

Hochschild cohomology ring of a cluster-tilted algebra of type \mathbb{D}_4

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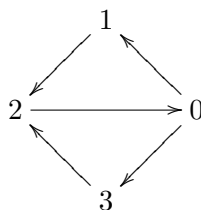
Abstract. In this paper we describe the Hochschild cohomology rings for algebras in a class of some special biserial algebras which contains a cluster-tilted algebra of Dynkin type \mathbb{D}_4 . In particular it is shown that the Hochschild cohomology rings modulo nilpotence for these algebras are isomorphic to the polynomial ring $K[x]$. As an application we prove that the cluster-tilted algebra of type \mathbb{D}_4 contained in this class satisfies the finiteness conditions **(Fg1)** and **(Fg2)** introduced in [EHSST].

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§1. Introduction

Let \mathcal{Q} be the following quiver with four vertices 0, 1, 2, 3 and five arrows:



Let e_i be the trivial path corresponding to the vertex i for $i = 0, 1, 2$, and let f_1 the trivial path corresponding to the vertex 3. For our convenience, f_i also denotes the trivial path corresponding to i for $i = 0, 2$. Hence we may write $e_j = f_j$ for $j = 0, 2$. Let a_i be the arrow from i to $i + 1$ for $i = 0, 1$, and let a_2 the arrow from 2 to 0. Moreover let b_0 the arrow from 0 to 3, and b_1 the

arrow from 3 to 2. For our convenience b_2 also denotes the arrow from 2 to 0. Hence we may write $a_2 = b_2$.

Throughout this paper, we always consider the indices i of e_i , f_i , a_i and b_i as modulo 3. Hence it follows that, for all integers i , a_i starts at e_i and ends with e_{i+1} , whereas b_i starts at f_i and ends with f_{i+1} . We write paths from left to right.

Let K be an algebraically closed field, and let n be a non-negative integer. We denote by I_n the ideal in the path algebra $K\mathcal{Q}$ generated by the elements

$$(a_0a_1a_2)^n a_0a_1 - b_0b_1, \quad (a_i a_{i+1} a_{i+2})^n a_i a_{i+1}, \quad b_i b_{i+1} \quad \text{for } i = 1, 2.$$

Denote the algebra $K\mathcal{Q}/I_n$ by Λ_n . Then the set

$$\{(a_i a_{i+1} a_{i+2})^j, (a_i a_{i+1} a_{i+2})^j a_i, (a_i a_{i+1} a_{i+2})^k a_i a_{i+1}, \\ f_2, b_0, b_1, b_0 b_1 \mid i = 0, 1, 2; j = 0, 1, \dots, n; k = 0, 1, \dots, n-1\}$$

is a K -basis of Λ_n , so that $\dim_K \Lambda_n = 9n + 10$. Also we easily see that Λ_n is a special biserial algebra and is not a selfinjective algebra. The purpose of this paper is to study the Hochschild cohomology of Λ_n .

In [BHL], Bastian, Holm and Ladkani introduced some finite quivers, called “standard forms,” to investigate a derived equivalence classification for cluster-tilted algebras of Dynkin type \mathbb{D} . We notice that the quiver \mathcal{Q} is one of these standard forms. Moreover, if $n = 0$, then the algebra Λ_0 is a Koszul cluster-tilted algebra of type \mathbb{D}_4 (see [ABS, F]), and hence is an algebra of finite representation type (see [BMR]). Also, Λ_0 appears in [BHL] as a representative of some derived equivalence class of cluster-tilted algebras of type \mathbb{D} .

In [F], we constructed a minimal projective bimodule resolution of Λ_n for all $n \geq 0$, and gave an explicit K -basis of the Hochschild cohomology groups of Λ_n . In this paper we use this K -basis to describe generators and relations of the Hochschild cohomology ring $\mathrm{HH}^*(\Lambda_n)$ of Λ_n , where the product is given by the Yoneda product. In [EHSST], the authors proved that if a finite-dimensional algebra satisfies certain reasonable finiteness conditions, denoted by **(Fg1)** and **(Fg2)**, then the support varieties have a lot of analogous properties of those for finite group algebras (see also [Sn]). In particular it is proved in [EHSST, Theorem 2.5] that if these conditions are satisfied, then the algebra is Gorenstein and a module has trivial support variety if and only if it has finite projective dimension. In this paper, we show that the cluster-tilted algebra Λ_0 of type \mathbb{D}_4 satisfies **(Fg1)** and **(Fg2)**, and consider a condition for the support variety of a Λ_0 -module to be trivial.

In [Sn], Snashall asked the following question: When is the Hochschild cohomology ring modulo nilpotence of a finite-dimensional algebra finitely generated as an algebra? It is known that the Hochschild cohomology rings modulo

nilpotence for the classes of the following algebras are finitely generated as algebras: group algebras of finite groups ([E, V]), monomial algebras ([GSS2]), selfinjective algebras of finite representation type ([GSS1]), and algebras of finite global dimension ([H]). But any definitive answer to this question has not been obtained yet. Our main theorem shows that both the Hochschild cohomology ring $\text{HH}^*(A_n)$ and the Hochschild cohomology ring modulo nilpotence $\text{HH}^*(A_n)/\mathcal{N}$ for $n \geq 0$ are finitely generated as algebras. Note that an example of the Hochschild cohomology ring modulo nilpotence which is not finitely generated appears in the papers [Sn, X].

