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A generalization and short proof of a theorem of Hano on affine vector fields

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Abstract. We prove that a bounded affine vector field on a complete Finsler manifold is a Killing vector field. This generalizes the analogous result of Hano for Riemannian manifolds [3]. Even though our result is more general, the proof is significantly simpler.

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§1. Introduction

Yano showed that affine vector fields on a compact orientable Riemannian manifold are Killing vector fields [7]; the proof was based on integral formulas. Hano found a generalization: bounded affine vector fields on a complete Riemannian manifold are Killing vector fields. The proof relied on the de Rham decomposition, and special properties of irreducible Riemannian manifolds. A similar proof can be found in [4]. We show that Hano's result is true for the much more general Finsler manifolds, using only the Euler–Lagrange equation.

§2. Definitions and prerequisites

Throughout, M is a second countable and smooth Hausdorff manifold; the tangent bundle is $\tau\colon TM \to M$, and we denote by $\mathring{T}M$ the tangent manifold with the zero vectors removed. If $\varphi\colon M \to N$ is a smooth mapping between manifolds, $\varphi_*\colon TM \to TN$ stands for its derivative.

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We are going to work on the tangent manifold, where we use two kinds of lifts of vector fields on the base manifold. The *vertical lift* X^{v} of a vector field $X \in \mathfrak{X}(M)$ is the velocity field of the global flow

$$(t, v) \in \mathbb{R} \times TM \longmapsto v + tX(\tau(v)) \in TM$$

on TM. If $\varphi^X : \mathcal{D}_X \subset \mathbb{R} \times M \to M$ is the maximal local flow of $X \in \mathfrak{X}(M)$ and

$$\widetilde{\mathcal{D}}_X := \{(t, v) \in \mathbb{R} \times TM \mid (t, \tau(v)) \in \mathcal{D}_X\},$$

then

$$(t, v) \in \widetilde{\mathcal{D}}_X \longmapsto (\varphi_t^X)_*(v) \in TM$$

is a local flow on TM, whose velocity field is called the *complete lift* of X, denoted by X^{c}. The *Liouville vector field* C on TM is the velocity field of the flow of positive dilations:

$$(t, v) \in \mathbb{R} \times TM \longmapsto e^t v \in TM.$$

It is clear that a smooth function f on $\mathring{T}M$ is k^+-homogeneous ($k \in \mathbb{Z}$) if and only if $Cf = kf$.

A continuous function F on TM is a *Finsler function* for M if it is smooth on $\mathring{T}M$, 1^+-homogeneous, $F \restriction \mathring{T}M > 0$, and for any $p \in M$ and $u \in \mathring{T}_pM$, the symmetric bilinear form $(E \restriction T_pM)''(u)$ is non-degenerate (hence positive definite), where $E := \frac{1}{2}F^2$. A *Finsler manifold* is a manifold together with a Finsler function.

If (M, F) is a Finsler manifold, then there exists a unique second-order vector field $S \in \mathfrak{X}(\mathring{T}M)$ such that a curve γ in M is a geodesic of (M, F) if and only if $S \circ \dot{\gamma} = \ddot{\gamma}$. This vector field S is usually called the *canonical spray* or *geodesic spray* of (M, F). Another characterization of S is that

$$S(X^{\mathsf{v}}E) - X^{\mathsf{c}}E = 0 \qquad \text{for all } X \in \mathfrak{X}(M). \tag{2.1}$$

This form of the Euler–Lagrange equation is due to Crampin (see, e.g., [2, p. 348] or [5, p. 16]). It can be derived directly from the elementary form $\left(\frac{\partial E}{\partial y^i} \circ \dot{\gamma}\right)' - \frac{\partial E}{\partial x^i} \circ \dot{\gamma} = 0$ using the local formulae for X^{v} and X^{c}.

A Finsler manifold is said to be *forward complete* if the domains of its maximal geodesics are not bounded from above, and *complete* if the domain of its maximal geodesics is $\mathbb{R}$. For many equivalent characterizations of completeness, see [1, §6.6].

A vector field X on a Finsler manifold (M, F) is *affine* if its flow preserves geodesics, and it is a *Killing vector field* if its flow preserves the Finsler function, i.e., $F \circ (\varphi_t^X)_* = F$ for all possible $t \in \mathbb{R}$. Both properties can be expressed in terms of the complete lift of X: X is affine if and only if $[X^{\mathsf{c}}, S] = 0$, and X is a Killing vector field if and only if $X^{\mathsf{c}}E = 0$.

§3. Proof of the result

The key of our argument is the following simple observation. It is in fact a disguised special case of Exercise 5.4.3 from [1], but we give a short direct proof.

