

Certain curvature properties of $N(k)$ -quasi Einstein manifolds

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Abstract. In this paper, we consider $N(k)$ -quasi Einstein manifolds satisfying the curvature conditions $\bar{P}(\xi, X) \cdot W_2 = 0$ and $\bar{P}(\xi, X) \cdot H = 0$, where \bar{P} , W_2 and H are the pseudo projective, W_2 and conharmonic curvature tensors respectively. We study pseudo projectively symmetric $N(k)$ -quasi Einstein manifolds and show that there does not exist a pseudo projectively semisymmetric $N(k)$ -quasi Einstein manifold. Also, we construct some examples to support the existence of such manifolds..

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§1. Introduction

An n -dimensional Riemannian or semi-Riemannian manifold $M(n > 2)$ is said to be an Einstein manifold if it satisfies

$$(1.1) \quad S = \frac{r}{n}g,$$

where S and r are the Ricci tensor and the scalar curvature respectively. Equation (1.1) is called the Einstein metric condition [1]. Einstein manifolds are important in the study of Riemannian geometry and in general theory of relativity. Also, Einstein manifolds form a natural subclass of various classes of Riemannian or semi-Riemannian manifolds by a curvature condition imposed on their Ricci tensor [1].

The notion of a quasi Einstein manifold was introduced during the study of exact solutions to the Einstein field equations and consideration of quasi-umbilical hypersurfaces [3]. A non-flat Riemannian manifold (M, g) is called

a quasi Einstein manifold its Ricci tensor S satisfies

$$(1.2) \quad S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),$$

$\forall X, Y \in T(M)$ where a and b are smooth functions, $b \neq 0$ called the associated scalars, η is a non zero 1-form defined by

$$(1.3) \quad g(X, \xi) = \eta(X), \quad g(\xi, \xi) = 1,$$

called the associated 1-form and the unit vector field ξ is called the generator of the manifold. Chaki [2], Guha [13], De and Ghosh [7, 8] and several other geometers continued the study of quasi Einstein manifolds. Özgür also studied generalized quasi Einstein manifolds [17] and super quasi Einstein manifolds [18]. The k -nullity distribution of a Riemannian manifold M is defined as

$$N(k) : p \rightarrow N_p(k) = \{Z \in T_p(M) : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]\}$$

for some smooth function k [24]. If the generator ξ in a quasi Einstein manifold M belongs to some k -nullity distribution, then M is called an $N(k)$ -quasi Einstein manifold [19].

In an n -dimensional $N(k)$ -quasi Einstein manifold, k is not arbitrary and is given by [19]

$$(1.4) \quad k = \frac{a+b}{n-1}.$$

Also, we have [19]

$$(1.5) \quad R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y],$$

$$(1.6) \quad R(X, \xi)Y = k[\eta(Y)X - g(X, Y)\xi] = -R(\xi, X)Y,$$

$$(1.7) \quad \eta(R(X, Y)Z) = k[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)].$$

In 2011, Hosseinzadeh and Taleshian [14] considered $N(k)$ -quasi Einstein manifolds satisfying the curvature conditions $C(\xi, X) \cdot S = 0$, $\tilde{C}(\xi, X) \cdot S = 0$, $\bar{P}(\xi, X) \cdot C = 0$, $P(\xi, X) \cdot \tilde{C} = 0$ and $\bar{P}(\xi, X) \cdot \tilde{C} = 0$, where C, \tilde{C}, P and \bar{P} denote the conformal, quasi conformal, projective and pseudo projective curvature tensor respectively. Hui and Lemence [15] studied $N(k)$ -quasi Einstein manifolds admitting W_2 -curvature tensor and generalized Ricci recurrent $N(k)$ -quasi Einstein manifolds. Chaubey [4] studied $N(k)$ -quasi Einstein manifolds satisfying the curvature conditions $P(\xi, X) \cdot W^* = 0$, $P(\xi, X) \cdot \tilde{C} = 0$

and pseudo Ricci symmetric quasi Einstein manifolds, where W^* is the m -projective curvature tensor and \tilde{C} is the concircular curvature tensor. $N(k)$ -quasi Einstein manifolds have been studied by several authors such as Tripathi and Kim [25], De and Mallick [9], Yildiz et al. [26] and others.