This paper is organized as follows. In Section 2, we compute the products in the graded subring $\text{HH}^{6*}(A_n) := \bigoplus_{i \geq 0} \text{HH}^{6i}(A_n)$ of $\text{HH}^*(A_n)$, and give generators and relations of $\text{HH}^{6*}(A_n)$ (Proposition 2.1). In Section 3, we compute the products in the even Hochschild cohomology ring $\text{HH}^{\text{ev}}(A_n) := \bigoplus_{i \geq 0} \text{HH}^{2i}(A_n)$, and find generators and relations of $\text{HH}^{\text{ev}}(A_n)$, explicitly (Theorem 1). In Section 4, we describe all products in the Hochschild cohomology ring $\text{HH}^*(A_n)$, and then as a main result we give the presentation of $\text{HH}^*(A_n)$ by generators and relations for all $n \geq 0$ (Theorem 2). Moreover we determine the Hochschild cohomology ring modulo nilpotence $\text{HH}^*(A_n)/\mathcal{N}$ for all $n \geq 0$. In section 5, as an application, we prove that the Ext algebra $E(A_0)$ of A_0 is finitely generated as a $\text{HH}^{6*}(A_0)$ -module, and consequently it is shown that A_0 satisfies **(Fg1)** and **(Fg2)** (Theorem 3). Finally we describe the support varieties for all indecomposable modules over A_0 (Corollary 5.1), and determine the structures of the Hochschild cohomology rings modulo nilpotence for all cluster-tilted algebras of type \mathbb{D}_4 (Corollary 5.2).

Throughout this paper, we denote the enveloping algebra $A_n^{\text{op}} \otimes_K A_n$ of A_n by A_n^e (hence each A_n - A_n -bimodule corresponds to a right A_n^e -module and vice versa), and write \otimes_K as \otimes , for simplicity. We always denote the minimal projective bimodule resolution of A_n given in [F] by (Q^\bullet, ∂) . For any $i \geq 0$ and right A_n^e -module homomorphism $\lambda : Q^i \rightarrow A_n$, we again write the element in $\text{HH}^i(A_n) := \text{Ext}_{A_n^e}^i(A_n, A_n)$ represented by λ as λ , for simplicity.

§2. The subring $\text{HH}^{6*}(A_n)$

In this section we investigate the products in the graded subring

$$\text{HH}^{6*}(A_n) := \bigoplus_{i \geq 0} \text{HH}^{6i}(A_n) = \bigoplus_{i \geq 0} \text{Ext}_{A_n^e}^{6i}(A_n, A_n)$$

of $\text{HH}^*(A_n)$, and then find generators and relations of $\text{HH}^{6*}(A_n)$.

We start by recalling the Yoneda product \times in $\text{HH}^*(A_n) := \bigoplus_{i \geq 0} \text{HH}^i(A_n) = \bigoplus_{i \geq 0} \text{Ext}_{A_n^e}^i(A_n, A_n)$. Let $\phi : Q^i \rightarrow A_n$ and $\psi : Q^j \rightarrow A_n$, ($i, j \geq 0$) be

right Λ_n^e -module homomorphisms. Then we have liftings $\sigma_0, \sigma_1, \dots, \sigma_i$ of ψ , namely, there is the following commutative diagram of right Λ_n^e -modules:

$$\begin{array}{ccccccccccc}
 Q^{i+j} & \xrightarrow{\partial^{i+j}} & Q^{i+j-1} & \xrightarrow{\partial^{i+j-1}} & \dots & \xrightarrow{\partial^{j+2}} & Q^{j+1} & \xrightarrow{\partial^{j+1}} & Q^j & & \\
 \downarrow \sigma^i & & \downarrow \sigma^{i-1} & & & & \downarrow \sigma^1 & & \downarrow \sigma^0 & \searrow \psi & \\
 Q^i & \xrightarrow{\partial^i} & Q^{i-1} & \xrightarrow{\partial^{i-1}} & \dots & \xrightarrow{\partial^2} & Q^1 & \xrightarrow{\partial^1} & Q^0 & \xrightarrow{\partial^0} & \Lambda_n \longrightarrow 0 \\
 & \searrow \phi & & & & & & & & & \\
 & & & & & & & & & & \Lambda_n
 \end{array}$$

We define the product $\phi \times \psi$ of the homogeneous elements $\phi \in \text{HH}^i(\Lambda_n)$ and $\psi \in \text{HH}^j(\Lambda_n)$ by $\phi\sigma^i \in \text{HH}^{i+j}(\Lambda_n)$. Then the Yoneda product \times in $\text{HH}^*(\Lambda_n)$ is defined by linearly extending these to the products in $\text{HH}^*(\Lambda_n)$.

