint operators with spectra in I, we consider the auxiliary function $\varphi_{(A,B)} : [0,1] \to \mathcal{SA}_I(H)$ defined by

$$\varphi_{(A,B)}(t) := f((1-t)A + tB). \tag{2.1}$$

For $x \in H$ we can also consider the auxiliary function $\varphi_{(A,B);x} : [0,1] \to \mathbb{R}$ defined by

$$\varphi_{(A,B);x}(t) := \left\langle \varphi_{(A,B)}(t)x, x \right\rangle = \langle f((1-t)A + tB)x, x\rangle. \tag{2.2}$$

We have the following basic fact:

Lemma 3. Let f be an operator convex function on I. For any A, $B \in \mathcal{SA}_I(H)$, $\varphi_{(A,B)}$ is well defined and convex in the operator order. For any $(A,B) \in \mathcal{SA}_I(H)$ and $x \in H$ the function $\varphi_{(A,B);x}$ is convex in the usual sense on $[0,1]$.

Proof. If $(A,B) \in \mathcal{SA}_I(H)$ and $t \in [0,1]$ the convex combination $(1-t)A+tB$ is a selfadjoint operator with the spectrum in I showing that $\mathcal{SA}_I(H)$ is convex in the Banach algebra $\mathcal{B}(H)$ of all bounded linear operators on H. By the continuous functional calculus of selfadjoint operator we also conclude that $f((1-t)A+tB)$ is a selfadjoint operator in $\mathcal{B}(H)$.

Let A, $B \in \mathcal{SA}_I(H)$ and t_1, $t_2 \in [0,1]$. If α, $\beta > 0$ with $\alpha + \beta = 1$, then

$$\begin{aligned}
\varphi_{(A,B)}(\alpha t_1 + \beta t_2) &:= f((1 - \alpha t_1 - \beta t_2)A + (\alpha t_1 + \beta t_2)B) \\
&= f((\alpha + \beta - \alpha t_1 - \beta t_2)A + (\alpha t_1 + \beta t_2)B) \\
&= f(\alpha[(1-t_1)A + t_1 B] + \beta[(1-t_2)A + t_2 B]) \\
&\le \alpha f((1-t_1)A + t_1 B) + \beta f((1-t_2)A + t_2 B) \\
&= \alpha\varphi_{(A,B)}(t_1) + \beta\varphi_{(A,B)}(t_2),
\end{aligned}$$

which proves the convexity $\varphi_{(A,B)}$ in the operator order.

Ley A, $B \in \mathcal{SA}_I(H)$ and $x \in H$. If t_1, $t_2 \in [0,1]$ and α, $\beta > 0$ with $\alpha + \beta = 1$, then

$$\begin{aligned}
\varphi_{(A,B);x}(\alpha t_1 + \beta t_2) &= \left\langle \varphi_{(A,B)}(\alpha t_1 + \beta t_2)x, x \right\rangle \\
&\le \left\langle \left[\alpha\varphi_{(A,B)}(t_1) + \beta\varphi_{(A,B)}(t_2)\right]x, x \right\rangle \\
&= \alpha\left\langle \varphi_{(A,B)}(t_1)x, x\right\rangle + \beta\left\langle \varphi_{(A,B)}(t_2)x, x\right\rangle \\
&= \alpha\varphi_{(A,B);x}(t_1) + \beta\varphi_{(A,B);x}(t_2),
\end{aligned}$$

which proves the convexity of $\varphi_{(A,B);x}$ on $[0,1]$. □

Lemma 4. Let f be an operator convex function on I and A, $B\in\mathcal{SA}_I(H)$, with $A\neq B$. If $f\in\mathcal{G}([A,B])$, then the auxiliary function $\varphi_{(A,B)}$ is differentiable on $(0,1)$ and

$$\varphi'_{(A,B)}(t)=\nabla f_{(1-t)A+tB}(B-A). \tag{2.3}$$

Also

$$\varphi'_{(A,B)}(0+)=\nabla f_A(B-A) \tag{2.4}$$

and

$$\varphi'_{(A,B)}(1-)=\nabla f_B(B-A). \tag{2.5}$$

Proof. Let $t\in(0,1)$ and $h\neq 0$ small enough such that $t+h\in(0,1)$. Then

$$\begin{aligned}&\frac{\varphi_{(A,B)}(t+h)-\varphi_{(A,B)}(t)}{h}\\&=\frac{f((1-t-h)A+(t+h)B)-f((1-t)A+tB)}{h}\\&=\frac{f((1-t)A+tB+h(B-A))-f((1-t)A+tB)}{h}.\end{aligned} \tag{2.6}$$