This paper is organized as: After the preliminaries in Section 2, we construct some examples of $N(k)$ -quasi Einstein manifolds in Section 3. In Section 4, we study certain properties of the pseudo projective curvature tensor in $N(k)$ -quasi Einstein manifolds. Also, it is shown that there is no pseudo projective-symmetric $N(k)$ -quasi Einstein manifold. Finally, we obtain a condition for an $N(k)$ -quasi Einstein manifold to be conharmonically pseudosymmetric.

§2. Preliminaries

From (1.2) and (1.3), we have

$$(2.1) \quad r = an + b, \quad QX = aX + b\eta(X)\xi,$$

$$(2.2) \quad S(X, \xi) = k(n - 1)\eta(X),$$

where r is the scalar curvature and Q is the Ricci operator.

The pseudo projective curvature tensor [21], the conharmonic curvature tensor [10] and the W_2 -curvature tensor [20] are defined as

$$(2.3) \quad \begin{aligned} \bar{P}(X, Y)Z &= \alpha R(X, Y)Z + \beta[S(Y, Z)X - S(X, Z)Y] \\ &\quad - \frac{r}{n} \left[\frac{\alpha}{(n-1)} + \beta \right] [g(Y, Z)X - g(X, Z)Y] \end{aligned}$$

where α and β are constants such that $\alpha, \beta \neq 0$,

$$(2.4) \quad \begin{aligned} H(X, Y)Z &= R(X, Y)Z - \frac{1}{(n-2)} [S(X, Z)Y - S(Y, Z)X \\ &\quad + g(Y, Z)QX - g(X, Z)QY], \end{aligned}$$

$$(2.5) \quad W_2(X, Y)Z = R(X, Y)Z - \frac{1}{(n-1)} [g(Y, Z)QX - g(X, Z)QY],$$

for arbitrary vector fields X, Y, Z .

Using equations (1.5), (1.6), (1.7), (2.1) and (2.2) in (2.3), (2.4), (2.5) we have

$$(2.6) \quad \bar{P}(X, Y)\xi = \left[\frac{\beta(n-1) + \alpha}{b} \right] \{ \eta(Y)X - \eta(X)Y \},$$

$$(2.7) \quad \begin{aligned} \bar{P}(\xi, X)Y &= \frac{(\alpha - \beta)}{n} \{g(X, Y)\xi - \eta(Y)X\} \\ &+ \beta b \{\eta(Y)\eta(X) - \eta(Y)X\}, \end{aligned}$$

$$(2.8) \quad \eta(\bar{P}(X, Y)Z) = \frac{(\alpha - \beta)}{n} \{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\},$$

$$(2.9) \quad H(X, Y)\xi = \frac{(na + b)}{(n - 1)(n - 2)} \{\eta(Y)X - \eta(X)Y\},$$

$$(2.10) \quad H(\xi, X)Y = \frac{(na + b)}{(n - 1)(n - 2)} \{g(X, Y)\xi - \eta(Y)X\},$$

$$(2.11) \quad \begin{aligned} \eta(H(X, Y)Z) &= \frac{(na + b)}{(n - 1)(n - 2)} \{g(Y, Z)\eta(X) \\ &- g(X, Z)\eta(Y)\}, \end{aligned}$$

$$(2.12) \quad \eta(W_2(X, Y)Z) = 0.$$

§3. Examples of $N(k)$ -quasi Einstein manifolds

Example 1. Consider a pseudo projectively flat quasi Einstein manifold. Then, from (2.3), we have

$$(3.1) \quad \begin{aligned} \alpha R(X, Y)Z &= -\beta [S(Y, Z)X - S(X, Z)Y] \\ &+ \frac{r}{n} \left[\frac{\alpha}{n-1} + \beta \right] [g(Y, Z)X - g(X, Z)Y] \end{aligned}$$