Now, for simplicity, we denote the basis elements of $\text{HH}^0(\Lambda_n)$ and $\text{HH}^{6j}(\Lambda_n)$ ($j \geq 1$) given in [F, Proposition 4.9] as follows:

$$\begin{aligned}
 X_{0,0} &:= \alpha_0^0 + \alpha_1^0 + \alpha_2^0 + \beta : Q^0 \rightarrow \Lambda_n; \\
 X_{0,m} &:= \alpha_0^m + \alpha_1^m + \alpha_2^m : Q^0 \rightarrow \Lambda_n \text{ for } m = 1, \dots, n, \text{ if } n > 0; \\
 X_{6j,0} &:= \phi_0^0 + \phi_1^0 + \phi_2^0 - \psi : Q^{6j} \rightarrow \Lambda_n; \\
 X_{6j,m} &:= \phi_0^m + \phi_1^m + \phi_2^m : Q^{6j} \rightarrow \Lambda_n \text{ for } m = 1, \dots, n, \text{ if } n > 0.
 \end{aligned}$$

Then we have by [F, Lemma 4.1] that: for $l = 0, 1, 2$

$$\begin{aligned}
 X_{0,0} : \begin{cases} e_l \otimes e_l & \mapsto e_l \\ f_1 \otimes f_1 & \mapsto f_1, \end{cases} & \quad X_{0,m} : \begin{cases} e_l \otimes e_l & \mapsto (a_l a_{l+1} a_{l+2})^m \\ f_1 \otimes f_1 & \mapsto 0, \end{cases} \\
 X_{6j,0} : \begin{cases} e_l \otimes e_l & \mapsto e_l \\ e_1 \otimes f_1 & \mapsto 0 \\ f_1 \otimes e_1 & \mapsto 0 \\ f_1 \otimes f_1 & \mapsto -f_1, \end{cases} & \quad X_{6j,m} : \begin{cases} e_l \otimes e_l & \mapsto (a_l a_{l+1} a_{l+2})^m \\ e_1 \otimes f_1 & \mapsto 0 \\ f_1 \otimes e_1 & \mapsto 0 \\ f_1 \otimes f_1 & \mapsto 0. \end{cases}
 \end{aligned}$$

Remark 2.1. *It is known that the map $\text{HH}^0(\Lambda_n) \rightarrow Z(\Lambda_n)$, $h \mapsto h((\sum_{l=0}^2 e_l \otimes e_l) + f_1 \otimes f_1)$ is an isomorphism of algebras, where the products in $\text{HH}^0(\Lambda_n)$ are given by the Yoneda products. Then using this isomorphism we have*

$$(2.1) \quad X_{0,s} \times X_{0,t} = \begin{cases} X_{0,s+t} & \text{if } s+t \leq n, \\ 0 & \text{if } s+t > n. \end{cases}$$

for integers s and t with $0 \leq s, t \leq n$. In particular, we see that $X_{0,0}$ is the identity of $\text{HH}^0(\Lambda_n)$.

For $u \geq 1$, we define the map $\sigma_{6u,0}^0 : Q^{6u} \rightarrow Q^0$ of Λ_n - Λ_n -bimodules by

$$\sigma_{6u,0}^0 : \begin{cases} e_l \otimes e_l & \mapsto e_l \otimes e_l & \text{for } l = 0, 1, 2 \\ e_1 \otimes f_1 & \mapsto 0 \\ f_1 \otimes e_1 & \mapsto 0 \\ f_1 \otimes f_1 & \mapsto -f_1 \otimes f_1, \end{cases}$$

and $\sigma_{6u,0}^i : Q^{i+6u} \rightarrow Q^i$ by the identity map id_{Q^i} for $i \geq 1$.

Now by direct observations we have the following lemma.

Lemma 2.1. *We have $X_{6u,0} = \partial^0 \sigma_{6u,0}^0$ and $\partial^i \sigma_{6u,0}^i = \sigma_{6u,0}^{i-1} \partial^{i+6}$ for all $u \geq 1$ and $i \geq 1$. Hence $\sigma_{6u,0}^i$ ($i \geq 0$) give liftings of $X_{6u,0}$.*

From the lemma above we can describe the products in $\text{HH}^{6*}(\Lambda_n)$.

Lemma 2.2. *We have the following products:*

- (a) $X_{6u,0} \times X_{0,s} = X_{6u,s}$ for integers $u \geq 1$ and s with $0 \leq s \leq n$.
- (b) $X_{6u,0} \times X_{6v,0} = X_{6(u+v),0}$ for integers $u \geq 1$ and $v \geq 1$.

Hence, for integers $u_1 \geq 0$, $u_2 \geq 0$, s_1 , and s_2 with $0 \leq s_1, s_2 \leq n$,

$$X_{6u_1,s_1} \times X_{6u_2,s_2} = \begin{cases} X_{6(u_1+u_2),s_1+s_2} & \text{if } s_1 + s_2 \leq n \\ 0 & \text{if } s_1 + s_2 > n. \end{cases}$$

Proof. (a) Let u be a positive integer. Then for $0 \leq s \leq n$ we get

$$X_{0,s} \sigma_{6u,0}^0 : \begin{cases} e_l \otimes e_l & \mapsto (a_l a_{l+1} a_{l+2})^s & \text{for } l = 0, 1, 2 \\ e_1 \otimes f_1 & \mapsto 0 \\ f_1 \otimes e_1 & \mapsto 0 \\ f_1 \otimes f_1 & \mapsto 0. \end{cases}$$

Therefore $X_{0,s} \sigma_{6u,0}^0 = X_{6u,s}$, which gives the desired product.

(b) Clearly $X_{6v,0} \sigma_{6u,0}^{6v} = X_{6(u+v),0}$ holds for $u \geq 1$ and $v \geq 1$, so that we have the desired product.

The last equality easily follows from (2.1), (a) and (b). □

Now we can find generators and relations of $\text{HH}^{6*}(\Lambda_n)$. Here we note that $\text{HH}^{6*}(\Lambda_n)$ is a commutative graded subring of $\text{HH}^*(\Lambda_n)$.

