SUT Journal of Mathematics
Vol. 56, No. 1 (2020), 1–19

L^2-properties for linearized KdV equation around small solutions

Masaki Kawamoto

(Received April 15, 2019)

Abstract. We consider the asymptotic behavior of a small solution to the linearized KdV equation. By rewriting this equation as a Hamiltonian system, the deduced Hamiltonian has unbounded, non-symmetric, and time-dependent potential. In this paper, we show the stableness of this solution to a linearized KdV equation in the L^2 sense and the decay estimates by analyzing this system.

AMS 2010 *Mathematics Subject Classification.* 35Q53, 47A45.

Key words and phrases. linearized KdV equation, non-selfadjoint operator, scattering theory, smoothing estimates.

§1. Introduction

The KdV equation with small nonlinear perturbation is written as:

$$\partial_t \phi + \partial_x^3 \phi + 6(\phi \partial_x \phi) = \varepsilon F(\phi), \tag{1.1}$$

where $t, x \in \mathbf{R}$, $\phi = \phi(t,x); \mathbf{R} \times \mathbf{R} \to \mathbf{C}$ is an unknown function, F; $\mathbf{C} \to \mathbf{C}$ is a given function and $\varepsilon > 0$ is a small constant. Here for the constants $\alpha > 0$ and $x_0 \in \mathbf{R}$ and the parameters $x \in \mathbf{R}$ and $t \in \mathbf{R}$, let us define the so-called *soliton* as

$$q = q(t,x) := 2\alpha^2 \mathrm{sech}^2(\alpha(x - 4\alpha^2 t - x_0)).$$

Then q satisfies $\partial_t q + \partial_x^3 q + 6q\partial_x q = 0$, where we note that even if we replace α with $-\alpha$, the quantity of q is equivalent and hence, it is sufficient to consider only the case where $\alpha > 0$. For some constant $\delta > 0$, by substituting the small solution $\phi = \delta q + \varepsilon \psi$, (1.1) will be

$$\delta \partial_t q + \varepsilon \partial_t \psi + (\delta \partial_x^3 q + \varepsilon \partial_x^3 \psi) + 6(\delta q + \varepsilon \psi)(\delta \partial_x q + \varepsilon \partial_x \psi) = \varepsilon F(\delta q + \varepsilon \psi)$$

and is equivalent to

$$\begin{aligned}&\delta\left(\partial_t q+\partial_x^3 q+6q\partial_x q\right)+\varepsilon\left(\partial_t\psi+\partial_x^3\psi+6\delta\partial_x(\psi q)\right)\\&=\varepsilon F(\delta q+\varepsilon\psi)-6\varepsilon^2\psi\psi_x+6\delta\left(1-\delta\right)q\partial_x q.\end{aligned}$$

Using the condition of q and dividing both terms by ε, $\psi=\psi(t,x)$ satisfies the equation;

$$\partial_t\psi+\partial_x^3\psi+6\delta\partial_x(\psi q)=F(\delta q+\varepsilon\psi)-6\varepsilon\psi\psi_x+6\delta\varepsilon^{-1}\left(1-\delta\right)q\partial_x q. \tag{1.2}$$

The aim of this paper is to reduce this equation to a simple form and consider some L^2-properties as solutions to the reduced equation. In the following, we consider the case where q is not only the soliton but also the generalized potential V. In particular, we consider the following time-independent linearized KdV equations with generalized potentials;

$$\begin{cases}\partial_t u_0(t,x)+6\delta\partial_x(2\alpha^2 V_0(t)u_0(t,x))+(\partial_x^3 u_0)(t,x)=0,\\ u_0(0,x)=u_{0,0}\in L^2(\mathbf{R}),\end{cases} \tag{1.3}$$

where $V_0(t)$ is the multiplication operator of $V(\alpha(x-4\alpha^2 t-x_0))$ and $V:\mathbf{R}\to\mathbf{R}$ is a generalized potential, which is defined later. Let $p=-i\partial_x$. Then, by substituting $\mathscr{J}(t)w=u_0$ with $\mathscr{J}(t):=e^{-i(4t\alpha^2+x_0)p}e^{i((x\cdot p+p\cdot x)\log\alpha)/2}$, (see §2), we obtain the reduced system

$$\begin{cases}i\partial_t w=\alpha^3 Hw,\\ w(0,x)=w_0=\mathscr{J}^{-1}(0)u_{0,0},\end{cases} \tag{1.4}$$

with

$$H=-p^3-4p+6\delta(pV+Vp)+i6\delta V',$$

where V and V' are the multiplication operators of $V(x)$ and $V'(x)$, respectively. As is seen in Proposition 2.1 and the comments after this proposition, (1.1) can be decomposed into a linear term (1.4) plus a nonlinear term; hence, the investigation of some of the properties of the solution to the linear equation will be a first step toward considering perturbation in the soliton (1.4). In particular, we prove the L^2-stableness of the solutions $w(t,x)$ and $u_0(t,x)$ in t, (see, Theorem 1.7 and 1.8). By decomposing H into $H=\hat{H}+i6\delta V'$, we notice that $\hat{H}$ is selfadjoint on $L^2(\mathbf{R})$ and hence, we determine that the propagator $e^{\mp it\hat{H}}$ is unitary, that is, for all $u\in L^2(\mathbf{R})$, $\|e^{\mp it\hat{H}}u\|_{L^2(\mathbf{R})}=\|u\|_{L^2(\mathbf{R})}$. We term this condition L^2- the conservation property of $e^{\mp it\hat{H}}$. Selfadjointness implies that $\hat{H}$ is real-valued and hence, we can expect $e^{\mp it\hat{H}}$ to be unitary. However, H has a complex component $i6\delta V'$ and, in general, e^{-itH} will not be a unitary operator. Moreover if H has the complex eigenvalues $z \in \mathbf{C}$, then by taking u as the eigenfunction of H, $\left\|e^{-itH}u\right\|_{L^2(\mathbf{R})} = \left\|e^{-itz}u\right\|_{L^2(\mathbf{R})}$ holds, and is equivalent to $e^{t\mathrm{Im}z}\left\|u\right\|_{L^2(\mathbf{R})}$. According to the sign of $\mathrm{Im}z$; this term diverges to ∞ or converges to 0 (we say e^{-itH} is unstable). Hence, we are interested to establish whether $e^{\mp itH}$ is stable or not and find that under the small condition of δ, $e^{\mp itH}$ is stable. As far as we know, such a result has not been observed yet; this result can be applied to nonlinear problems and so on.

Remark 1.1. In the usual sense, a linearized operator is written as $L = -p^3 - 4p + 6(pV + Vp) + i6V'$, which is obtained by the insertion of $\phi = q + \varepsilon\psi$ in (1.1), see e.g., Sachs [16], Mann [6], Kato-Kawamoto-Nanbu [10] and references therein. In this case, the situation changes significantly, and it is difficult even to prove the nonexistence of the $\hat{L} = L - i6V'$ eigenvalues. Besides this issue, we have to deal with non-selfadjoint perturbation $6iV'$. Unfortunately, a non-small coefficient 6 and the condition of V' so that V' is not always positive or negative make it difficult to apply the previous approaches in the scattering theory for non-selfadjoint perturbation. There are some studies associated with these issues (for the nonexistence of eigenvalues: Froese, Herbst, M. H. Ostenhof, and T. H. Ostenhof [4] and Sigal [17]; for the scattering theory for non-small complex perturbation: Mochizuki [8], Nakazawa [9], Royer [14], and Wang [19])). However, to apply these approaches to L is not easy since V' is not always positive and it is difficult to obtain the particulars of the $\hat{L}$.

Remark 1.2. Consider the KdV equation with generalized coefficients

$$\partial_t P(t,x) + aP(t,x)\partial_x P(t,x) + \gamma\partial_x^3 P(t,x) = 0.$$

Then the soliton of this equation can be written as

$$Q(t,x) = c\cosh^{-2}(b(x - dt - x_0)),$$

where $x_0 \in \mathbf{R}$, $abcd\gamma \neq 0$ are the given constants and these ratios satisfy

$$ac = 12b^2\gamma = 3d.$$

When we consider the perturbation of solitons, we use the substitution $P = \delta Q + \varepsilon R$. Then, the deduced linearized operator coincides with H. Hence, it is sufficient only to consider (1.5) to consider the perturbation of solitons.