Using (1.1) in (3.1), we get

$$(3.2) \quad \begin{aligned} \alpha R(X, Y)Z &= -\left\{ \beta a - \frac{r}{n} \left(\frac{\alpha}{n-1} + \beta \right) \right\} [g(Y, Z)X - g(X, Z)Y] \\ &- \beta b [\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \end{aligned}$$

Replacing Z by ξ in (3.2) we have,

$$R(X, Y)\xi = \left[\frac{r}{n(n-1)} - \frac{\beta b}{n} \left(\frac{n-1}{n} \right) \right] [\eta(Y)X - \eta(X)Y],$$

which shows that ξ belongs to the $\left(\frac{r}{n(n-1)} - \frac{\beta b}{n} \left(\frac{n-1}{n} \right) \right)$ -nullity distribution. Therefore, we can state:

Theorem 3.1. *A pseudo projectively flat quasi Einstein manifold is an $N\left(\frac{r}{n(n-1)} - \frac{\beta b}{n}\left(\frac{n-1}{n}\right)\right)$ -quasi Einstein manifold.*

Example 2. Consider a quasi Einstein manifold which is conharmonically flat. Then by equation (2.4), we have

$$(3.3) \quad R(X, Y)Z = \frac{1}{(n-2)} \left[S(X, Z)Y - S(Y, Z)X + g(Y, Z)QX - g(X, Z)QY \right].$$

Using (1.1) and (2.1) in (3.3), we have

$$(3.4) \quad R(X, Y)Z = \frac{1}{(n-2)} \left[2a\{g(Y, Z)X - g(X, Z)Y\} + b\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi\} \right].$$

Substituting $Z = \xi$, equation (3.4) reduces to

$$R(X, Y)\xi = \left(\frac{2a+b}{n-2}\right) [\eta(Y)X - \eta(X)Y],$$

showing that the manifold is an $N\left(\frac{2a+b}{n-2}\right)$ -quasi Einstein manifold. Thus, we have the following theorem:

Theorem 3.2. *A conharmonically flat quasi Einstein manifold is an $N\left(\frac{2a+b}{n-2}\right)$ -quasi Einstein manifold.*

Example 3. Consider the 4-dimensional Riemannian space (\mathbb{R}^4, g) endowed with the metric g given by

$$(3.5) \quad ds^2 = g_{ij}dx^i dx^j = (1+2q)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2],$$

$(i, j = 1, 2, 3, 4)$ where $q = \frac{e^{x^1}}{k^2}$ and k is a non zero constant. Then, (\mathbb{R}^4, g) is an $N\left(\frac{q}{(1+2q)^3}\right)$ -quasi Einstein manifold [11].

Example 4. Consider a Riemannian metric g on $\mathbb{R}^4 = \{(x^1, x^2, x^3, x^4) : x^i \in \mathbb{R}\}$ defined by

$$(3.6) \quad ds^2 = g_{ij}dx^i dx^j = (x^4)^{\frac{4}{3}}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] - (dx^4)^2,$$

$(i, j = 1, 2, 3, 4)$. Then, (\mathbb{R}^4, g) is an $N\left(\frac{[1-18(x^4)^4]}{9(x^4)^2}\right)$ -quasi Einstein manifold [6].

Example 5: A special Para-Sasakian manifold with vanishing D -concurvature tensor is an $N(k)$ -quasi Einstein manifold [6].

Example 6: A four-dimensional conformally flat perfect fluid (M^4, g) is an $N\left(\frac{1}{3}\left\{\frac{1}{2}r + f(T) + 8\pi\rho + 2(\rho + p)f'(T)\right\}\right)$ -quasi Einstein manifold, where ρ is the energy density and p is the pressure [5].

Example 7: An n -dimensional conformally flat Kenmotsu manifold equipped with Ricci soliton is an $N(k)$ -quasi Einstein manifold [5].