Proposition 2.1. *There is the following isomorphism of graded rings:*

- (a) *If $n = 0$, then $\text{HH}^{6*}(\Lambda_n) \simeq K[y_6]$, where $\text{deg } y_6 = 6$.*

(b) If $n > 0$, then $\mathrm{HH}^{6*}(\Lambda_n) \simeq K[y_0, y_6]/(y_0^{n+1})$, where $\deg y_0 = 0$ and $\deg y_6 = 6$.

Proof. We put $y_6 = X_{6,0}$. If $n = 0$, then by Lemma 2.2 we get the desired isomorphism in (a).

Now suppose that $n > 0$, and put $y_0 = X_{0,1}$. Then we see by (2.1) and Lemma 2.2 that $X_{6u,s} = y_0^s \times y_6^u$ for $u \geq 0$ and $s = 0, 1, \dots, n$, so that $\{y_0, y_6\}$ is a generator of $\mathrm{HH}^{6*}(\Lambda_n)$. Also by (2.1) we have the relation $y_0^{n+1} = 0$. Therefore we get the desired isomorphism in (b). \square

§3. The even Hochschild cohomology ring $\mathrm{HH}^{\mathrm{ev}}(\Lambda_n)$

In this section we compute all products of basis elements in even degrees, and then determine the structure of the even Hochschild cohomology ring

$$\mathrm{HH}^{\mathrm{ev}}(\Lambda_n) = \bigoplus_{i \geq 0} \mathrm{HH}^{2i}(\Lambda_n).$$

Note that $\mathrm{HH}^{\mathrm{ev}}(\Lambda_n)$ is a commutative graded subring of $\mathrm{HH}^*(\Lambda_n)$. Throughout this section we keep the notations from Section 2.

For simplicity we denote the basis elements of $\mathrm{HH}^{6j+2}(\Lambda_n)$ and $\mathrm{HH}^{6j+4}(\Lambda_n)$ given in [F, Proposition 4.9] as follows: for $j \geq 0$

$$\begin{aligned} X_{6j+2,m} &:= \theta_0^m + \theta_1^m + \theta_2^m : Q^{6j+2} \rightarrow \Lambda_n \text{ for } m = 0, 1, \dots, n-1, \text{ if } n > 0; \\ X_{6j+4,m} &:= \mu_0^m + \mu_1^m + \mu_2^m : Q^{6j+4} \rightarrow \Lambda_n \text{ for } m = 0, 1, \dots, n-1, \text{ if } n > 0; \\ X_{6j+4,n} &:= \mu_0^n + \mu_1^n + \mu_2^n : Q^{6j+4} \rightarrow \Lambda_n \text{ if } \mathrm{char} K \mid 3n + 2. \end{aligned}$$

Note that, by [F, Lemma 4.1], for $s = 0, 1, \dots, n-1$

$$X_{6j+2,s} : \begin{cases} e_l \otimes e_{l+2} & \mapsto (a_l a_{l+1} a_{l+2})^s a_l a_{l+1} & \text{for } l = 0, 1, 2 \\ f_r \otimes f_{r+2} & \mapsto 0 & \text{for } r = 1, 2, \end{cases}$$

and, for $t = 0, 1, \dots, n$,

$$X_{6j+4,t} : \begin{cases} e_l \otimes e_{l+1} & \mapsto (a_l a_{l+1} a_{l+2})^t a_l & \text{for } l = 0, 1, 2 \\ f_r \otimes f_{r+1} & \mapsto 0 & \text{for } r = 0, 1. \end{cases}$$

hold.

3.1. Liftings of $X_{6u+2,0}$ and $X_{6u+4,0}$

To compute Yoneda products in $\mathrm{HH}^{\mathrm{ev}}(\Lambda_n)$ we find liftings of $X_{6u+2,0}$ and $X_{6u+4,0}$ for $u \geq 0$.

For $u \geq 0$ we define a homomorphism $\sigma_{6u+2,0}^0 : Q^{6u+2} \rightarrow Q^0$ as Λ_n - Λ_n -bimodules by

$$\sigma_{6u+2,0}^0 : \begin{cases} e_l \otimes e_{l+2} & \mapsto & a_l a_{l+1} \otimes e_{l+2} & \text{for } l = 0, 1, 2 \\ f_r \otimes f_{r+2} & \mapsto & 0 & \text{for } r = 1, 2. \end{cases}$$

Also, for $u \geq 0$ and $i \geq 1$, define homomorphisms $\sigma_{6u+2,0}^i : Q^{6u+i+2} \rightarrow Q^i$ as Λ_n - Λ_n -bimodules by the following: For $j \geq 0$

$$\begin{aligned} \sigma_{6u+2,0}^{3j+1} &: \begin{cases} e_l \otimes e_l & \mapsto & a_l a_{l+1} \otimes e_l & \text{for } l = 0, 1, 2 \\ e_1 \otimes f_1 & \mapsto & a_1 a_2 \otimes f_1 \\ f_1 \otimes e_1 & \mapsto & 0 \\ f_1 \otimes f_1 & \mapsto & 0, \end{cases} \\ \sigma_{6u+2,0}^{3j+2} &: \begin{cases} e_l \otimes e_{l+1} & \mapsto & a_l a_{l+1} \otimes e_{l+1} & \text{for } l = 0, 1, 2 \\ f_r \otimes f_{r+1} & \mapsto & \begin{cases} a_0 a_1 \otimes f_1 & \text{for } r = 0 \\ 0 & \text{for } r = 1, \end{cases} \end{cases} \\ \sigma_{6u+2,0}^{3j+3} &: \begin{cases} e_l \otimes e_{l+2} & \mapsto & a_l a_{l+1} \otimes e_{l+2} & \text{for } l = 0, 1, 2 \\ f_r \otimes f_{r+2} & \mapsto & \begin{cases} 0 & \text{for } r = 1 \\ a_2 a_0 \otimes f_1 & \text{for } r = 2. \end{cases} \end{cases} \end{aligned}$$

Then by direct computations we have the following lemma.