The aim of this paper is to prove the stableness of e^{-itH} and its inverse e^{itH} by using the scattering theory. Kato [11] considered the scattering theory for non-selfadjoint operators written in the form $T = T_0 + iW$ with sufficiently small W, and established the Kato methods to prove such issues. To replicate

Kato's approach, we assume that $\delta > 0$ is a small constant. Throughout, we put

$$\beta = 6\delta$$

and suppose $|\beta| \ll 1$,; we also assume that V decays faster than $\langle x \rangle^{-2}$. Specifically, we assume the following:

Assumption 1.3. Assume that $V; \mathbf{R} \to \mathbf{R}$ satisfies $V \in C^3(\mathbf{R})$ and the following decaying condition: for all integers $l \in \mathbf{N} \cup \{0\}$ with $l \leq 3$ and for some constants where $s > 1$, there exist constants $C_{l,s} > 0$ such that

$$\sup_{y \in \mathbf{R}} \left| \langle y \rangle^{2s+l} (\partial^l V)(y) \right| \leq C_{l,s} \tag{1.5}$$

holds, where $\langle \tau \rangle = (1 + \tau^2)^{1/2}$. Moreover, $\delta > 0$ is sufficiently small.

Throughout, if we write s, it is always equivalent to that in Assumption 1.3.

Remark 1.4. The usual soliton $V(y) = \mathrm{sech}^2 y$ satisfies both conditions $V \in C^3(\mathbf{R})$ and (1.5).

Under this assumption, we have the smoothing estimates for $e^{-it\alpha^3 H} w_0$ and $u_0(t,x)$.

Theorem 1.5. Under the assumption 1.3, for all $0 \leq \theta < 1$, the estimates

$$\int_{-\infty}^{\infty} \left\| \langle x \rangle^{-s} \langle p \rangle^{\theta} e^{-it\alpha^3 H} w_0 \right\|^2_{L^2(\mathbf{R}_x)} dt \leq C\alpha^{-3} \|w_0\|^2_{L^2(\mathbf{R})} = C\alpha^{-3} \|u_{0,0}\|^2_{L^2(\mathbf{R})}, \tag{1.6}$$

and

$$\int_{-\infty}^{\infty} \left\| \left\langle \alpha(x - 4\alpha^2 t - x_0) \right\rangle^{-s} \langle p \rangle^{\theta} u_0(t,x) \right\|^2_{L^2(\mathbf{R}_x)} dt \leq C\alpha^{-3} \|u_{0,0}\|^2_{L^2(\mathbf{R})} \tag{1.7}$$

hold, where $w_0(x) = (e^{-iA \log \alpha} e^{ix_0 p} u_{0,0})(x)$ defined in (1.4) and $u_0(t,x)$ is the solution to (1.3).

As an application to the smoothing estimate, we can prove the existence of wave operators and these inverses:

Theorem 1.6. Define $H_0 = -p^3 - 4p$. Suppose Assumption 1.3. Then, the wave operators

$$W^{\pm} = \mathrm{s-}\lim_{t \to \pm\infty} e^{it\alpha^3 H} e^{-it\alpha^3 H_0}$$

exist; these inverses

$$W_{\mathrm{In}}^{\pm} = \mathrm{s}-\lim_{t\to\pm\infty} e^{it\alpha^3 H_0} e^{-it\alpha^3 H}$$

also exist. Moreover, the adjoints of wave operators and these inverses

$$\left(W^{\pm}\right)^* = \mathrm{s}-\lim_{t\to\pm\infty} e^{it\alpha^3 H_0} e^{-it\alpha^3 H^*}$$

and

$$\left(W_{\mathrm{In}}^{\pm}\right)^* = \mathrm{s}-\lim_{t\to\pm\infty} e^{it\alpha^3 H^*} e^{-it\alpha^3 H_0}$$

exist.

By imitating the approach of Kato [11], the existence of wave operators and these inverses provides the L^2-stableness theorem for propagators $e^{-it\alpha^3 H}$ and $e^{-it\alpha^3 H^*}$:

Theorem 1.7. Suppose Assumption 1.3 holds. Then, for all $t \in \mathbf{R}$, there exist (t, α, δ)-independent constants $0 < c_0 \le C_0$ and $0 < c_0^* \le C_0^*$ such that for all $t \in \mathbf{R}$

$$c_0 \|w_0\|_{L^2(\mathbf{R})}^2 \le \|e^{-it\alpha^3 H} w_0\|_{L^2(\mathbf{R})}^2 \le C_0 \|w_0\|_{L^2(\mathbf{R})}^2,$$

and

$$c_0^* \|w_0\|_{L^2(\mathbf{R})}^2 \le \|e^{-it\alpha^3 H^*} w_0\|_{L^2(\mathbf{R})}^2 \le C_0^* \|w_0\|_{L^2(\mathbf{R})}^2$$

hold.

Using this theorem, we finally obtain the stableness of the solutions to (1.3);

Theorem 1.8. Let $u(t,x)$ be a solution to (1.3) and $0 < c_0 \le C_0$ be equivalent to those in Theorem 1.7. Then, for all $t \in \mathbf{R}$

$$c_0 \|u_{0,0}\|_{L^2(\mathbf{R})} \le \|u_0(t,\cdot)\|_{L^2(\mathbf{R})} \le C_0 \|u_{0,0}\|_{L^2(\mathbf{R})}$$

holds.

As for the asymptotic behavior of the solution to (1.3), asymptotic expansion was recently obtained by Guo Quang-Can, Guo Guo-Ping, Hao, Tao and Wang [5]. However, as far as we know, it has not yet been shown that L^2 ensures stableness and smoothing estimates. Our result may apply to the so-called soliton perturbation nonlinear problem.

§2. Reduction steps

Remarking (1.2), let us start by considering the linearized equations written in the form;

$$\begin{cases} \partial_t u(t,x) + 6\delta\partial_x(2\alpha^2 q_0(\alpha(x-4\alpha^2 t - x_0))u(t,x)) + \partial_x^3 u(t,x) \\ \qquad\qquad = -iG(t,x), \\ u(0,x) = u_0 \in L^2(\mathbf{R}), \end{cases}$$

where for $y \in \mathbf{R}$, $q_0(y) = \operatorname{sech}^2(y)$, $G(t,x)$ is defined as

$$G(t,x) = 24i\alpha^5(1-\delta)\delta\varepsilon^{-1} q_0(\alpha(x-4\alpha^2 t - x_0))q_0'(\alpha(x-4\alpha^2 t - x_0)),$$

where $q_0'(y) = (d(\operatorname{sech}^2(\tau))/d\tau)|_{\tau=y}$; reduce this equation to the simplified form. By defining $p = -i\partial_x$ with $i = \sqrt{-1}$, this equation can be written as a Hamiltonian system;

$$i\partial_t u = K(t)u + G(t,x)$$

with

$$K(t) = -p^3 + 6\delta\alpha^2(pQ_0(t) + Q_0(t)p) + 6\delta\alpha^3 iQ_0'(t),$$

where $Q_0(t)$ and $Q_0'(t)$ are the multiplication operators of $q_0(\alpha(x-4\alpha^2 t - x_0))$ and $q_0'(\alpha(x-4\alpha^2t-x_0))$, respectively. Since the operator $K(t)$ depends on time and is non-symmetric, it would be difficult to apply resolvent estimates or spectral theory to it. To avoid the difficulties arising from time-dependence, we use a Galilean transformation and reduce $K(t)$ to the time-independent operator. Because p is selfadjoint on $L^2(\mathbf{R})$, the unitary operator $e^{-i(4\alpha^2 t + x_0)p}$ is well defined; it is called the Galilean transformation. The Galilean transformation $e^{-i(4\alpha^2 t+x_0)p}$ satisfies $e^{i(4\alpha^2t+x_0)p}q_0(\alpha(x-4\alpha^2t-x_0))e^{-i(4\alpha^2t+x_0)p} = q_0(\alpha x)$, $e^{i(4\alpha^2t+x_0)p}p^3e^{-i(4\alpha^2t+x_0)p} = p^3$. Moreover for $u \in \mathscr{S}(\mathbf{R})$, using $\hat{u}(\xi)$, the Fourier transform of u,