Since $f\in\mathcal{G}([A,B])$, hence by taking the limit over $h\to 0$ in (2.6) we get

$$\begin{aligned}\varphi'_{(A,B)}(t)&=\lim_{h\to 0}\frac{\varphi_{(A,B)}(t+h)-\varphi_{(A,B)}(t)}{h}\\&=\lim_{h\to 0}\frac{f((1-t)A+tB+h(B-A))-f((1-t)A+tB)}{h}\\&=\nabla f_{(1-t)A+tB}(B-A),\end{aligned}$$

which proves (2.7).

Also, we have

$$\begin{aligned}\varphi'_{(A,B)}(0+)&=\lim_{h\to 0+}\frac{\varphi_{(A,B)}(h)-\varphi_{(A,B)}(0)}{h}\\&=\lim_{h\to 0+}\frac{f((1-h)A+hB)-f(A)}{h}\\&=\lim_{h\to 0+}\frac{f(A+h(B-A))-f(A)}{h}\\&=\nabla f_A(B-A)\end{aligned}$$

since f is assumed to be Gâteaux differentiable in A. This proves (2.4).

The equality (2.5) follows in a similar way. $\square$

Lemma 5. Let f be an operator convex function on I and A, $B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$, then for $0 < t_1 < t_2 < 1$

$$\nabla f_{(1-t_1)A+t_1B}(B-A) \leq \nabla f_{(1-t_2)A+t_2B}(B-A) \tag{2.7}$$

in the operator order.

Moreover

$$\nabla f_A(B-A) \leq \nabla f_{(1-t_1)A+t_1B}(B-A) \tag{2.8}$$

and

$$\nabla f_{(1-t_2)A+t_2B}(B-A) \leq \nabla f_B(B-A). \tag{2.9}$$

Proof. Let $x \in H$. The auxiliary function $\varphi_{(A,B);x}$ is convex in the usual sense on $[0,1]$ and differentiable on $(0,1)$ and for $t \in (0,1)$

$$\begin{aligned}
\varphi'_{(A,B);x}(t) &= \lim_{h\to 0} \frac{\varphi_{(A,B),x}(t+h) - \varphi_{(A,B),x}(t)}{h} \\
&= \lim_{h\to 0} \left\langle \frac{\varphi_{(A,B)}(t+h) - \varphi_{(A,B)}(t)}{h} x, x \right\rangle \\
&= \left\langle \lim_{h\to 0} \frac{\varphi_{(A,B)}(t+h) - \varphi_{(A,B)}(t)}{h} x, x \right\rangle \\
&= \left\langle \nabla f_{(1-t)A+tB}(B-A)x, x \right\rangle.
\end{aligned}$$

Since for $0 < t_1 < t_2 < 1$ we have by the gadient inequality for scalar convex functions that

$$\varphi'_{(A,B);x}(t_1) \leq \varphi'_{(A,B);x}(t_2),$$

then we get

$$\left\langle \nabla f_{(1-t_1)A+t_1B}(B-A)x, x \right\rangle \leq \left\langle \nabla f_{(1-t_2)A+t_2B}(B-A)x, x \right\rangle \tag{2.10}$$

for all $x \in H$, which is equivalent to the inequality (2.7) in the operator order.

Let $0 < t_1 < 1$. By the gadient inequality for scalar convex functions we also have

$$\varphi'_{(A,B);x}(0+) \leq \varphi'_{(A,B);x}(t_1),$$

which, as above, implies that

$$\langle \nabla f_A(B-A)x, x \rangle \leq \left\langle \nabla f_{(1-t_1)A+t_1B}(B-A)x, x \right\rangle$$

for all $x \in H$, that is equivalent to the operator inequality (2.8).