Lemma 3.1. *We have $X_{6u+2,0} = \partial^0 \sigma_{6u+2,0}^0$ and $\partial^i \sigma_{6u+2,0}^i = \sigma_{6u+2,0}^{i-1} \partial^{6u+i+2}$ for all $u \geq 0$ and $i \geq 1$. Thus the map $\sigma_{6u+2,0}^i : Q^{6u+i+2} \rightarrow Q^i$ ($i \geq 0$) is a lifting of $X_{6u+2,0}$.*

Next for $u \geq 0$ we define a homomorphism $\sigma_{6u+4,0}^0 : Q^{6u+4} \rightarrow Q^0$ as Λ_n - Λ_n -bimodules by

$$\sigma_{6u+4,0}^0 : \begin{cases} e_l \otimes e_{l+1} & \mapsto & a_l \otimes e_{l+1} & \text{for } l = 0, 1, 2 \\ f_r \otimes f_{r+1} & \mapsto & 0 & \text{for } r = 0, 1. \end{cases}$$

Moreover for $u \geq 0$ and $i \geq 1$ we define homomorphisms $\sigma_{6u+4,0}^i : Q^{6u+i+4} \rightarrow$

Q^i as Λ_n - Λ_n -bimodules by: For $j \geq 0$

$$\begin{aligned} \sigma_{6u+4,0}^{3j+1} : & \begin{cases} e_l \otimes e_{l+2} & \mapsto a_l \otimes e_{l+2} & \text{for } l = 0, 1, 2 \\ f_r \otimes f_{r+2} & \mapsto \begin{cases} 0 & \text{for } r = 1 \\ a_2 \otimes f_1 & \text{for } r = 2, \end{cases} \end{cases} \\ \sigma_{6u+4,0}^{3j+2} : & \begin{cases} e_l \otimes e_l & \mapsto a_l \otimes e_l & \text{for } l = 0, 1, 2 \\ e_1 \otimes f_1 & \mapsto a_1 \otimes f_1 \\ f_1 \otimes e_1 & \mapsto 0 \\ f_1 \otimes f_1 & \mapsto 0, \end{cases} \\ \sigma_{6u+4,0}^{3j+3} : & \begin{cases} e_l \otimes e_{l+1} & \mapsto a_l \otimes e_{l+1} & \text{for } l = 0, 1, 2 \\ f_r \otimes f_{r+1} & \mapsto \begin{cases} a_0 \otimes f_1 & \text{for } r = 0 \\ 0 & \text{for } r = 1. \end{cases} \end{cases} \end{aligned}$$

We also have the following lemma.

Lemma 3.2. *We have $X_{6u+4,0} = \partial^0 \sigma_{6u+4,0}^0$ and $\partial^i \sigma_{6u+4,0}^i = \sigma_{6u+4,0}^{i-1} \partial^{6u+i+4}$ for all $u \geq 0$ and $i \geq 1$. Hence the map $\sigma_{6u+4,0}^i : Q^{6u+i+4} \rightarrow Q^i$ ($i \geq 0$) is a lifting of $X_{6u+4,0}$.*

3.2. The products in $\text{HH}^{6u}(\Lambda_n) \times \text{HH}^{6v+2}(\Lambda_n)$

Now we investigate the products of elements in $\text{HH}^{6u}(\Lambda_n)$ and $\text{HH}^{6v+2}(\Lambda_n)$ for $u \geq 0$ and $v \geq 0$.

Lemma 3.3. *Suppose that $n > 0$ (so that $\text{HH}^{6j+2}(\Lambda_n) \neq 0$). We have the following products:*

(a) *For any integers $u \geq 0$ and s with $0 \leq s \leq n$,*

$$X_{6u+2,0} \times X_{0,s} = \begin{cases} X_{6u+2,s} & \text{if } 0 \leq s \leq n - 1 \\ 0 & \text{if } s = n. \end{cases}$$

(b) $X_{6u,0} \times X_{2,0} = X_{6u+2,0}$ *for any integer $u \geq 1$.*

Consequently, for any integers $u_1 \geq 0$, $u_2 \geq 0$, s_1 with $0 \leq s_1 \leq n$, and s_2 with $0 \leq s_2 < n$, we have

$$X_{6u_1,s_1} \times X_{6u_2+2,s_2} = \begin{cases} X_{6(u_1+u_2)+2,s_1+s_2} & \text{if } 0 \leq s_1 + s_2 \leq n - 1 \\ 0 & \text{if } s_1 + s_2 \geq n. \end{cases}$$