$$(e^{i\theta p}u)(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} e^{ix\xi}e^{i\theta\xi}\hat{u}(\xi)d\xi = u(x+\theta)$$

holds for all $\theta \in \mathbf{R}$. Hence, by substituting $u(t,x) = e^{-i(4\alpha^2t+x_0)p}v(t,x)$ for some $v(t,x)$ with $v(0,x) = e^{ix_0p}u_0$, $v(t,x)$ satisfies the differential equations

$$i\partial_t v = \tilde{K}v + \tilde{G}(x), \quad v(0,x) = e^{ix_0p}u_0(x) = u_0(x+x_0)$$

with

$$\tilde{K} = -p^3 - 4\alpha^2 p + 6\delta\alpha^2(Q_0p + pQ_0) + 6\delta i\alpha^3 Q_0',$$

and

$$\tilde{G}(x) = e^{i(4\alpha^2 t + x_0)p} G(t,x) = 24i\alpha^5 \delta\varepsilon^{-1} (1-\delta)\, q_0(\alpha x) q_0'(\alpha x)$$

where Q_0 and Q_0' are multiplication operators of $q_0(\alpha x)$ and $q_0'(\alpha x)$, respectively. We easily see that the operator $\tilde{K}$ is not a symmetric operator of $L^2(\mathbf{R})$ but is independent of time. Next, we introduce the unitary operator $U := e^{iA\log\alpha}$ with $A = (x\cdot p + p\cdot x)/2$ acting on $L^2(\mathbf{R})$; this operator satisfies

$$U^{-1}\begin{pmatrix} x \\ p \end{pmatrix} U = \begin{pmatrix} x/\alpha \\ \alpha p \end{pmatrix}, \quad (U^{-1}f)(x) = \frac{1}{\alpha^{1/2}} f(x/\alpha)$$

on $\mathscr{S}(\mathbf{R})$ and $f \in \mathscr{S}(\mathbf{R})$, respectively. Hence, $U^{-1}\tilde{K}U$ can be written as

$$K := U^{-1}\tilde{K}U, \quad K = \alpha^3(-p^3 - 4p + 6\delta(Qp + pQ) + i6\delta Q'),$$

where Q and Q' are the multiplication operators of $q_0(x)$ and $q_0'(x)$, respectively. Hence, for $\tilde{w} = U^{-1}v$, we obtain an equation

$$i\partial_t \tilde{w} = K\tilde{w} + (U^{-1}\tilde{G})(x).$$

Then, we have the system

$$\begin{cases} i\partial_t \tilde{w} = \alpha^3 \tilde{H}\tilde{w} + G(x), \\ \tilde{w}(0,x) = (U^{-1}e^{ix_0 p}u_0)(x) = (\alpha)^{-1/2} u_0(x/\alpha + x_0), \end{cases}$$
$$\tilde{H} := -p^3 - 4p + 6\delta(pQ + Qp) + i6\delta Q',$$
$$G := 24i\alpha^{9/2}\delta\varepsilon^{-1}(1-\delta)\, q_0(x) q_0'(x).$$

Now we reduce (1.2). If we consider the power type nonlinear term $F(y) = |y|^\rho y$ with $\rho \geq 1$. Then, in (1.2), substituting $\mathscr{J}(t)z = \psi$ with $\mathscr{J}(t) := e^{-i(4t\alpha^2 + x_0)p}U$, we have the equation

$$\begin{aligned} &i\partial_t z(t,x) - \alpha^3(\tilde{H}z)(t,x) \\ &= -6i\varepsilon\alpha^{3/2} z(t,x)\partial_x z(t,x) + 6i\alpha^{9/2}\delta\varepsilon^{-1}(1-\delta)\tilde{q}(x)\tilde{q}'(x) \\ &\quad + i\alpha^{\rho/2}|\varepsilon z(t,x) + \delta\alpha^{3/2}\tilde{q}(x)|^\rho \left(\varepsilon z(t,x) + \delta\alpha^{3/2}\tilde{q}(x)\right) \end{aligned}$$

where $\tilde{q}(x) = 2\mathrm{sech}^2 x$, and we use $(\mathscr{J}(t)z)(t,x) = \sqrt{\alpha} z(t, \alpha(x - 4t\alpha^2 - x_0))$,

$$\begin{aligned} \mathscr{J}^{-1}(t) 2\psi\psi_x &= \mathscr{J}^{-1}\partial_x \mathscr{J}(t) \cdot \mathscr{J}(t)^{-1}\left((\mathscr{J}(t)z\mathscr{J}(t)z)\right) \\ &= \alpha^2 \partial_x \mathscr{J}(t)^{-1}\left(z(t,\alpha(x - 4t\alpha^2 - x_0)) z(t,\alpha(x - 4t\alpha^2 - x_0))\right) \\ &= 2\alpha^{3/2} z(t,x)\partial_x z(t,x) \end{aligned}$$

and

$$
\begin{aligned}
&\mathscr{J}(t)^{-1}|\varepsilon\psi+\delta q|^{\rho}(\varepsilon\psi+\delta q)\\
&=\mathscr{J}(t)^{-1}\left(\alpha^{1/2}\varepsilon z(t,\alpha(x-4t\alpha^2-x_0)+2\delta\alpha^2\mathrm{sech}^2(\alpha(x-4t\alpha^2-x_0))\right)\\
&\quad\times\left|\alpha^{1/2}\varepsilon z(t,\alpha(x-4t\alpha^2-x_0)+2\delta\alpha^2\mathrm{sech}^2(\alpha(x-4t\alpha^2-x_0))\right|^{\rho}\\
&=\alpha^{\rho/2}(\varepsilon z(t,x)+\delta\alpha^{3/2}\tilde{q}(x))|\varepsilon z(t,x)+\delta\alpha^{3/2}\tilde{q}(x)|^{\rho}.
\end{aligned}
$$

Hence, we have the following proposition.

Proposition 2.1. For some $\rho>0$, let $F(\theta)=|\theta|^{\rho}\theta$. Then, the KdV equation (1.2) can be reduced to

$$
\begin{aligned}
&\partial_t z(t,x)+\alpha^3\partial_x^3 z(t,x)-4\alpha^3\partial_x z(t,x)+6\delta\alpha^3\partial_x(\tilde{q}(x)z(t,x))\\
&=-6\varepsilon\alpha^{3/2}z(t,x)\partial_x z(t,x)+6\alpha^{9/2}\delta\varepsilon^{-1}(1-\delta)\tilde{q}(x)\tilde{q}'(x)\\
&\quad+\alpha^{\rho/2}F\left(\varepsilon z(t,x)+\delta\alpha^{3/2}\tilde{q}(x)\right)
\end{aligned}
\tag{2.1}
$$

by substituting $\mathscr{J}(t)z=\psi$.