The inequality (2.9) follows in a similar way. □

Corollary 6. Let f be an operator convex function on I and A, $B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$, then for all $t \in (0,1)$ we have

$$\nabla f_A(B-A) \leq \nabla f_{(1-t)A+tB}(B-A) \leq \nabla f_B(B-A). \tag{2.11}$$

§3. Main results

We start to the following identity that is of interest in itself as well:

Lemma 7. Let f be an operator convex function on I and $A, B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$ and $g : [0,1] \to \mathbb{C}$ is a Lebesgue integrable function, then we have the equality

$$\begin{aligned}(3.1)\qquad & \int_0^1 g(\tau)\,\varphi_{(A,B)}(\tau)\,d\tau - \int_0^1 g(\tau)\,d\tau \int_0^1 \varphi_{(A,B)}(\tau)\,d\tau \\ &= \int_0^1 \left(\int_\tau^1 g(s)\,ds\right) \tau\varphi'_{(A,B)}(\tau)\,d\tau \\ &+ \int_0^1 \left(\int_0^\tau g(s)\,ds\right)(\tau-1)\,\varphi'_{(A,B)}(\tau)\,d\tau.\end{aligned}$$

Proof. Integrating by parts in the Bochner's integral, we have

$$\begin{aligned}&\int_0^\tau t\varphi'_{(A,B)}(t)\,dt + \int_\tau^1 (t-1)\,\varphi'_{(A,B)}(t)\,dt \\ &= \tau\varphi_{(A,B)}(\tau) - \int_0^\tau \varphi_{(A,B)}(t)\,dt - (\tau-1)\,\varphi_{(A,B)}(\tau) - \int_\tau^1 \varphi_{(A,B)}(t)\,dt \\ &= \varphi_{(A,B)}(\tau) - \int_0^1 \varphi_{(A,B)}(t)\,dt\end{aligned}$$

that holds for all $\tau \in [0,1]$.

If we multiply this identity by $g(\tau)$ and integrate over τ in $[0,1]$, then we get

(3.2)
$$\begin{aligned}&\int_0^1 g(\tau)\,\varphi_{(A,B)}(\tau)\,d\tau - \int_0^1 g(\tau)\,d\tau \int_0^1 \varphi_{(A,B)}(t)\,dt \\ &= \int_0^1 g(\tau)\left(\int_0^\tau t\varphi'_{(A,B)}(t)\,dt\right)d\tau + \int_0^1 g(\tau)\left(\int_\tau^1 (t-1)\,\varphi'_{(A,B)}(t)\,dt\right)d\tau.\end{aligned}$$

From the theory of Lebesgue-Stieltjes integral for scalar-valued functions, see for instance [3, Theorem 1] or [14, Theorem 9], if u is absolutely continous in $[0,1]$ and h is a bounded Borel measurable function on $[0,1]$, then the Lebesgue-Stieltjes integral reduces to the usual Lebesgue integral, namely

$$(3.3)\qquad \int_0^1 h(s)\,du(s) = \int_0^1 h(s)\,u'(s)\,ds,$$

where $u'(s)$ is the a.e. derivative of u.

Since $u(\tau) := \int_0^\tau g(s)\,ds$ is absolutely continuous on $[0,1]$, then by (3.3) we have

$$\int_0^1 \left(\int_0^\tau t \left\langle \varphi'_{(A,B)}(t)\,x, x \right\rangle dt \right) d\left(\int_0^\tau g(s)\,ds \right)$$
$$= \int_0^1 g(\tau) \left(\int_0^\tau t \left\langle \varphi'_{(A,B)}(t)\,x, x \right\rangle dt \right) d\tau,$$

for all $x \in H$, which gives the operator identity

$$\int_0^1 \left(\int_0^\tau t\varphi'_{(A,B)}(t)\,dt \right) d\left(\int_0^\tau g(s)\,ds \right) = \int_0^1 g(\tau) \left(\int_0^\tau t\varphi'_{(A,B)}(t)\,dt \right) d\tau.$$

Using integration by parts, we derive

$$\int_0^1 \left(\int_0^\tau t\varphi'_{(A,B)}(t)\,dt \right) d\left(\int_0^\tau g(s)\,ds \right) \tag{3.4}$$
$$= \left(\int_0^\tau g(s)\,ds \right) \left(\int_0^\tau t\varphi'_{(A,B)}(t)\,dt \right) \Bigg|_0^1$$
$$- \int_0^1 \left(\int_0^\tau g(s)\,ds \right) \tau\varphi'_{(A,B)}(\tau)\,d\tau$$
$$= \left(\int_0^1 g(s)\,ds \right) \left(\int_0^1 t\varphi'_{(A,B)}(t)\,dt \right)$$
$$- \int_0^1 \left(\int_0^\tau g(s)\,ds \right) \tau\varphi'_{(A,B)}(\tau)\,d\tau$$
$$= \int_0^1 \left(\int_0^1 g(s)\,ds - \int_0^\tau g(s)\,ds \right) \tau\varphi'_{(A,B)}(\tau)\,d\tau$$
$$= \int_0^1 \left(\int_\tau^1 g(s)\,ds \right) \tau\varphi'_{(A,B)}(\tau)\,d\tau$$