Proof. (a) Let u and s be integers with $u \geq 0$ and $0 \leq s \leq n$. Then we have

$$X_{0,s}\sigma_{6u+2,0}^0 : \begin{cases} e_l \otimes e_{l+2} & \mapsto \begin{cases} (a_l a_{l+1} a_{l+2})^s a_l a_{l+1} & \text{if } 0 \leq s \leq n-1, \\ & \text{or if } s = n \text{ and } l = 0 \\ 0 & \text{if } s = n \text{ and } l \neq 0 \end{cases} \\ f_r \otimes f_{r+2} & \mapsto 0 \end{cases}$$

for $l = 0, 1, 2$ and $r = 1, 2$. This shows that

$$X_{0,s}\sigma_{6u+2,0}^0 = \begin{cases} X_{6u+2,s} & \text{if } 0 \leq s \leq n-1 \\ \eta & \text{if } s = n. \end{cases}$$

However $\eta \in \text{Im Hom}_{\Lambda_n^e}(\partial^{6u+2}, \Lambda_n)$ by [F, Lemma 4.5 (b)], and so $X_{0,n}\sigma_{6u+2,0}^0 = 0$ in $\text{HH}^{6u+2}(\Lambda_n)$. Thus we get the desired equality.

(b) Clearly $X_{2,0}\sigma_{6u,0}^2 = X_{6u+2,0}$ for all $u > 0$. So we have the required equality in (b).

The last equality follows from (a), (b), and Lemma 2.2. \square

3.3. The products in $\text{HH}^{6u}(\Lambda_n) \times \text{HH}^{6v+4}(\Lambda_n)$

Now we describe the products of elements in $\text{HH}^{6u}(\Lambda_n)$ and $\text{HH}^{6v+4}(\Lambda_n)$ for $u \geq 0$ and $v \geq 0$.

Lemma 3.4. *Suppose that $n > 0$ or $\text{char } K \mid 3n+2$ (hence $\text{HH}^{6j+4}(\Lambda_n) \neq 0$). We have the following products:*

(a) *For any integers $u \geq 0$ and $0 \leq s \leq n$,*

$$\begin{aligned} X_{6u+4,0} \times X_{0,s} \\ = \begin{cases} X_{6u+4,s} & \text{if } 0 \leq s \leq n-1, \text{ or if } s = n \text{ and } \text{char } K \mid 3n+2 \\ 0 & \text{if } s = n \text{ and } \text{char } K \nmid 3n+2. \end{cases} \end{aligned}$$

(b) $X_{6u,0} \times X_{4,0} = X_{6u+4,0}$ for any integer $u \geq 1$.

So, for any integers $u_1 \geq 0$, $u_2 \geq 0$, s_1 with $0 \leq s_1 \leq n$, and s_2 with $0 \leq s_2 \leq n$ (if $\text{char } K \mid 3n+2$) or $0 \leq s_2 \leq n-1$ (if $\text{char } K \nmid 3n+2$), we have

$$\begin{aligned} X_{6u_1,s_1} \times X_{6u_2+4,s_2} \\ = \begin{cases} X_{6(u_1+u_2)+4,s_1+s_2} & \text{if } s_1 + s_2 \leq n-1, \\ & \text{or if } \text{char } K \mid 3n+2 \text{ and } s_1 + s_2 = n \\ 0 & \text{if } \text{char } K \nmid 3n+2 \text{ and } s_1 + s_2 = n, \\ & \text{or if } s_1 + s_2 > n. \end{cases} \end{aligned}$$

Proof. (a) Let u and s be integers with $u \geq 0$ and $1 \leq s \leq n$. We get

$$X_{0,s}\sigma_{6u+4,0}^0 : \begin{cases} e_l \otimes e_{l+1} & \mapsto (a_l a_{l+1} a_{l+2})^s a_l & \text{for } l = 0, 1, 2 \\ f_r \otimes f_{r+1} & \mapsto 0 & \text{for } r = 0, 1, \end{cases}$$

Hence $X_{0,s}\sigma_{6u+4,0}^0 = \mu_0^s + \mu_1^s + \mu_2^s$. Therefore if $0 \leq s \leq n - 1$, or if $s = n$ and $\text{char } K \mid 3n + 2$, then $X_{0,s}\sigma_{6u+4,0}^0 = X_{6u+4,s}$. Also if $s = n$ and $\text{char } K \nmid 3n + 2$, then $X_{0,n}\sigma_{6u+4,0}^0 \in \text{Im Hom}_{A_n^e}(\partial^{6j+4}, A_n)$ by [F, Lemma 4.5 (d)]. Therefore (a) is proved.

(b) We have $X_{4,0}\sigma_{6u,0} = X_{6u+4,0}$ for all $u \geq 0$. This yields the required equality in (b).

The last equality follows from (a), (b), and Lemma 2.2. □

3.4. The products in $\text{HH}^{6u+2}(A_n) \times \text{HH}^{6v+2}(A_n)$

Now we describe the products in $\text{HH}^{6u+2}(A_n) \times \text{HH}^{6v+2}(A_n)$ for $u \geq 0$ and $v \geq 0$.