Here, let us consider the case where $\delta=1$. Then, as said before, the possible existence of complex-valued eigenvalues of $\tilde{H}$ cannot be discounted and if the imaginary part of such eigenvalues is positive, letting $\tilde{H}\phi=(\lambda_R+i\lambda_I)\phi$ with $\lambda_R\in\mathbf{R}$ and $\lambda_I>0$, for $t\geq 0$, the exponential growth

$$\|e^{-it\tilde{H}}\phi\|_2=e^{\lambda_I t}\|\phi\|_2$$

may make it difficult to analyze (2.1). Conversely , if $\delta\ll 1$, the linear equation will be

$$
\begin{cases}
\partial_t W(t,x)+\alpha^3\partial_x^3 W(t,x)-4\alpha^3\partial_x W(t,x)+6\delta\alpha^3\partial_x(\tilde{q}(x)W(t,x))\\
\qquad -6\alpha^{9/2}\delta\varepsilon^{-1}(1-\delta)\tilde{q}(x)\tilde{q}'(x)=0,\\
W(0,x)=W_0,
\end{cases}
$$

this yields

$$W(t,x)=e^{-it\alpha^3\tilde{H}}W_0+6\alpha^{9/2}\delta\varepsilon^{-1}(1-\delta)\int_0^t e^{-i\alpha^3(t-s)H}\tilde{q}(x)\tilde{q}'(x)ds.$$

Hence, we can find the growth order $\|W(t,x)\|_2=O(t)$; this may enable us to analyze (2.1) more easily. This is the merit of considering the small perturbation of solitons .

§3. Stableness of $e^{-it\alpha^3 H}$ and $e^{-it\alpha^3 H^*}$

In this section, we shall prove Theorem 1.7. We let $6\delta = \beta$ and

$$H_0 := -p^3 - 4p, \quad H := H_0 + \beta(pV + Vp) + i\beta V'.$$

The norm of $L^q(\mathbf{R})$, $1 \le q \le \infty$, is denoted as $\|\cdot\|_q$ and the inner product of $L^2(\mathbf{R})$ is denoted as $(\cdot,\cdot)$, i.e., for $u, v \in L^2(\mathbf{R})$,

$$(u, v) := \int_{\mathbf{R}} u(x)\overline{v(x)}dx.$$

The operator norm of $L^2(\mathbf{R})$ is denoted as $\|\cdot\|$, i.e., for some bounded operator A, $\|A\| := \sup_{\|u\|_2=1} \|Au\|_2$.

3.1. Uniform resolvent estimate for H_0

The main objective of this section is to prove the uniform resolvent estimate for the weighted resolvent $\langle x\rangle^{-s} \langle p\rangle^{\theta} (H - \lambda \mp i\mu)^{-1} \langle p\rangle^{\theta} \langle x\rangle^{-s}$, where $\lambda \in \mathbf{R}$, $s > 1$, $0 \le \theta < 1$ and $\mu > 0$. Because H is non-selfadjoint, it may be difficult to deduce a uniform resolvent estimate by using the conventional approaches, e.g., Mourre's theory, to calculate integral kernels and so on. Hence, we initially prove the weighted uniform resolvent estimate for H_0 and extend this result to H.

Lemma 3.1 (Weighted uniform resolvent estimate for H_0)**.** For all $0 \le \theta < 1$ and for all $\phi \in L^2(\mathbf{R})$, there exists a constant $C > 0$ so that

$$\sup_{\lambda\in\mathbf{R},\mu>0} \left\| \langle x\rangle^{-s} \langle p\rangle^{\theta} (H_0 - \lambda \mp i\mu)^{-1} \langle p\rangle^{\theta} \langle x\rangle^{-s} \phi \right\|_2 \le C\|\phi\|_2 \tag{3.1}$$

holds.

Remark 3.2. The smoothing estimate for H_0 has been the focus of many studies, for example, Theorem 5.4. of Ruzhansky–Sugimoto [15] (this paper deals with H_0 and the more general (dispersive) hamiltonian). However, we must deal with the non-selfadjoint operator H and, as far as we know, the scheme for deducing smoothing estimates for generalized operators including non-selfadjoint operators has not yet been obtained . Hence, we must extend the smoothing estimates for H_0 to H; however, this may be difficult even if the perturbation is sufficiently small. The typical strategy to overcome this issue is first to prove the uniform resolvent estimate for H_0 and extend this to H. However, the uniform resolvent estimate for H_0 provides super-smoothness for H_0 and is more powerful than the smoothing estimate. As far as we know, super-smoothness for a generalized operator has not been obtained yet (the high-energy case has been studied by Kawamoto [13]); hence, we must prove this type of estimate for H_0.

Remark 3.3. As for generalized elliptic operators including the Schrödinger operator, the uniform resolvent estimate is proven by stationary scattering theories, such as by the Agmon–Kato–Kuroda theorems (see, e.g., Chihara [1] and references therein). Conversely, our energy $p^3 + 4p$ satisfies $\partial_p(p^3 + 4p) = 3p^2 + 4$, and $(3p^2 + 4)^{-1}$ is the bounded operator. For this case, the time-dependent approach due to [11] §6 (but with some different aspects) works well; hence, we demonstrate the uniform resolvent by using a time-dependent approach.

Proof. We prove (3.1) for all $\phi \in C_0^\infty(\mathbf{R})$ and, thereafter, using the density argument, we deduce (3.1) for all $\phi \in L^2(\mathbf{R})$. Let $\phi \in C_0^\infty(\mathbf{R})$ and $\mu > 0$. For $a > 0$, define $\chi(\cdot \le a)$ as the cut-off function so that $\chi(s \le a) = 1$ for all $s \le a$ and $= 0$ for all $s > a$. Moreover, we denote that $\chi(\cdot > a) = 1 - \chi(\cdot \le a)$. By the Laplace and Fourier transforms, we have

$$\begin{aligned}
&\left\| \langle x\rangle^{-s} \langle p\rangle^{2\theta} (H_0 - \lambda - i\mu)^{-1} \langle x\rangle^{-s} \phi \right\|_2 \\
&= \left\| \langle x\rangle^{-s} \langle p\rangle^{2\theta} \int_0^\infty e^{-it(H_0-\lambda-i\mu)} \langle x\rangle^{-s} \phi dt \right\|_2 \\
&= C_F \left\| \int_0^\infty \int_{\mathbf{R}} \langle x\rangle^{-s} e^{ix\xi} \langle \xi\rangle^{2\theta} e^{it(\xi^3+4\xi+\lambda+i\mu)} \mathscr{F}\left[\langle\cdot\rangle^{-s} \phi\right](\xi) d\xi dt \right\|_2 \\
&\le C_F(I + J)
\end{aligned}$$

with

$$I := \left\| \int_1^\infty \int_{\mathbf{R}} \langle x\rangle^{-s} e^{ix\xi} \langle \xi\rangle^{2\theta} e^{it(\xi^3+4\xi+\lambda+i\mu)} \mathscr{F}\left[\langle\cdot\rangle^{-s} \phi\right](\xi) d\xi dt \right\|_2$$

and

$$J := \left\| \int_0^1 \int_{\mathbf{R}} \langle x\rangle^{-s} e^{ix\xi} \langle \xi\rangle^{2\theta} e^{it(\xi^3+4\xi+\lambda+i\mu)} \mathscr{F}\left[\langle\cdot\rangle^{-s} \phi\right](\xi) d\xi dt \right\|_2,$$

where $C_F = (2\pi)^{-1/2}$ and $\mathscr{F}$ indicates the Fourier transform. For simplicity, we put $E(t, \xi) := e^{it(\xi^3+4\xi+\lambda+i\mu)}$ and $K(\xi) := \langle \xi\rangle^{2\theta} (3\xi^2 + 4)^{-1}$. Since $\phi \in$