Example 8: Every m -projectively flat quasi Einstein manifold is an $N\left(\frac{2a+b}{2(n-1)}\right)$ -quasi Einstein manifold [4].

§4. Pseudo projective curvature tensor in an $N(k)$ -quasi Einstein manifold

Theorem 4.1. *An n -dimensional $N(k)$ -quasi Einstein manifold M satisfies the curvature condition $\bar{P}(\xi, X) \cdot W_2 = 0$ provided $\alpha = \beta$ or the manifold is W_2 -flat.*

Proof. Suppose M satisfies the curvature condition $\bar{P}(\xi, X) \cdot W_2 = 0$. Then,

$$(4.1) \quad \begin{aligned} \bar{P}(\xi, X)W_2(U, V)Z &- W_2(\bar{P}(\xi, X)U, V)Z - W_2(U, \bar{P}(\xi, X)V)Z \\ &- W_2(U, V)\bar{P}(\xi, X)Z = 0, \end{aligned}$$

for all vector fields $U, V, Z, X \in M$.

Using (2.7) in (4.1), we have

$$(4.2) \quad \begin{aligned} &b \left[\frac{(\alpha - \beta)}{n} \left\{ W_2'(U, V, Z, X)\xi - \eta(W_2(U, V)Z)X \right. \right. \\ &\quad - g(X, U)W_2(\xi, V)Z + \eta(U)W_2(X, V)Z \\ &\quad - g(X, V)W_2(U, \xi)Z + \eta(V)W_2(U, X)Z \\ &\quad \left. \left. - g(X, Z)W_2(U, V)\xi + \eta(Z)W_2(U, V)X \right\} \right. \\ &\quad + \beta \left\{ \eta(X)\eta(W_2(U, V)Z)\xi - \eta(W_2(U, V)Z)X \right. \\ &\quad - \eta(X)\eta(U)W_2(\xi, V)Z + \eta(U)W_2(X, V)Z \\ &\quad - \eta(X)\eta(V)W_2(U, \xi)Z + \eta(V)W_2(U, X)Z \\ &\quad \left. \left. - \eta(X)\eta(Z)W_2(U, V)\xi + \eta(Z)W_2(U, V)X \right\} \right] = 0. \end{aligned}$$

Since $b \neq 0$, equation (4.2) can be written as

$$(4.3) \quad \begin{aligned} & \frac{(\alpha - \beta)}{n} \left\{ W_2'(U, V, Z, X)\xi - \eta(W_2(U, V)Z)X \right. \\ & - g(X, U)W_2(\xi, V)Z + \eta(U)W_2(X, V)Z \\ & - g(X, V)W_2(U, \xi)Z + \eta(V)W_2(U, X)Z \\ & - g(X, Z)W_2(U, V)\xi + \eta(Z)W_2(U, V)X \left. \right\} \\ & + \beta \left\{ \eta(X)\eta(W_2(U, V)Z)\xi - \eta(W_2(U, V)Z)X \right. \\ & - \eta(X)\eta(U)W_2(\xi, V)Z + \eta(U)W_2(X, V)Z \\ & - \eta(X)\eta(V)W_2(U, \xi)Z + \eta(V)W_2(U, X)Z \\ & \left. - \eta(X)\eta(Z)W_2(U, V)\xi + \eta(Z)W_2(U, V)X \right\} = 0. \end{aligned}$$

Taking inner product of (4.3) with respect to ξ , we have

$$(4.4) \quad \begin{aligned} & \frac{(\alpha - \beta)}{n} \left\{ W_2'(U, V, Z, X) - \eta(W_2(U, V)Z)\eta(X) \right. \\ & - g(X, U)\eta(W_2(\xi, V)Z) + \eta(U)\eta(W_2(X, V)Z) \\ & - g(X, V)\eta(W_2(U, \xi)Z) + \eta(V)\eta(W_2(U, X)Z) \\ & \left. - g(X, Z)\eta(W_2(U, V)\xi) + \eta(Z)\eta(W_2(U, V)X) \right\} \\ & + \beta \left\{ \eta(X)\eta(W_2(U, V)Z) - \eta(W_2(U, V)Z)\eta(X) \right. \\ & - \eta(X)\eta(U)\eta(W_2(\xi, V)Z) + \eta(U)\eta(W_2(X, V)Z) \\ & - \eta(X)\eta(V)\eta(W_2(U, \xi)Z) + \eta(V)\eta(W_2(U, X)Z) \\ & \left. - \eta(X)\eta(Z)\eta(W_2(U, V)\xi) + \eta(Z)\eta(W_2(U, V)X) \right\} = 0. \end{aligned}$$