and, similarly

$$\int_0^1 g(\tau) \left(\int_\tau^1 (t-1)\,\varphi'_{(A,B)}(t)\,dt \right) d\tau \tag{3.5}$$
$$= \int_0^1 \left(\int_\tau^1 (t-1)\,\varphi'_{(A,B)}(t)\,dt \right) d\left(\int_0^\tau g(s)\,ds \right)$$
$$= \left(\int_\tau^1 (t-1)\,\varphi'_{(A,B)}(t)\,dt \right) \left(\int_0^\tau g(s)\,ds \right) \Bigg|_0^1$$
$$+ \int_0^1 \left(\int_0^\tau g(s)\,ds \right) (\tau-1)\,\varphi'_{(A,B)}(\tau)\,d\tau$$
$$= \int_0^1 \left(\int_0^\tau g(s)\,ds \right) (\tau-1)\,\varphi'_{(A,B)}(\tau)\,d\tau,$$

which proves the identity in (3.1). □

Theorem 8. Let f be an operator convex function on I and $A, B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$ and $p : [0,1] \to \mathbb{R}$ is a Lebesgue integrable function such that

$$0 \leq \int_0^\tau p(s)\,ds \leq \int_0^1 p(s)\,ds \text{ for all } \tau \in [0,1], \tag{3.6}$$

then we have the inequalities

$$\begin{aligned} &\int_0^1 \left(\int_\tau^1 p(s)\,ds\right) \tau d\tau \nabla f_A(B-A) \\ &- \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \nabla f_B(B-A) \\ &\leq \int_0^1 p(\tau) f((1-\tau)A+\tau B)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)A+\tau B)\,d\tau \\ &\leq \int_0^1 \left(\int_\tau^1 p(s)\,ds\right) \tau d\tau \nabla f_B(B-A) \\ &- \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \nabla f_A(B-A) \end{aligned} \tag{3.7}$$

or, equivalently,

$$\begin{aligned} &\int_0^1 (1-\tau)\left(\int_0^\tau [p(1-s)\nabla f_B(B-A) - p(s)\nabla f_A(B-A)]\,ds\right) d\tau \\ &\leq \int_0^1 p(\tau) f((1-\tau)A+\tau B)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)A+\tau B)\,d\tau \\ &\leq \int_0^1 (1-\tau)\left(\int_0^\tau [p(1-s)\nabla f_A(B-A) - p(s)\nabla f_B(B-A)]\,ds\right) d\tau. \end{aligned} \tag{3.8}$$

Proof. We have for $\varphi_{(A,B)}$ and $p : [0,1] \to \mathbb{R}$ a Lebesgue integrable function that

$$\begin{aligned} &\int_0^1 p(\tau)\varphi_{(A,B)}(\tau)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 \varphi_{(A,B)}(\tau)\,d\tau \\ &= \int_0^1 \left(\int_\tau^1 p(s)\,ds\right)(\tau)\varphi'_{(A,B)}(\tau)\,d\tau \\ &- \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\varphi'_{(A,B)}(\tau)\,d\tau. \end{aligned} \tag{3.9}$$

By the properties of $\varphi_{(A,B)}$ from the above section, we have in the operator order that

$$\tau\varphi'_{(A,B)}(1-)\geq\tau\varphi'_{(A,B)}(\tau)\geq\tau\varphi'_{(A,B)}(0+) \tag{3.10}$$

and

$$(1-\tau)\varphi'_{(A,B)}(1-)\geq(1-\tau)\varphi'_{(A,B)}(\tau)\geq(1-\tau)\varphi'_{(A,B)}(0+) \tag{3.11}$$

for all $\tau\in(0,1)$.

From

$$\int_0^\tau p(s)\,ds\leq\int_0^1 p(s)\,ds=\int_0^\tau p(s)\,ds+\int_\tau^1 p(s)\,ds,$$

we get that $\int_\tau^1 p(s)\,ds\geq 0$ for all $\tau\in(0,1)$.