$C_0^\infty(\mathbf{R})$, we can use integration by parts with respect to ξ. Then, it holds that

$$\begin{aligned} I &= \left\| \int_1^\infty \int_{\mathbf{R}} \langle x\rangle^{-s} e^{ix\xi} K(\xi) \left(\partial_\xi E(t,\xi)\right) \mathscr{F}\left[\langle\cdot\rangle^{-s}\phi\right](\xi) d\xi \frac{dt}{t} \right\|_2 \\ &\le \left\| \int_1^\infty \int_{\mathbf{R}} \langle x\rangle^{-s} x K(\xi) e^{ix\xi} E(t,\xi) \mathscr{F}\left[\langle\cdot\rangle^{-s}\phi\right](\xi) d\xi \frac{dt}{t} \right\|_2 \\ &\quad + \left\| \int_1^\infty \int_{\mathbf{R}} \langle x\rangle^{-s} K(\xi) e^{ix\xi} E(t,\xi) \mathscr{F}\left[-iy\langle y\rangle^{-s}\phi(y)\right](\xi) d\xi \frac{dt}{t} \right\|_2 \\ &\quad\quad + \left\| \int_1^\infty \int_{\mathbf{R}} \langle x\rangle^{-s} e^{ix\xi} \left(\partial_\xi K(\xi)\right) \left(E(t,\xi)\right) \mathscr{F}\left[\langle\cdot\rangle^{-s}\phi\right](\xi) d\xi \frac{dt}{t} \right\|_2 \\ &\le I_1 + I_2 + I_3 + I_4 + I_5, \end{aligned}$$

where, for $j \in \{1,2,3,4\}$,

$$I_j := \left\| \int_1^\infty \int_{\mathbf{R}} \chi_j(x,\xi) K(\xi) e^{ix\xi} E(t,\xi) d\xi \frac{dt}{t} \right\|_2$$

with

$$\begin{aligned} \chi_1(x,\xi) &:= \langle x\rangle^{-s} x \chi(|x| > t^\delta) \mathscr{F}\left[\langle\cdot\rangle^{-s}\phi\right](\xi), \\ \chi_2(x,\xi) &:= \langle x\rangle^{-s} x \chi(|x| \le t^\delta) \mathscr{F}\left[\langle\cdot\rangle^{-s}\phi\right](\xi), \\ \chi_3(x,\xi) &:= \langle x\rangle^{-s} \mathscr{F}\left[-iy\langle y\rangle^{-s}\chi(|y| > t^\delta)\phi(y)\right](\xi) \\ \chi_4(x,\xi) &:= \langle x\rangle^{-s} \mathscr{F}\left[-iy\langle y\rangle^{-s}\chi(|y| \le t^\delta)\phi(y)\right](\xi) \end{aligned}$$

for some $\delta > 0$, and

$$I_5 := \left\| \langle x\rangle^{-s} \int_1^\infty \int_{\mathbf{R}} e^{ix\xi} \left(\partial_\xi K(\xi)\right) \left(E(t,\xi)\right) \mathscr{F}\left[\langle\cdot\rangle^{-s}\phi\right](\xi) d\xi \frac{dt}{t} \right\|_2$$

Now we show $I_1 \le C\|\phi\|_2$ and $I_3 \le C\|\phi\|_2$. The estimation for I_1 is similar to that for I_3, and hence, we only estimate for I_3, and get

$$\begin{aligned} I_3 &\le \int_1^\infty \left\| \langle x\rangle^{-s} e^{-it(H_0-\lambda-i\mu)} K(p) x \langle x\rangle^{-s} \chi(|x| > t^\delta)\phi \right\|_2 \frac{dt}{t} \\ &\le C \int_1^\infty t^{-1-\delta(s-1)} dt \, \|\phi\|_2 \le C\|\phi\|_2, \end{aligned}$$

where we use $s > 1$, $\delta > 0$ and $K(p)$ is bounded since $\theta < 1$. Next, we show $I_2 \le C\|\phi\|_2$ and $I_4 \le C\|\phi\|_2$. For the same reason, we only estimate about I_4. We note that

$$E(t,\xi) = \frac{-i}{t(3\xi^2+4)} \partial_\xi E(t,\xi), \quad \frac{1}{t(3\xi^2+4)} \le Ct^{-1}. \tag{3.2}$$

Then I_4 can be estimated as

$$\begin{aligned}
I_4 &\le \int_1^\infty \left\| \int_{\mathbf{R}} (\partial_\xi \chi_4(x,\xi)) e^{ix\xi} \frac{\langle \xi \rangle^{2\theta}}{(3\xi^2+4)^2} E(t,\xi) d\xi \right\|_2 \frac{dt}{t^2} \\
&\quad + \int_1^\infty \left\| \int_{\mathbf{R}} \chi_4(x,\xi) x e^{ix\xi} \frac{\langle \xi \rangle^{2\theta}}{(3\xi^2+4)^2} E(t,\xi) d\xi \right\|_2 \frac{dt}{t^2} \\
&\quad + \int_1^\infty \left\| \int_{\mathbf{R}} \chi_4(x,\xi) e^{ix\xi} \left(\partial_\xi \frac{\langle \xi \rangle^{2\theta}}{(3\xi^2+4)^2} \right) E(t,\xi) d\xi \right\|_2 \frac{dt}{t^2} \\
&\le C \int_1^\infty \left(t^{-2+\delta(2-s)} + t^{-2} \right) dt \, \|\phi\|_2 \le C \|\phi\|_2 ,
\end{aligned}$$

by taking $\delta > 0$ to be sufficiently small, where we use

$$\begin{aligned}
&\left\| \int_{\mathbf{R}} (\partial_\xi \chi_4(x,\xi)) e^{ix\xi} \frac{\langle \xi \rangle^{2\theta}}{(3\xi^2+4)^2} E(t,\xi) d\xi \right\|_2 \\
&\le C \left\| \langle x \rangle^{-s} \frac{\langle p \rangle^{2\theta}}{(3p^2+4)^2} e^{-it(H_0-\lambda-i\mu)} x^2 \langle x \rangle^{-s} \chi(|x| \le t^\delta) \phi \right\|_2 \\
&\le C \left\| \langle x \rangle^{-s+2} \chi(|x| \le t^\delta) \phi \right\|_2 \le C t^{\delta(2-s)} \|\phi\|_2.
\end{aligned}$$

By the smoothness and boundedness of $\partial_\xi K(\xi)$, and (3.2), that the smooth and bounded function A exists so that

$$\begin{aligned}
I_5 &\le \left\| \langle x \rangle^{-s} \int_1^\infty \int_{\mathbf{R}} e^{ix\xi} \left(-ix \frac{\partial_\xi K(\xi)}{3\xi^2+4} + A(\xi) \right) (E(t,\xi)) \mathscr{F} \left[\langle \cdot \rangle^{-s} \phi \right](\xi) d\xi \frac{dt}{t^2} \right\|_2 \\
&\quad + \left\| \langle x \rangle^{-s} \int_1^\infty \int_{\mathbf{R}} e^{ix\xi} \frac{(\partial_\xi K(\xi))}{3\xi^2+4} (E(t,\xi)) \mathscr{F} \left[-i \cdot \langle \cdot \rangle^{-s} \phi \right](\xi) d\xi \frac{dt}{t^2} \right\|_2 \\
&\le \int_1^\infty \left\| \langle x \rangle^{-s} \left(-ix \frac{(\partial_\xi K)(p)}{3p^2+4} + A(p) \right) e^{-it(H_0-\lambda-i\mu)} \langle x \rangle^{-s} \phi \right\| \frac{dt}{t^2} \\
&\quad + \int_1^\infty \left\| \langle x \rangle^{-s} \frac{(\partial_\xi K)(p)}{3p^2+4} e^{-it(H_0-\lambda-i\mu)} x \langle x \rangle^{-s} \phi \right\| \frac{dt}{t^2},
\end{aligned}$$

and we see that I_5 is bounded by $C\|\phi\|_2$.