From (2.12) and (4.4), it follows that

$$\frac{(\alpha - \beta)}{n} W_2'(U, V, Z, X) = 0.$$

Since $n > 2$ this implies that

$$\alpha = \beta \quad \text{or} \quad W_2 = 0.$$

This completes the proof. \square

Theorem 4.2. *Let M be an n -dimensional $N(k)$ -quasi Einstein manifold. Then M satisfies the curvature condition $\bar{P}(\xi, X) \cdot H = 0$ if $\alpha = \beta$ or the manifold is conharmonically flat.*

Proof. Suppose M satisfies the curvature condition $\bar{P}(\xi, X) \cdot H = 0$. Then, we can write

$$(4.5) \quad \begin{aligned} \bar{P}(\xi, X)H(U, V)Z &- H(\bar{P}(\xi, X)U, V)Z - H(U, \bar{P}(\xi, X)V)Z \\ &- H(U, V)\bar{P}(\xi, X)Z = 0. \end{aligned}$$

Using (2.8) in (4.5), we have

$$(4.6) \quad \begin{aligned} &b \left[\frac{(\alpha - \beta)}{n} \left\{ H'(U, V, Z, X)\xi - \eta(H(U, V)Z)X \right. \right. \\ &\quad - g(X, U)H(\xi, V)Z + \eta(U)H(X, V)Z \\ &\quad - g(X, V)H(U, \xi)Z + \eta(V)H(U, X)Z \\ &\quad \left. - g(X, Z)H(U, V)\xi + \eta(Z)H(U, V)X \right\} \\ &\quad + \beta \left\{ \eta(X)\eta(H(U, V)Z)\xi - \eta(H(U, V)Z)X \right. \\ &\quad - \eta(X)\eta(U)H(\xi, V)Z + \eta(U)H(X, V)Z \\ &\quad - \eta(X)\eta(V)H(U, \xi)Z + \eta(V)H(U, X)Z \\ &\quad \left. \left. - \eta(X)\eta(Z)H(U, V)\xi + \eta(Z)H(U, V)X \right\} \right] = 0. \end{aligned}$$

Since $b \neq 0$, (4.6) can be written as

$$(4.7) \quad \begin{aligned} &\frac{(\alpha - \beta)}{n} \left\{ H'(U, V, Z, X)\xi - \eta(H(U, V)Z)X \right. \\ &\quad - g(X, U)H(\xi, V)Z + \eta(U)H(X, V)Z \\ &\quad - g(X, V)H(U, \xi)Z + \eta(V)H(U, X)Z \\ &\quad \left. - g(X, Z)H(U, V)\xi + \eta(Z)H(U, V)X \right\} \\ &\quad + \beta \left\{ \eta(X)\eta(H(U, V)Z)\xi - \eta(H(U, V)Z)X \right. \\ &\quad - \eta(X)\eta(U)H(\xi, V)Z + \eta(U)H(X, V)Z \\ &\quad - \eta(X)\eta(V)H(U, \xi)Z + \eta(V)H(U, X)Z \\ &\quad \left. - \eta(X)\eta(Z)H(U, V)\xi + \eta(Z)H(U, V)X \right\} = 0. \end{aligned}$$