From (3.10) we get that

$$\begin{aligned}\left(\int_\tau^1 p(s)\,ds\right)\tau\varphi'_{(A,B)}(1-)&\geq\left(\int_\tau^1 p(s)\,ds\right)\tau\varphi'_{(A,B)}(\tau)\\&\geq\left(\int_\tau^1 p(s)\,ds\right)\tau\varphi'_{(A,B)}(0+)\end{aligned}$$

and from (3.11) that

$$\begin{aligned}-\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\varphi'_{(A,B)}(0+)&\leq-\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\varphi'_{(A,B)}(\tau)\\&\leq-\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\varphi'_{(A,B)}(1-)\end{aligned}$$

all $\tau\in(0,1)$.

If we integrate these inequalities over $\tau\in[0,1]$ and add the obtained results, then we get

$$\begin{aligned}&\int_0^1\left(\int_\tau^1 p(s)\,ds\right)\tau d\tau\varphi'_{(A,B)}(1-)-\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\varphi'_{(A,B)+}(0)\\&\geq\int_0^1\left(\int_\tau^1 p(s)\,ds\right)\tau\varphi'_{(A,B)}(\tau)\,d\tau-\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\varphi'_{(A,B)}(\tau)\,d\tau\\&\geq\int_0^1\left(\int_\tau^1 p(s)\,ds\right)\tau d\tau\varphi'_{(A,B)}(0+)\\&-\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\varphi'_{(A,B)}(1-).\end{aligned}$$

By using the equality (3.1) we get

$$\begin{aligned}
&\int_0^1 \left(\int_\tau^1 p(s)\,ds\right) \tau d\tau \varphi'_{(A,B)}(0+) \\
&- \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \varphi'_{(A,B)}(1-) \\
&\le \int_0^1 p(\tau)\,\varphi_{(A,B)}(\tau)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 \varphi_{(A,B)}(\tau)\,d\tau \\
&\le \int_0^1 \left(\int_\tau^1 p(s)\,ds\right) \tau d\tau \varphi'_{(A,B)}(1-) \\
&- \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \varphi'_{(A,B)}(0+),
\end{aligned} \tag{3.12}$$

and since $\varphi'_{(A,B)}(1-) = \nabla f_B(B-A)$ and $\varphi'_{(A,B)}(0+) = \nabla f_B(B-A)$ hence we obtain (3.7).

If we change the variable $y = 1-\tau$, then we have

$$\int_0^1 \left(\int_\tau^1 p(s)\,ds\right) \tau d\tau = \int_0^1 \left(\int_{1-y}^1 p(s)\,ds\right)(1-y)\,dy.$$

Also by the change of variable $u = 1-s$, we get

$$\int_{1-y}^1 p(s)\,ds = \int_0^y p(1-u)\,du,$$

which implies that

$$\int_0^1 \left(\int_\tau^1 p(s)\,ds\right) \tau d\tau = \int_0^1 \left(\int_0^\tau p(1-s)\,ds\right)(1-\tau)\,d\tau.$$

Therefore

$$\begin{aligned}
&\int_0^1 \left(\int_\tau^1 p(s)\,ds\right) \tau d\tau \varphi'_{(A,B)}(1-) \\
&- \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \varphi'_{(A,B)}(0+) \\
&= \int_0^1 \left(\int_0^\tau p(1-s)\,ds\right)(1-\tau)\,d\tau \varphi'_{(A,B)}(1-) \\
&- \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \varphi'_{(A,B)}(0+) \\
&= \int_0^1 (1-\tau)\left(\int_0^\tau \left[p(1-s)\,\varphi'_{(A,B)}(1-) - p(s)\,\varphi'_{(A,B)+}(0+)\right] ds\right) d\tau
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \left(\int_\tau^1 p(s)\,ds\right) \tau d\tau \varphi'_{(A,B)}(0+) \\
&- \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \varphi'_{(A,B)}(1-) \\
&= \int_0^1 \left(\int_0^\tau p(1-s)\,ds\right)(1-\tau)\,d\tau \varphi'_{(A,B)}(0+) \\
&- \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \varphi'_{(A,B)}(1-) \\
&= \int_0^1 (1-\tau)\left(\int_0^\tau \left[p(1-s)\,\varphi'_{(A,B)}(0+) - p(s)\,\varphi'_{(A,B)}(1-)\right] ds\right) d\tau,
\end{aligned}$$