Lemma 3.5. *Suppose that $n > 0$. We get*

$$X_{2,0}^2 = \begin{cases} X_{4,1} & \text{if } n = 1 \text{ and } \text{char } K \mid 3n + 2, \text{ or if } n > 1 \\ 0 & \text{if } n = 1 \text{ and } \text{char } K \nmid 3n + 2. \end{cases}$$

Thus, for any integers $u_1 \geq 0, u_2 \geq 0, s_1,$ and s_2 with $0 \leq s_1, s_2 \leq n - 1,$ we have

$$\begin{aligned} & X_{6u_1+2,s_1} \times X_{6u_2+2,s_2} \\ &= \begin{cases} X_{6(u_1+u_2)+4,s_1+s_2+1} & \text{if } s_1 + s_2 < n - 1, \\ & \text{or if } \text{char } K \mid 3n + 2 \text{ and } s_1 + s_2 = n - 1 \\ 0 & \text{if } s_1 + s_2 > n - 1, \\ & \text{or if } \text{char } K \nmid 3n + 2 \text{ and } s_1 + s_2 = n - 1. \end{cases} \end{aligned}$$

Proof. We have $X_{2,0}\sigma_{2,0}^2 = \mu_0^1 + \mu_1^1 + \mu_2^1$. Therefore if $n = 1$ and $\text{char } K \nmid 3n + 2$, then $X_{2,0}\sigma_{2,0}^2 \in \text{Im Hom}_{A_n^e}(\partial^4, A_n)$ by [F, Lemma 4.5 (d)]. So $X_{2,0}\sigma_{2,0}^2 = 0$ in $\text{HH}^4(A_n)$. On the other hand if $n = 1$ and $\text{char } K \mid 3n + 2$ or if $n > 1$, then $X_{2,0}\sigma_{2,0}^2 = X_{4,1}$. This shows that the first equality holds.

The second equality follows from the first equality and Lemmas 2.2, 3.3, and 3.4. □

3.5. The products in $\mathrm{HH}^{6u+2}(\Lambda_n) \times \mathrm{HH}^{6v+4}(\Lambda_n)$

Now we investigate the products of elements in $\mathrm{HH}^{6u+2}(\Lambda_n)$ and $\mathrm{HH}^{6v+4}(\Lambda_n)$ for $u \geq 0$ and $v \geq 0$.

Lemma 3.6. *Suppose that $n > 0$. Then we get $X_{2,0} \times X_{4,0} = X_{6,1}$. Hence, for any integers $u_1 \geq 0, u_2 \geq 0, s_1$ with $0 \leq s_1 \leq n - 1$, and s_2 with $0 \leq s_2 \leq n$ (if $\mathrm{char} K \mid 3n + 2$) or $0 \leq s_2 \leq n - 1$ (if $\mathrm{char} K \nmid 3n + 2$), we have*

$$X_{6u_1+2,s_1} \times X_{6u_2+4,s_2} = \begin{cases} X_{6(u_1+u_2+1),s_1+s_2+1} & \text{if } s_1 + s_2 \leq n - 1, \\ 0 & \text{if } s_1 + s_2 \geq n. \end{cases}$$

Proof. We have $X_{2,0}\sigma_{4,0}^2 = \phi_0^1 + \phi_1^1 + \phi_2^1 = X_{6,1}$. Also, by this equality and Lemmas 2.2, 3.3, and 3.4, we have the second equality. \square

3.6. The products in $\mathrm{HH}^{6u+4}(\Lambda_n) \times \mathrm{HH}^{6v+4}(\Lambda_n)$

Finally we consider the products of elements in $\mathrm{HH}^{6u+4}(\Lambda_n)$ and $\mathrm{HH}^{6v+4}(\Lambda_n)$ for $u \geq 0$ and $v \geq 0$.

Lemma 3.7. *Suppose that $n > 0$ or $\mathrm{char} K \mid 3n + 2$. We have*

$$X_{4,0}^2 = \begin{cases} 0 & \text{if } n = 0 \text{ and } \mathrm{char} K \mid 3n + 2, \\ X_{8,0} & \text{if } n > 0. \end{cases}$$

Thus, for any integers $u_1 \geq 0, u_2 \geq 0, s_1$, and s_2 with $0 \leq s_1, s_2 \leq n$ (if $\mathrm{char} K \mid 3n + 2$) or $0 \leq s_1, s_2 \leq n - 1$ (if $\mathrm{char} K \nmid 3n + 2$), we have

$$X_{6u_1+4,s_1} \times X_{6u_2+4,s_2} = \begin{cases} X_{6(u_1+u_2+1)+2,s_1+s_2} & \text{if } s_1 + s_2 < n \\ 0 & \text{if } s_1 + s_2 \geq n. \end{cases}$$

Proof. We have

$$X_{4,0}\sigma_{4,0}^4 : \begin{cases} e_l \otimes e_{l+2} \mapsto a_l a_{l+1} & \text{for } l = 0, 1, 2 \\ f_r \otimes f_{r+2} \mapsto 0 & \text{for } r = 1, 2. \end{cases}$$

Thus if $n = 0$ and $\mathrm{char} K \mid 3n + 2$, then $X_{4,0}\sigma_{4,0}^4 = \eta \in \mathrm{Im} \mathrm{Hom}_{A_0^e}(\partial^8, \Lambda_0)$ by [F, Lemma 4.5 (b)], so that $X_{4,0}\sigma_{4,0}^4 = 0$ in $\mathrm{HH}^8(\Lambda_0)$. Moreover if $n > 0$ then $X_{4,0}\sigma_{4,0}^4 = \theta_0^0 + \theta_1^0 + \theta_2^0 = X_{8,0}$. Therefore the first equality holds.

The second equality follows from the first equality and Lemmas 2.2, 3.3, and 3.4. \square

3.7. Generators and relations of $\mathrm{HH}^{\mathrm{ev}}(\Lambda_n)$

Now we can provide generators and relations of the even Hochschild cohomology ring $\mathrm{HH}^{\mathrm{ev}}(\Lambda_n)$ of Λ_n .