Next, we estimate J. Let $a > 0$ and $\tilde{\chi}(\cdot \le a)$ be a smooth cut-off function so that $0 \le \tilde{\chi}(\cdot \le a) \le 1$, $\tilde{\chi}(s \le a) = 1$ for all $s \le a/2$ and $\tilde{\chi}(s \le a) = 0$ for all $s \ge a$; we also define $\tilde{\chi}(\cdot > a) = 1 - \tilde{\chi}(\cdot \le a)$. For some $0 < \varepsilon < 1/2$, divide J into $J_1 + J_2$ with

$$J_1 := \left\| \int_0^1 \int_{\mathbf{R}} \langle x \rangle^{-s} e^{ix\xi} \langle \xi \rangle^{2\theta} \tilde{\chi}(|\xi| > t^{-\varepsilon}) e^{-it(H_0-\lambda-i\mu)} \mathscr{F} \left[\langle \cdot \rangle^{-s} \phi \right](\xi) d\xi dt \right\|_2 .$$

and

$$J_2 := \left\| \int_0^1 \int_{\mathbf{R}} \langle x \rangle^{-s} e^{ix\xi} \langle \xi \rangle^{2\theta} \tilde{\chi}(|\xi| \le t^{-\varepsilon}) e^{-it(H_0 - \lambda - i\mu)} \mathscr{F} \left[\langle \cdot \rangle^{-s} \phi \right] (\xi) d\xi dt \right\|_2 .$$

Here, by using $e^{it\xi^3} = (3t|\xi|^2)^{-1}\partial_\xi e^{it\xi^3}$, $|\xi|^{-1} < 2t^{\varepsilon}$ and applying integration by parts, we can easily get $J_1 \le C\|\phi\|_2$. By the simple calculation, J_2 can be estimated as

$$J_2 = \left\| \langle x \rangle^{-s} \int_0^1 \langle p \rangle^{2\theta} \tilde{\chi}(|p| \le t^{-\varepsilon}) e^{-it(H_0 - \lambda - i\mu)} \langle x \rangle^{-s} \phi dt \right\|_2 \le C\|\phi\|_2,$$

where we use $\theta < 1$ and $\varepsilon < 1/2$. All constants in the estimates for I and J can be taken independently into ϕ, and hence, we can use the density argument and get Lemma 3.1. □

3.2. Proof of Theorem 1.6

Now, we shall prove Theorem 1.6. In the proofs, we employ the extended commutator calculation in various places, where the extended commutator is defined as follows: Let A be a selfadjoint operator and suppose that $D = \mathscr{D}(H) \cap \mathscr{D}(A) \subset \mathscr{H}$ is dense. We define the form $q_{H,A}(\cdot,\cdot)$ in D as $q_{H,A}(u,v) := i(Au, Hv) - i(Hu, Av)$ for $u, v \in D$. Then, if a bounded selfadjoint operator T exists such that the closure of $q_{H,A}(\cdot,\cdot)$, $\tilde{q}_{H,A}(\cdot,\cdot)$ satisfies $\tilde{q}_{H,A}(u,v) = (Tu, v)$, $u, v \in \mathscr{H}$, then we denote this by $T = i[H, A]^0$. We further employ the commutator expansion lemma.

Lemma 3.4. For some integer $2 \le j$, let A_0 and B_0 be the selfadjoint operators with

$$\|i[A_0, B_0]^0\| < \infty, \quad \|\mathrm{ad}^j_{A_0}(B_0)\| < \infty,$$

where $\mathrm{ad}^1_A(H) = i[H, A]^0$ and $\mathrm{ad}^j_A(H) = i[\mathrm{ad}^{j-1}_A(H), A]^0$. For $0 \le \rho \le 1$, suppose that $f \in C^j(\mathbf{R})$ satisfies $|\partial_s^k f(s)| \le C_k \langle s \rangle^{\rho - k}$, $0 \le k \le j$. Then,

$$i[f(A_0), B_0]^0 = \sum_{k=1}^{j-1} \frac{1}{k!} f^{(k)}(A_0) \mathrm{ad}^k_{A_0}(B_0) + R_j(f, A_0, B_0)$$

where $R_j(f, A_0, B_0)$ satisfies

$$\|(A_0 + i)^{j-1} R_j(f, A_0, B_0)\| \le C(f^{(j)}) \|\mathrm{ad}^j_{A_0}(B_0)\|.$$

In particular, let f satisfy the condition stated in Lemma 3.4, then

$$i[f(p), x]^0 = f'(p) \tag{3.3}$$

holds on $\mathscr{D}(f'(p))$.

The proof of this lemma can be seen in Sigal–Soffer [18] and as Lemma C.3.1 in Dereziński and Gérard [3].

To extend the uniform resolvent estimate for H_0 to H, we must prove the boundedness of the operator in the following (3.4); we do this by employing the approach of [18];

Lemma 3.5. Let $1/2 \leq \theta < 1$ and $s > 1$. Then, an operator acting on $\mathscr{S}(\mathbf{R})$,

$$\langle x\rangle^s \langle p\rangle^{-\theta} (pV + Vp + iV') \langle p\rangle^{-\theta} \langle x\rangle^s \tag{3.4}$$

can be extended to the bounded operator.

Proof. It suffices to prove that

$$\langle x\rangle^s \langle p\rangle^{-\theta} Vp \langle p\rangle^{-\theta} \langle x\rangle^s$$

can be extended to the bounded operator. On $\mathscr{S}(\mathbf{R})$, this operator can be divided into

$$\begin{aligned}
&\langle x\rangle^s [\langle p\rangle^{-\theta}, V]p \langle p\rangle^{-\theta} \langle x\rangle^s + \langle x\rangle^s V \langle p\rangle^{-\theta} p \langle p\rangle^{-\theta} \langle x\rangle^s \\
&= \langle x\rangle^s \langle p\rangle^{-\theta} [V, \langle p\rangle^{\theta}]p \langle p\rangle^{-2\theta} \langle x\rangle^s + \langle x\rangle^s Vp \langle p\rangle^{-2\theta} \langle x\rangle^s \\
&= \langle x\rangle^s \langle p\rangle^{-\theta} \langle x\rangle^{-s} \cdot \langle x\rangle^s [V, \langle p\rangle^{\theta}] \langle x\rangle^s \\
&\qquad \times \langle x\rangle^{-s} p \langle p\rangle^{-2\theta} \langle x\rangle^s + \langle x\rangle^s Vp \langle p\rangle^{-2\theta} \langle x\rangle^s .
\end{aligned}$$

We first estimate $\langle x\rangle^s \langle p\rangle^{-\theta} \langle x\rangle^{-s}$. By

$$\begin{aligned}
&\langle x\rangle^s \langle p\rangle^{-\theta} \langle x\rangle^{-s} \\
&= \langle x\rangle^s (x+i)^{-1}[x, \langle p\rangle^{-\theta}] \langle x\rangle^{-s} + \langle x\rangle^s (x+i)^{-1} \langle p\rangle^{-\theta} (x+i) \langle x\rangle^{-s} \\
&= -\theta \langle x\rangle^s (x+i)^{-1}p \langle p\rangle^{-\theta-2} \langle x\rangle^{-s} \\
&\quad + \langle x\rangle^s (x+i)^{-1} \langle p\rangle^{-\theta} (x+i) \langle x\rangle^{-s} .
\end{aligned} \tag{3.5}$$

Since $[x, p \langle p\rangle^{-\theta-2}]^0$ is bounded, by employing Lemma 3.4 as $A_0 = x$, $B_0 = p \langle p\rangle^{-\theta-2}$ and $f(t) = \langle t\rangle^s (t+i)^{-1}$, we find that

$$\begin{aligned}
&\langle x\rangle^s (x+i)^{-1}p \langle p\rangle^{-\theta-2} \langle x\rangle^{-s} \\
&= p \langle p\rangle^{-\theta-2} (x+i)^{-1} + [\langle x\rangle^s (x+i)^{-1}, p \langle p\rangle^{-\theta-2}] \langle x\rangle^{-s}
\end{aligned}$$

can be extended to the bounded operator. The second term on the right-hand side of (3.5) can be estimated in the same way and will be bounded. The boundedness of $\langle x\rangle^{-s} p \langle p\rangle^{-2\theta} \langle x\rangle^s$ similarly can be proven. The term

$$\begin{aligned}
&\langle x\rangle^s Vp \langle p\rangle^{-2\theta} \langle x\rangle^s \\
&= \langle x\rangle^s V[p \langle p\rangle^{-2\theta}, x+i](x+i)^{-1} \langle x\rangle^s + \langle x\rangle^s V(x+i)p \langle p\rangle^{-2\theta} (x+i)^{-1} \langle x\rangle^s \\
&= -i \langle x\rangle^s V \left(\langle p\rangle^{-2\theta} - 2\theta p^2 \langle p\rangle^{-2\theta-1}\right) (x+i)^{-1} \langle x\rangle^s \\
&\quad + \langle x\rangle^s V(x+i)p \langle p\rangle^{-2\theta} (x+i)^{-1} \langle x\rangle^s
\end{aligned}$$

similarly can also be estimated, where we use (3.3). Hence, the proof is completed if we have the boundedness for $\langle x\rangle^s [V, \langle p\rangle^\theta] \langle x\rangle^s$. By simple calculation, we have

$$\langle x\rangle^s [V, \langle p\rangle^\theta] \langle x\rangle^s = \langle x\rangle^s V[\langle p\rangle^\theta, \langle x\rangle^s] + \langle x\rangle^s [\langle x\rangle^s V, \langle p\rangle^\theta]$$