Lemma 1. *If X is an affine vector field on a Finsler manifold (M, F) and γ is a geodesic, then for all t and t_0 in the domain of γ we have*

$$X^{\mathsf{v}}E(\dot\gamma(t)) = X^{\mathsf{v}}E(\dot\gamma(t_0)) + (t - t_0)X^{\mathsf{c}}E(\dot\gamma(t_0)).$$

Proof. Since X is affine, we have $[X^{\mathsf{c}}, S] = 0$. Geodesics have constant speed, hence $SE = 0$. From these we get

$$0 = [X^{\mathsf{c}}, S]E = X^{\mathsf{c}}(SE) - S(X^{\mathsf{c}}E) = -S(X^{\mathsf{c}}E).$$

Since γ is a geodesic, $S \circ \dot\gamma = \ddot\gamma$, and we have

$$(X^{\mathsf{v}}E \circ \dot\gamma)' = S(X^{\mathsf{v}}E) \circ \dot\gamma \overset{(2.1)}{=} X^{\mathsf{c}}E \circ \dot\gamma,$$
$$(X^{\mathsf{v}}E \circ \dot\gamma)'' = (X^{\mathsf{c}}E \circ \dot\gamma)' = S(X^{\mathsf{c}}E) \circ \dot\gamma = 0.$$

Therefore $X^{\mathsf{v}}E \circ \dot\gamma$ is an affine function, and our claim follows. □

Theorem 2. *Let (M, F) be a Finsler manifold, X an affine vector field, and suppose that one of the following conditions holds:*

(1) $F \circ X$ is bounded, and (M, F) is complete;

(2) $F \circ X$ and $F \circ (-X)$ are bounded, and (M, F) is forward complete.

Then X is a Killing vector field.

Proof. First we prove that $X^{\mathsf{v}}E$ is bounded from above on the set $U(TM) := F^{-1}(\{1\})$ if (1) holds, and it is bounded from above and from below if (2) holds. For any $v \in U(TM)$, setting $p := \tau(v)$, we have

$$X^{\mathsf{v}}E(v) = F(v)X^{\mathsf{v}}F(v) = X^{\mathsf{v}}F(v) = (F \restriction T_pM)'(v)(X(p)) \le F(X(p)),$$

where in the last step we used the fundamental inequality (see [1, p. 7] or [6, Proposition 9.1.37]). In a similar way, we obtain

$$X^{\mathsf{v}}E(v) = (F \restriction T_pM)'(v)(X(p)) = -(F \restriction T_pM)'(v)(-X(p)) \ge -F(-X(p)).$$

Over $U(TM)$ these two inequalities give $-F \circ (-X) \circ \tau \le X^{\mathsf{v}}E \le F \circ X \circ \tau$, from which it follows that $X^{\mathsf{v}}E$ has the desired boundedness property.

Now we show that $X^{\mathsf{c}}E = 0$, and hence X is a Killing vector field. It suffices to prove it on $U(TM)$, because $X^{\mathsf{c}}E$ is 2^+-homogeneous. Indeed, the flows of X^{c} and C clearly commute, hence $[X^{\mathsf{c}}, C] = 0$, and we have

$$C(X^{\mathsf{c}}E) = [C, X^{\mathsf{c}}]E + X^{\mathsf{c}}(CE) = 2X^{\mathsf{c}}E. \tag{3.1}$$

So fix $v \in U(TM)$ and let γ be the maximal geodesic with $\dot{\gamma}(0) = v$. Then Lemma 1 gives

$$X^{\mathsf{v}}E(\dot{\gamma}(t)) = X^{\mathsf{v}}E(\dot{\gamma}(0)) + tX^{\mathsf{c}}E(\dot{\gamma}(0)) = X^{\mathsf{v}}E(v) + tX^{\mathsf{c}}E(v)$$

for any real number t in case (1) and for any positive real number t in case (2). Geodesics have constant speed, hence $\dot{\gamma}$ remains inside $U(TM)$, and the left-hand side of the above formula has to be bounded from above in case (1), and it has to be bounded from above and below in case (2), which is possible only if $X^{\mathsf{c}}E(v) = 0$. Thus $X^{\mathsf{c}}E = 0$ on $U(TM)$. This together with (3.1) implies $X^{\mathsf{c}}E = 0$, that is, X is a Killing vector field. □

As a corollary we have

Theorem 3 (Hano)**.** *Let (M, g) be a complete Riemannian manifold, and X an affine vector field on M such that the function $g(X, X)$ is bounded. Then X is a Killing vector field.*

The proof is immediate if we apply Theorem 2 to the Finsler function given by $F(v) := \sqrt{g(v, v)}$, $v \in TM$. Since compact Finsler manifolds are complete, we also have

Theorem 4. *An affine vector field on a compact Finsler manifold is a Killing vector field.*

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