and by (3.12) we get (3.8). □

We say that the function $p : [0,1] \to \mathbb{R}$ is symmetric on $[0,1]$ if

$$p(1-t) = p(t) \text{ for all } t \in [0,1].$$

Corollary 9. Let f be an operator convex function on I and $A, B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$ and $p : [0,1] \to \mathbb{R}$ a Lebesgue integrable and symmetric function such that the condition (3.6) holds, then we have

$$\begin{aligned}
&-\frac{1}{2}\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right] \\
&\leq -\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \\
&\times \left[\nabla f_B(B-A) - \nabla f_A(B-A)\right] \\
&\leq \frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 p(\tau)\, f((1-\tau)A + \tau B)\,d\tau \\
&- \int_0^1 f((1-\tau)A + \tau B)\,d\tau \\
&\leq \frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \\
&\times \left[\nabla f_B(B-A) - \nabla f_A(B-A)\right] \\
&\leq \frac{1}{2}\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right].
\end{aligned} \tag{3.13}$$

Proof. Since p is symmetric, then $p(1-s) = p(s)$ for all $s \in [0,1]$ and by

(3.8) we get

$$\begin{aligned}
&\int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \left[\varphi'_{(A,B)}(0+)-\varphi'_{(A,B)}(1-)\right] \\
&\le \int_0^1 p(\tau)\,\varphi_{(A,B)}(\tau)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 \varphi_{(A,B)}(\tau)\,d\tau \\
&\le \left[\varphi'_{(A,B)}(1-)-\varphi'_{(A,B)}(0+)\right]\int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau,
\end{aligned}$$

which is equivalent to the second and third inequalities (3.13).

Since $0 \le \int_0^\tau p(s)\,ds \le \int_0^1 p(\tau)\,d\tau$, hence

$$\int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \le \int_0^1 p(\tau)\,d\tau \int_0^1 (1-\tau)\,d\tau = \frac{1}{2}\int_0^1 p(\tau)\,d\tau$$

and the last part of (3.13) is proved. □

Remark 1. If the function p is nonnegative and symmetric then the inequality (3.13) holds true.

Remark 2. It is well known that, if f is a C^1-function defined on an open interval, then the operator function $f(X)$ is Fréchet differentiable and the derivative $Df(A)(B)$ equals the Gâteaux derivative $\nabla f_A(B)$. So for operator convex functions f that are of class C^1 on I and $p:[0,1]\to\mathbb{R}$ is a Lebesgue integrable and symmetric weight on $[0,1]$ such that the condition (3.6) holds, we have the inequalities

$$\begin{aligned}
&-\frac{1}{2}\left[Df(B)(B-A)-Df(A)(B-A)\right] \\
&\le -\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \\
&\times \left[Df(B)(B-A)-Df(A)(B-A)\right] \\
&\le \frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 p(\tau)\,f((1-\tau)A+\tau B)\,d\tau \\
&- \int_0^1 f((1-\tau)A+\tau B)\,d\tau \\
&\le \frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \\
&\times \left[Df(B)(B-A)-Df(A)(B-A)\right] \\
&\le \frac{1}{2}\left[Df(B)(B-A)-Df(A)(B-A)\right]
\end{aligned} \tag{3.14}$$

for $A,\ B\in\mathcal{SA}_I(H)$.

If we consider the weight $p:[0,1]\to[0,\infty)$, $p(s)=\left|s-\frac{1}{2}\right|$, then

$$\begin{aligned}
&\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\\
&=\int_0^1\left(\int_0^\tau\left|s-\frac{1}{2}\right|ds\right)(1-\tau)\,d\tau\\
&=\int_0^{\frac{1}{2}}\left(\int_0^\tau\left|s-\frac{1}{2}\right|ds\right)(1-\tau)\,d\tau\\
&+\int_{\frac{1}{2}}^1\left(\int_0^\tau\left|s-\frac{1}{2}\right|ds\right)(1-\tau)\,d\tau\\
&=\int_0^{\frac{1}{2}}\left(\int_0^\tau\left(\frac{1}{2}-s\right)ds\right)(1-\tau)\,d\tau\\
&+\int_{\frac{1}{2}}^1\left(\int_0^{\frac{1}{2}}\left(\frac{1}{2}-s\right)ds+\int_{\frac{1}{2}}^\tau\left(s-\frac{1}{2}\right)\right)(1-\tau)\,d\tau\\
&=\int_0^{\frac{1}{2}}\left(\frac{1}{2}\tau-\frac{\tau^2}{2}\right)(1-\tau)\,d\tau\\
&+\int_{\frac{1}{2}}^1\left(\int_0^{\frac{1}{2}}\left(\frac{1}{2}-s\right)ds+\int_{\frac{1}{2}}^\tau\left(s-\frac{1}{2}\right)ds\right)(1-\tau)\,d\tau.
\end{aligned}$$