Taking inner product of (4.7) with respect to ξ , we have

$$\begin{aligned}
 (4.8) \quad & \frac{(\alpha - \beta)}{n} \left\{ H'(U, V, Z, X) - \eta(H(U, V)Z)\eta(X) \right. \\
 & - g(X, U)\eta(H(\xi, V)Z) + \eta(U)\eta(H(X, V)Z) \\
 & - g(X, V)\eta(H(U, \xi)Z) + \eta(V)\eta(H(U, X)Z) \\
 & \left. - g(X, Z)\eta(H(U, V)\xi) + \eta(Z)\eta(H(U, V)X) \right\} \\
 & + \beta \left\{ \eta(X)\eta(H(U, V)Z) - \eta(H(U, V)Z)\eta(X) \right. \\
 & - \eta(X)\eta(U)\eta(H(\xi, V)Z) + \eta(U)\eta(H(X, V)Z) \\
 & - \eta(X)\eta(V)\eta(H(U, \xi)Z) + \eta(V)\eta(H(U, X)Z) \\
 & \left. - \eta(X)\eta(Z)\eta(H(U, V)\xi) + \eta(Z)\eta(H(U, V)X) \right\} = 0.
 \end{aligned}$$

Using (2.10) in (4.8), we get

$$\begin{aligned}
 (4.9) \quad & \frac{(\alpha - \beta)}{n} \left[H'(U, V, Z, X) + \frac{(na + b)}{(n - 1)(n - 2)} \{g(X, U)g(V, Z) \right. \\
 & \left. - g(X, V)g(U, Z)\} \right] - \beta \frac{(na + b)}{(n - 1)(n - 2)} \left[\eta(X)\eta(U)g(V, Z) \right. \\
 & \left. - \eta(X)\eta(V)g(U, Z) \right] = 0.
 \end{aligned}$$

Making use of (2.4) in (4.9), we obtain

$$\begin{aligned}
 (4.10) \quad & \frac{(\alpha - \beta)}{n} \left[R'(U, V, Z, X) - \frac{1}{(n - 2)} \{S(V, Z)g(X, U) \right. \\
 & \left. - S(U, Z)g(X, V) + g(V, Z)S(X, U) - g(U, Z)S(X, V)\} \right. \\
 & \left. + \frac{(na + b)}{(n - 1)(n - 2)} \{g(V, Z)g(X, U) - g(U, Z)g(X, V)\} \right] \\
 & + \beta \frac{(na + b)}{(n - 1)(n - 2)} \left[\eta(V)\eta(Z)g(X, U) - \eta(U)\eta(Z)g(X, V) \right] = 0.
 \end{aligned}$$

Taking $U = X = e_i$ in (4.10) and summing over $i, 1 \leq i \leq n$, we get

$$\beta \left(\frac{na + b}{n - 2} \right) \eta(V)\eta(Z) = 0.$$

Since $n > 2$, $\beta \neq 0$ and $\eta \neq 0$, we have

$$na + b = 0.$$

Using this in (4.9), it follows that

$$\begin{aligned} \frac{(\alpha - \beta)}{n} H'(U, V, Z, X) &= 0, \\ \Rightarrow \alpha = \beta = 0 \quad \text{or} \quad H'(U, V, Z, X) &= 0, \end{aligned}$$

which proves the theorem. \square

Definition 4.1. A Riemannian manifold is said to be semisymmetric [22, 23] if

$$(4.11) \quad R \cdot R = 0.$$

Consider an $N(k)$ -quasi Einstein manifold which is pseudo projectively semisymmetric. Then,

$$(4.12) \quad \begin{aligned} (R(X, Y) \cdot \bar{P})(U, V)W &= 0, \\ R(X, Y)\bar{P}(U, V)W - \bar{P}(R(X, Y)U, V)W - \bar{P}(U, R(X, Y)V)W \\ &\quad - \bar{P}(U, V)R(X, Y)W = 0, \end{aligned}$$

Taking inner product of (4.12) with respect to ξ , we have

$$\begin{aligned} g(R(X, Y)\bar{P}(U, V)W, \xi) - g(\bar{P}(R(X, Y)U, V)W, \xi) - g(\bar{P}(U, R(X, Y)V)W, \xi) \\ - g(\bar{P}(U, V)R(X, Y)W, \xi) = 0, \end{aligned}$$