Since $[x, \langle p\rangle^\theta]^0$ is bounded, we can employ Lemma 3.4 and get

$$\begin{aligned}&\langle x\rangle^s [\langle x\rangle^s V, \langle p\rangle^\theta]\\ &= \langle x\rangle^s \left(F^{(1)}(x)[x, \langle p\rangle^\theta] + F^{(2)}(x)\mathrm{ad}_x^2(\langle p\rangle^\theta)/2\right)\\ &\quad + \langle x\rangle^s (x+i)^{-2} \cdot (x+i)^2 R_3(F, x, \langle p\rangle^\theta),\end{aligned}$$

where $F(t) = \langle t\rangle^s V(t)$. This operator can be extended to the bounded operator. Conversely, the term $\langle x\rangle^s V[\langle p\rangle^\theta, \langle x\rangle^s]$ satisfies

$$\begin{aligned}&\langle x\rangle^s V[\langle p\rangle^\theta, \langle x\rangle^s]\\ &= \langle x\rangle^s V \left([\langle p\rangle^\theta, (x+i)](x+i)^{-1} \langle x\rangle^s + (x+i)[\langle p\rangle^\theta, (x+i)^{-1} \langle x\rangle^s]\right)\\ &= -i \langle x\rangle^s V p \langle p\rangle^{\theta-2} (x+i)^{-1} \langle x\rangle^s + \langle x\rangle^s V(x+i)[\langle p\rangle^\theta, (x+i)^{-1} \langle x\rangle^s]\\ &= -i \langle x\rangle^s V[p \langle p\rangle^{\theta-2}, (x+i)^{-1} \langle x\rangle^s] - i \langle x\rangle^{2s} (x+i)^{-1} V p \langle p\rangle^{\theta-2}\\ &\quad + \langle x\rangle^s V(x+i)[\langle p\rangle^\theta, (x+i)^{-1} \langle x\rangle^s]\end{aligned}$$

and by using Lemma 3.4 again, we notice that each of the aforementioned operators also can be extended to bounded operators. These complete the proof. □

To define the resolvent of H, we first demonstrate the following Lemma;

Lemma 3.6. For all $\lambda \in \mathbf{R}$ and $\mu > 0$,

$$(H_0 - \lambda \mp i\mu)^{-1} \mathscr{S}(\mathbf{R}) \subset \mathscr{S}(\mathbf{R}) \tag{3.6}$$

holds.

Proof. By applying the Fourier transform, for all $\phi \in \mathscr{S}(\mathbf{R})$,

$$(H_0 - \lambda \mp i\mu)^{-1}\phi = (2\pi)^{-1/2} \int_{\mathbf{R}} \frac{e^{ix\xi}}{-\xi^3 - 4\xi - \lambda \mp i\mu} \mathscr{F}[\phi](\xi) d\xi,$$

where $\mathscr{F}$ indicates the Fourier transform; this immediately proves (3.6). □

Now we prove the resolvent estimate for H. Define $\mathcal{Z}_s(\cdot) := \langle x \rangle^s \langle p \rangle^{-\theta} (\cdot - \lambda \mp i\mu) \langle p \rangle^{-\theta} \langle x \rangle^s$. By the definition of H, $\mathcal{Z}_s(H)$ satisfies

$$\begin{aligned}(3.7) \qquad \mathcal{Z}_s(H) &:= \Big(1 + \beta \langle x \rangle^s \langle p \rangle^{-\theta} \mathcal{V}(H_0 - \lambda \mp i\mu)^{-1} \langle p \rangle^{\theta} \langle x \rangle^{-s} \Big) \mathcal{Z}_s(H_0) \\ &:= \mathcal{I}\mathcal{Z}_s(H_0),\end{aligned}$$

where $\mathcal{V} = (pV + Vp) + iV'$. Hence, from Lemma 3.6, we can see that $\mathcal{Z}_s(H)$ is well defined on $\mathscr{S}(\mathbf{R})$. Conversely, the operator $\mathcal{I}$ satisfies for all $\phi \in \mathscr{S}(\mathbf{R})$,

$$\mathcal{I}\phi = \phi + \beta \langle x \rangle^s \langle p \rangle^{-\theta} \mathcal{V} \langle p \rangle^{-\theta} \langle x \rangle^s \cdot \langle x \rangle^{-s} \langle p \rangle^{\theta} (H_0 - \lambda \mp i\mu)^{-1} \langle p \rangle^{\theta} \langle x \rangle^{-s} \phi.$$

Together with Lemma 3.1 and 3.5, we find that $\mathcal{I}$ can be extended to the bounded operator. Moreover by the smallness of β, $\mathcal{I}$ has a certain bounded inverse $\mathcal{I}^{-1}$. By Lemma 3.1, we notice that $\mathcal{Z}_s(H_0)$ has a certain bounded inverse $\mathcal{Z}_s(H_0)^{-1}$. Hence, we obtain $\mathcal{I}\mathcal{Z}_s(H_0)$ that has its inverse $\mathcal{Z}_s(H_0)^{-1}\mathcal{I}^{-1}$; this implies that the operator $\mathcal{Z}_s(H)$ has a certain bounded inverse that is written in the form

$$\mathcal{Z}_s(H)^{-1} = \langle x \rangle^{-s} \langle p \rangle^{\theta} (H - \lambda \mp i\mu)^{-1} \langle p \rangle^{\theta} \langle x \rangle^{-s}.$$

Hence, we obtain the resolvent estimate

$$(3.8) \qquad \sup_{\lambda \in \mathbf{R}, \mu > 0} \left\| \langle x \rangle^{-s} \langle p \rangle^{\theta} (H - \lambda \mp i\mu)^{-1} \langle p \rangle^{\theta} \langle x \rangle^{-s} \phi \right\|_2 \leq C\|\phi\|_2.$$

Every approach to prove (3.8) works well for the operators $-H$, H^* and $-H^*$. Hence, let $\mathcal{H}$ be any one of the following $\pm H$, $\pm H^*$ and $\pm H_0$. Then

$$\sup_{\lambda \in \mathbf{R}, \mu > 0} \left\| \langle x \rangle^{-s} \langle p \rangle^{\theta} (\mathcal{H} - \lambda \mp i\mu)^{-1} \langle p \rangle^{\theta} \langle x \rangle^{-s} \phi \right\|_2 \leq C\|\phi\|_2$$

holds.