We have

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\left(\frac{1}{2}\tau-\frac{\tau^2}{2}\right)(1-\tau)\,d\tau=\frac{1}{2}\int_0^{\frac{1}{2}}(1-\tau)\,\tau\,(1-\tau)\,d\tau\\
&=\frac{1}{2}\int_0^{\frac{1}{2}}(1-\tau)^2\,\tau d\tau=\frac{11}{384}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{1}{2}}^1\left(\int_0^{\frac{1}{2}}\left(\frac{1}{2}-s\right)ds+\int_{\frac{1}{2}}^\tau\left(s-\frac{1}{2}\right)ds\right)(1-\tau)\,d\tau\\
&=\int_{\frac{1}{2}}^1\left(\frac{1}{8}+\frac{1}{2}\left(\tau-\frac{1}{2}\right)^2\right)(1-\tau)\,d\tau\\
&=\frac{1}{8}\int_{\frac{1}{2}}^1(1-\tau)\,d\tau+\frac{1}{2}\int_{\frac{1}{2}}^1\left(\tau-\frac{1}{2}\right)^2(1-\tau)\,d\tau=\frac{7}{384}.
\end{aligned}$$

Therefore

$$\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau=\frac{3}{64}.$$

Since $\int_0^1 \left|\tau - \frac{1}{2}\right| d\tau = \frac{1}{4}$, hence

$$\frac{1}{\int_0^1 p(\tau)\,d\tau} \int_0^1 \left(\int_0^\tau p(s)\,ds \right) (1-\tau)\,d\tau = \frac{3}{16}.$$

Utilising (3.13) for symmetric weight $p : [0,1] \to [0,\infty)$, $p(s) = \left|s - \frac{1}{2}\right|$, we get

$$\begin{aligned} &-\frac{3}{16}\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right] \\ &\le 4\int_0^1 \left|\tau - \frac{1}{2}\right| f((1-\tau)A + \tau B)\,d\tau - \int_0^1 f((1-\tau)A+\tau B)\,d\tau \\ &\le \frac{3}{16}\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right], \end{aligned} \tag{3.15}$$

where f is an operator convex function on I, A, $B \in \mathcal{SA}_I(H)$, with $A \neq B$ and $f \in \mathcal{G}([A,B])$.

Consider now the symmetric function $p(s) = (1-s)s$, $x \in [0,1]$. Then

$$\int_0^\tau p(s)\,ds = \int_a^\tau (1-s)\,sds = -\frac{1}{6}\tau^2(2\tau - 3),\ \tau \in [0,1]$$

and

$$\int_0^1 \left(\int_0^\tau p(s)\,ds \right) (1-\tau)\,d\tau = -\frac{1}{6}\int_0^1 \tau^2(2\tau-3)(1-\tau)\,d\tau = \frac{1}{40}.$$

Also

$$\int_0^1 p(\tau)\,d\tau = \int_0^1 (1-\tau)\,\tau d\tau = \frac{1}{6}$$

and

$$\frac{1}{\int_0^1 p(\tau)\,d\tau} \int_0^1 \left(\int_0^\tau p(s)\,ds \right)(1-\tau)\,d\tau = \frac{3}{20}$$

and by (3.13) we obtain

$$\begin{aligned} &-\frac{3}{20}\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right] \\ &\le 6\int_0^1 (1-\tau)\,\tau f((1-\tau)A + \tau B)\,d\tau - \int_0^1 f((1-\tau)A+\tau B)\,d\tau \\ &\le \frac{3}{20}\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right], \end{aligned} \tag{3.16}$$

where f is an operator convex function on I, A, $B \in \mathcal{SA}_I(H)$, with $A \neq B$ and $f \in \mathcal{G}([A,B])$.

§4. Some examples

The function $f(x) = x^{-1}$ is operator convex on $(0,\infty)$, operator Gâteaux differentiable and

$$\nabla f_T(S) = -T^{-1}ST^{-1}$$

for T, $S > 0$.