Substituting $X = \xi$, the above equation becomes

$$\begin{aligned} g(R(\xi, Y)\bar{P}(U, V)W, \xi) - g(\bar{P}(R(\xi, Y)U, V)W, \xi) - g(\bar{P}(U, R(\xi, Y)V)W, \xi) \\ - g(\bar{P}(U, V)R(\xi, Y)W, \xi) = 0. \end{aligned}$$

Using (1.6), we have

$$(4.13) \quad k \left[\bar{P}'(U, V, W, Y) - \frac{(\alpha - \beta)}{n} \left\{ g(V, W)g(U, Y) - g(U, W)g(V, Y) \right\} \right] = 0.$$

Assuming $k \neq 0$ and making use of (2.3), (4.13) becomes

$$(4.14) \quad \begin{aligned} \alpha R'(U, V, W, Y) &= -\beta[S(V, W)g(U, Y) - S(U, W)g(V, Y)] \\ &\quad + \left\{ \frac{r}{n} \left(\frac{\alpha}{n-1} + \beta \right) + \frac{(\alpha - \beta)b}{n} \right\} [g(V, W)g(U, Y) \\ &\quad - g(U, W)g(V, Y)]. \end{aligned}$$

Contracting (4.14) with respect to U and Y , we get

$$S(V, W) = \left[\frac{r}{n(n-1)} + \frac{(\alpha - \beta)b}{n(\alpha + \beta(n-1))} \right] g(V, W)$$

showing that the manifold is Einstein, which is not possible since M is an $N(k)$ -quasi Einstein manifold. Thus, we can state:

Theorem 4.3. *There does not exist a pseudo projectively semisymmetric $N(k)$ -quasi Einstein manifold.*

Definition 4.2. *A Riemannian manifold is called a symmetric manifold [12, 16] if*

$$(4.15) \quad (\nabla_X R)(Y, Z)V = 0,$$

where ∇ is the operator of covariant differentiation with respect to the metric g .

Consider a pseudo projectively symmetric $N(k)$ -quasi Einstein manifold. Then,

$$(\nabla_X \bar{P}')(Z, U, V, W) = 0.$$

By virtue of (2.3), the above equation can be written as

$$(4.16) \quad \alpha(\nabla_X R')(Z, U, V, W) + \beta[(\nabla_X S)((U, V)g(Z, W) \\ - (\nabla_X S)(Z, V)g(U, W)] - \frac{dr(X)}{n} \left(\frac{\alpha}{n-1} + \beta \right) [g(U, V)g(Z, W) \\ - g(Z, V)g(U, W)] = 0.$$

Contracting (4.16) with respect to Z and W , we obtain

$$(4.17) \quad (\nabla_X S)(U, V) = \frac{dr(X)}{n} g(U, V).$$

Using (1.2) in equation (4.8), we have

$$(4.18) \quad da(X)g(U, V) + db(X)\eta(U)\eta(V) + b[(\nabla_X \eta)(U)\eta(V) \\ + (\nabla_X \eta)(V)\eta(U)] = \frac{dr(X)}{n} g(U, V).$$

Substituting $U = V = \xi$, (4.18) becomes

$$(4.19) \quad dr(X) = n[da(X) + db(X)].$$

Also, taking covariant derivative of (2.1) with respect to X , we obtain

$$(4.20) \quad dr(X) = nda(X) + db(X).$$

From equations (4.19) and (4.20), it follows that

$$(n - 1)db(X) = 0,$$

or

$$db(X) = 0.$$

which implies that b is constant. This leads to the following theorem:

Theorem 4.4. *An $N(k)$ -quasi Einstein manifold is pseudo projectively symmetric provided the associated scalar b is non-zero constant.*