As the direct consequence of Kato's method (see, Theorem 1.5 of [11]). This inequality implies that $\langle x \rangle^{-s} \langle p \rangle^{\theta}$ is $\mathcal{H}$-smooth. Here, for all $k > 0$, we can see that $\lim_{t \to \infty} \|e^{-kt} e^{-itH_0}\| = 0$ holds (it is said to be $H_0 \in \mathscr{G}(\mathfrak{H})$ in terms of Kato's notation). Hence, from Theorem 3.9 of [11], we obtain $\mathcal{H} \in \mathscr{G}(\mathfrak{H})$ that implies $\langle x \rangle^{-s} \langle p \rangle^{\theta}$ is $\mathcal{H}$-smooth, and then Lemma 3.6 of [11] provides the decay (smoothing) estimate

$$\int_{-\infty}^{\infty} \left\| \langle x \rangle^{-s} \langle p \rangle^{\theta} e^{-i\tau\mathcal{H}} \phi \right\|_2^2 d\tau \leq C\|\phi\|_2^2.$$

A change of variable $\tau \to \alpha^3 t$ yields

$$(3.9) \qquad \int_{-\infty}^{\infty} \left\| \langle x \rangle^{-s} \langle p \rangle^{\theta} e^{-it\alpha^3\mathcal{H}} \phi \right\|_2^2 dt \leq C\alpha^{-3}\|\phi\|_2^2.$$

The decay estimates immediately prove Theorem 1.6, i.e., the limits

$$\text{s-}\lim_{t\to\pm\infty} e^{it\alpha^3 H}e^{-it\alpha^3 H_0}, \quad \text{s-}\lim_{t\to\pm\infty} e^{it\alpha^3 H_0}e^{-it\alpha^3 H},$$
$$\text{s-}\lim_{t\to\pm\infty} e^{it\alpha^3 H^*}e^{-it\alpha^3 H_0}, \quad \text{s-}\lim_{t\to\pm\infty} e^{it\alpha^3 H_0}e^{-it\alpha^3 H^*}$$

exist.

3.3. Proof of theorems

We now prove Theorem 1.5, 1.7 and 1.8. First, we prove Theorem 1.5. The inequality (1.6) is already proven in (3.9). Moreover, by noting

$$\begin{aligned}\left\|\langle x\rangle^{-s}\langle p\rangle e^{-it\alpha^3 H}w_0\right\|_2 &= \left\|\langle x\rangle^{-s}\langle p\rangle^{\theta}\mathscr{J}(t)^{-1}\mathscr{J}(t)e^{-it\alpha^3 H}w_0\right\|_2\\ &= \left\|\left\langle \alpha(x-4\alpha^2 t - x_0)\right\rangle^{-s}\langle p/\alpha\rangle^{\theta} u_0(t,x)\right\|_2,\end{aligned}$$

the inequality (1.7) also can be proven.

Now, we prove Theorem 1.7 and 1.8. Let $\mathcal{V} = (pV + Vp) + iV'$ and $\phi, \psi \in L^2(\mathbf{R})$. Using (3.9), we estimate for all t,

$$\begin{aligned}&\left|\left(e^{it\alpha^3 H_0}e^{-it\alpha^3 H}\phi, \psi\right)\right| - |(\phi,\psi)|\\ &\le \left|\left(6\alpha^3\delta\int_0^t e^{i\tau\alpha^3 H_0}\mathcal{V}e^{-i\tau\alpha^3 H}\phi d\tau, \psi\right)\right|\\ &\le C\alpha^3\delta\left(\int_0^t \left\|\langle x\rangle^{-s}\langle p\rangle^{1/2} e^{-i\tau\alpha^3 H_0}\psi\right\|_2^2 d\tau\right)^{1/2}\\ &\qquad\times\left(\int_0^t \left\|\langle x\rangle^{-s}\langle p\rangle^{1/2} e^{-i\tau\alpha^3 H}\phi\right\|_2^2 d\tau\right)^{1/2}\\ &\le C\delta\|\phi\|_2\|\psi\|_2,\end{aligned}$$

and with this, we find that (t,α,δ)-independent constant $C > 0$ is such that

$$\left\|e^{-it\alpha^3 H}\phi\right\|_2 = \left\|e^{it\alpha^3 H_0}e^{-it\alpha^3 H}\phi\right\|_2 \le C\|\phi\|_2.$$

Similarly, we also find

$$\left\|e^{it\alpha^3 H}\phi\right\|_2 \le C\|\phi\|_2$$

and this yields

$$\|e^{-itH}\phi\|_2 \ge C\left\|e^{itH}e^{-itH}\phi\right\|_2 = C\left\|\phi\right\|_2.$$

Similarly, we have that (t, α, δ)-independent constants $C > c > 0$ are such that for all $\phi \in L^2(\mathbf{R})$,

$$c \|\phi\|_2 \leq \left\| e^{-it\mathcal{H}} \phi \right\|_2 \leq C \|\phi\|_2$$

holds. Hence, the proof of Theorem 1.7 is completed. Using

$$u_0(t,x) = e^{-i(4t\alpha^2 + x_0)p} U e^{-it\alpha^3 H} w_0 = \mathscr{J}(t) e^{-it\alpha^3 H} w_0$$

and $\mathscr{J}(t)$ is the unitary operator on $L^2(\mathbf{R})$, we also have Theorem 1.8.

Acknowledgement

The author would like to thank the anonymous referee for the careful reading of the manuscript and for useful suggestions.

References

[1] Chihara, H.: Resolvent estimates related with a class of dispersive equations, J. Fourier Anal. and Appl. **14** (2008) 301–325.

[2] Cycon, H.L., Froese, R.G., Kirsch, W., Simon, B.: Schrödinger operators, Text and Monographs in Physics, Springer (2007)

[3] Dereziński, J., Gérard, C.: Scattering theory of classical and quantum N-particle systems, Text Monographs. Phys., Springer, Berlin, (1997).

[4] Froese, R., Herbst, I., Hoffmann-Ostenhof, M., Hoffmann-Ostenhof, T.: On the absence of positive eigenvalues for one-body Schrödinger operators, J. D'Analyse Math., **41** (1982), 272–284.

[5] Guo, Q.C., Guo, G.P., Hao, X.J., Tao, T., Wang, L.J.: Renormalization group method for soliton evolution in a perturbed KdV equation, Chinese Physical Letters, **26** (2009), 060501 (3 pages).

[6] Mann, E.: The perturbed Korteweg-de Vries equation considered anew, J. Math. Phys., **38** (1997), 3772–3785.

[7] Mochizuki, K.: On the large perturbation by a class of non-selfadjoint operators, J. Math. Soc. Japan, **19** (1967), 123–158.

[8] Mourre, E.: Absence of singular continuous spectrum for certain selfadjoint operators, Comm. Math. Phys., **78** (1981), 391–408.

[9] Nakazawa, H.: The principle of limiting absorption for the non-selfadjoint Schrödinger operator with energy dependent potential, Tokyo J. Math. (2000)

[10] Kato, K., Kawamoto, M., Nanbu, K.: Singularity for solutions of linearized KdV equations, J. Math. Phys., **61** (2020).

[11] Kato, T.: Wave operators and similarity for some non-selfadjoint operators, Math. Ann., **162** (1966), 258–279.

[12] Kato, T., Yajima, K.: Some examples of smoothing operators and the associated smoothing effect, Rev. Math. Phys., **1** (1989), 481–496.

[13] Kawamoto, M.: High-energy uniform resolvent estimates for selfadjoint operators, arXiv 1811.02853v1

[14] Royer, J.: Limiting absorption principle for the dissipative Helmholtz equation, Comm. P. D. E., **35** (2010), 1458–1489.

[15] Ruzhansky, M., Sugimoto, M.: Smoothing properties of evolution equations via canonical transforms and comparison principle, Proceedings of the London Math. Soc., **105** (2012), 393-423.

[16] Sachs, L. R.: Completeness of derivatives of squared Schrödinger eigenfunctions and explicit solutions of the linearized $\mathrm{K}_D\mathrm{V}$ equation, SIAM J. Math. Anal., **14** (1983), 674–683.

[17] Sigal, I.M.: Stark effect in multielectron systems: Non-existence of bound states, Comm. Math. Phys., **122** (1989), 1–22.

[18] Sigal, I.M., Soffer, A.: Local decay and propagation estimates for time-dependent and independent Hamiltonians, Preprint Princeton University.

[19] Wang, X.P.: Time-decay of semigroups generated by dissipative Schrödinger operators, J. Dif. Eqn., **253** (2012), 3523–3542.

Masaki Kawamoto
Department of Engineering for Production, Graduate School of Science and Engineering, Ehime University, 3 Bunkyo-cho Matsuyama, Ehime, 790-8577. Japan
E-mail: kawamoto.masaki.zs@ehime-u.ac.jp