If $p : [0,1] \to \mathbb{R}$ is a Lebesgue integrable and symmetric function such that the condition (3.6) holds, then we have

$$\begin{aligned}(4.1)\quad & -\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \\ & \times\left[A^{-1}(B-A)A^{-1} - B^{-1}(B-A)B^{-1}\right] \\ & \le \frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 p(\tau)\left((1-\tau)A+\tau B\right)^{-1}d\tau - \int_0^1\left((1-\tau)A+\tau B\right)^{-1}d\tau \\ & \le \frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \\ & \times\left[A^{-1}(B-A)A^{-1} - B^{-1}(B-A)B^{-1}\right]\end{aligned}$$

for all A, $B > 0$.

In particular,

$$\begin{aligned}(4.2)\quad & -\frac{3}{16}\left[A^{-1}(B-A)A^{-1} - B^{-1}(B-A)B^{-1}\right] \\ & \le 4\int_0^1\left|\tau-\frac{1}{2}\right|\left((1-\tau)A+\tau B\right)^{-1}d\tau - \int_0^1\left((1-\tau)A+\tau B\right)^{-1}d\tau \\ & \le \frac{3}{16}\left[A^{-1}(B-A)A^{-1} - B^{-1}(B-A)B^{-1}\right],\end{aligned}$$

and

$$\begin{aligned}(4.3)\quad & -\frac{3}{20}\left[A^{-1}(B-A)A^{-1} - B^{-1}(B-A)B^{-1}\right] \\ & \le 6\int_0^1(1-\tau)\tau\left((1-\tau)A+\tau B\right)^{-1}d\tau - \int_0^1\left((1-\tau)A+\tau B\right)^{-1}d\tau \\ & \le \frac{3}{20}\left[A^{-1}(B-A)A^{-1} - B^{-1}(B-A)B^{-1}\right]\end{aligned}$$

for all A, $B > 0$.

We note that the function $f(x) = -\ln x$ is operator convex on $(0,\infty)$. The ln function is operator Gâteaux differentiable with the following explicit formula for the derivative (cf. Pedersen [15, p. 155]):

$$(4.4)\qquad \nabla \ln_T(S) = \int_0^\infty (s1_H+T)^{-1}S(s1_H+T)^{-1}\,ds$$

for $T,\, S > 0$.

If we write the inequality (3.6) for $-\ln$ and $p : [0,1] \to \mathbb{R}$ is Lebesgue integrable and symmetric function such that the condition (3.6) holds, then we get

$$\begin{aligned} &-\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \\ &\times\left[\int_0^\infty (s1_H+A)^{-1}(B-A)(s1_H+A)^{-1}\,ds\right. \\ &\left.-\int_0^\infty (s1_H+B)^{-1}(B-A)(s1_H+B)^{-1}\,ds\right] \\ &\le \int_0^1 \ln((1-\tau)A+\tau B)\,d\tau \\ &-\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 p(\tau)\ln((1-\tau)A+\tau B)\,d\tau \\ &\le \frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \\ &\times\left[\int_0^\infty (s1_H+A)^{-1}(B-A)(s1_H+A)^{-1}\,ds\right. \\ &\left.-\int_0^\infty (s1_H+B)^{-1}(B-A)(s1_H+B)^{-1}\,ds\right] \end{aligned} \tag{4.5}$$

for all $A,\, B > 0$.

If we take in (4.5) $p(\tau) = \left|\tau - \frac{1}{2}\right|$, $\tau \in [0,1]$, then we get

$$\begin{aligned} &-\frac{3}{16}\left[\int_0^\infty (s1_H+A)^{-1}(B-A)(s1_H+A)^{-1}\,ds\right. \\ &\left.-\int_0^\infty (s1_H+B)^{-1}(B-A)(s1_H+B)^{-1}\,ds\right] \\ &\le \int_0^1 \ln((1-\tau)A+\tau B)\,d\tau \\ &-4\int_0^1\left|\tau-\frac{1}{2}\right|\ln((1-\tau)A+\tau B)\,d\tau \\ &\le \frac{3}{16}\left[\int_0^\infty (s1_H+A)^{-1}(B-A)(s1_H+A)^{-1}\,ds\right. \\ &\left.-\int_0^\infty (s1_H+B)^{-1}(B-A)(s1_H+B)^{-1}\,ds\right] \end{aligned} \tag{4.6}$$

for all $A,\, B > 0$.

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