§5. Conharmonically pseudosymmetric $N(k)$ -quasi Einstein manifolds

Definition 5.1. *An $N(k)$ -quasi Einstein manifold is said to be conharmonically pseudosymmetric if the tensors*

$$\begin{aligned} (R(X, Y) \cdot H)(Z, W)U &= R(X, Y)H(Z, W)U - H(R(X, Y)Z, W)U \\ &\quad - H(Z, R(X, Y)W)U - H(Z, W)R(X, Y)U \end{aligned}$$

and

$$\begin{aligned} Q(g, H)(Z, W, U, X, Y) &= (X \wedge_H Y)H(Z, W)U - H((X \wedge_H Y)Z, W)U \\ &\quad - H(Z, (X \wedge_H Y)W)U - H(Z, W)(X \wedge_H Y)U \end{aligned}$$

are linearly dependent, i. e.,

$$(5.1) \quad (R(X, Y) \cdot H)(Z, W)U = L_H \{Q(g, H)(Z, W, U, X, Y)\}$$

for a smooth function $L_H \in A_H = \{x \in M : H \neq 0 \text{ at } x\}$, where X, Y, Z, W, U are arbitrary.

Consider an $N(k)$ -quasi Einstein manifold which is conharmonically pseudosymmetric. Then, from equation (5.1) we have

$$\begin{aligned} (5.2) \quad &R(X, Y)H(Z, W)U - H(R(X, Y)Z, W)U \\ &\quad - H(Z, R(X, Y)W)U - H(Z, W)R(X, Y)U \\ &= L_H \{(X \wedge_H Y)H(Z, W)U - H((X \wedge_H Y)Z, W)U \\ &\quad - H(Z, (X \wedge_H Y)W)U - H(Z, W)(X \wedge_H Y)U\}. \end{aligned}$$

Putting $X = \xi$, (5.2) becomes

$$\begin{aligned} (5.3) \quad &R(\xi, Y)H(Z, W)U - H(R(\xi, Y)Z, W)U \\ &\quad - H(Z, R(\xi, Y)W)U - H(Z, W)R(\xi, Y)U \\ &= L_H \{(\xi \wedge_H Y)H(Z, W)U - H((\xi \wedge_H Y)Z, W)U \\ &\quad - H(Z, (\xi \wedge_H Y)W)U - H(Z, W)(\xi \wedge_H Y)U\}. \end{aligned}$$

Making use of equation (1.6) in (5.3) and

$$(X \wedge_H Y)Z = g(Y, Z)X - g(X, Z)Y,$$

we have

$$\begin{aligned} & (k - L_H)[H'(Z, W, U, Y)\xi - \eta(H(Z, W)U)Y - g(Y, Z)H(\xi, W)U \\ & + \eta(Z)H(Y, W)U - g(Y, W)H(Z, \xi)U + \eta(W)H(Z, Y)U \\ & - g(Y, U)H(Z, W)\xi + \eta(U)H(Z, W)Y] = 0. \end{aligned}$$

Assuming $k \neq L_H$ and taking inner product of the above equation with respect to ξ , we have

$$(5.4) \quad \begin{aligned} H'(Z, W, U, Y) + \frac{na + b}{(n - 1)(n - 2)} [g(Z, Y)g(W, U) \\ - g(Z, U)g(W, Y)] = 0. \end{aligned}$$

From equations (2.4) and (5.4), we obtain

$$\begin{aligned} R'(Z, W, U, Y) &= a_1[g(Z, Y)g(W, U) - g(Z, U)g(W, Y)] \\ &+ a_2[\eta(W)\eta(U)g(Y, Z) - \eta(Z)\eta(U)g(W, Y) \\ &+ \eta(Z)\eta(Y)g(W, U) - \eta(W)\eta(Y)g(Z, U)], \end{aligned}$$

where $a_1 = \frac{a}{n - 1} - \frac{b}{(n - 1)(n - 2)}$ and $a_2 = \frac{b}{n - 2}$. This leads to the following theorem:

Theorem 5.1. *An n -dimensional $N(k)$ -quasi Einstein manifold which is conharmonically pseudosymmetric and $k \neq L_H$ is of quasi-constant curvature.*

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