Theorem 1. *We have the following isomorphism of commutative graded algebras:*

(a) *The case $\mathrm{char} K \mid 3n + 2$:*

(1) *If $n = 0$ (hence $\mathrm{char} K = 2$), then*

$$\mathrm{HH}^{\mathrm{ev}}(\Lambda_0) \simeq K[y_4, y_6]/(y_4^2),$$

where $\deg y_i = i$ ($i = 4, 6$).

(2) *If $n > 0$, then*

$$\mathrm{HH}^{\mathrm{ev}}(\Lambda_n) \simeq K[y_0, y_2, y_4, y_6]/(y_0^{n+1}, y_0^n y_2, y_2^2 - y_0 y_4, y_4^2 - y_2 y_6),$$

where $\deg y_i = i$ ($i = 0, 2, 4, 6$).

(b) *The case $\mathrm{char} K \nmid 3n + 2$:*

(1) *If $n = 0$, then*

$$\mathrm{HH}^{\mathrm{ev}}(\Lambda_0) \simeq K[y_6], \quad \text{where } \deg y_6 = 6.$$

(2) *If $n = 1$, then*

$$\begin{aligned} \mathrm{HH}^{\mathrm{ev}}(\Lambda_1) &\simeq K[y_0, y_2, y_4, y_6] \\ &\quad / (y_0^2, y_0 y_2, y_0 y_4, y_2^2, y_2 y_4 - y_0 y_6, y_4^2 - y_2 y_6), \end{aligned}$$

where $\deg y_i = i$ ($i = 0, 2, 4, 6$).

(3) *If $n > 1$, then*

$$\begin{aligned} \mathrm{HH}^{\mathrm{ev}}(\Lambda_n) &\simeq K[y_0, y_2, y_4, y_6] \\ &\quad / (y_0^{n+1}, y_0^n y_2, y_0^n y_4, y_2^2 - y_0 y_4, y_2 y_4 - y_0 y_6, y_4^2 - y_2 y_6), \end{aligned}$$

where $\deg y_i = i$ ($i = 0, 2, 4, 6$).

Proof. We put

$$y_0 := X_{0,1}, \quad y_2 := X_{2,0}, \quad y_4 := X_{4,0}, \quad y_6 := X_{6,0}.$$

If $\mathrm{char} K \mid 3n + 2$ and $n = 0$ (hence $\mathrm{char} K = 2$), then note that $\mathrm{HH}^{6u+s}(\Lambda_0) = K$ and $\mathrm{HH}^{6u+2}(\Lambda_0) = 0$ hold for all $u \geq 0$ and $s = 0, 4$. Since $X_{6u,0} = y_6^u$ and $X_{6u+4,0} = y_4 y_6^u$ hold for all $u \geq 0$, $\mathrm{HH}^{\mathrm{ev}}(\Lambda_0)$ is multiplicatively generated by y_4 and y_6 . Moreover the equation $y_4^2 = 0$ holds.

If $\text{char } K \nmid 3n + 2$ and $n = 0$, then $\text{HH}^{\text{ev}}(\Lambda_0) = \text{HH}^{6*}(\Lambda_0)$. So by Proposition 2.1 we have the desired isomorphism.

If $n > 0$, from Lemmas 2.2 and Lemmas 3.3 through 3.7, we have the following relations:

$$\begin{aligned} y_0^{n+1} &= 0, & y_0^n y_2 &= 0, \\ y_2^2 &= \begin{cases} y_0 y_4 & \text{if } n = 1 \text{ and } \text{char } K \mid 3n + 2, \text{ or if } n > 1 \\ 0 & \text{if } n = 1 \text{ and } \text{char } K \nmid 3n + 2, \end{cases} \\ y_0^n y_4 &= \begin{cases} X_{4,n} & \text{if } \text{char } K \mid 3n + 2 \\ 0 & \text{if } \text{char } K \nmid 3n + 2, \end{cases} \\ y_4^2 &= y_2 y_6. \end{aligned}$$

Furthermore we have that

$$\begin{aligned} X_{6u,s} &= y_6^u y_0^s && \text{for } u \geq 0 \text{ and } 0 \leq s \leq n, \\ X_{6u+2,s} &= y_6^u y_2 y_0^s && \text{for } u \geq 0 \text{ and } 0 \leq s < n, \\ X_{6u+4,s} &= y_6^u y_4 y_0^s && \text{for } u \geq 0 \text{ and } 0 \leq s < n, \\ X_{6u+4,n} &= y_6^u y_4 y_0^s && \text{for } u \geq 0 \text{ (if } \text{char } K \mid 3n + 2). \end{aligned}$$

Therefore we take $\{y_0, y_2, y_4, y_6\}$ as algebra generators of $\text{HH}^{\text{ev}}(\Lambda_n)$. This completes the proof. □

§4. The Hochschild cohomology ring $\text{HH}^*(\Lambda_n)$

In this section we describe all products of basis elements of whole Hochschild cohomology ring $\text{HH}^*(\Lambda_n)$, and then describe its ring structure of the Hochschild, completely. Throughout this section, we keep the notations from Sections 2 and 3.

For simplicity, we denote the basis elements of $\text{HH}^{6j+1}(\Lambda_n)$ given in [F, Lemma 4.1]) as follows: for $j \geq 0$

$$\begin{aligned} X_{6j+1,0} &:= \mu_0^0 + (n + 1)\nu_0 : Q^{6j+1} \rightarrow \Lambda_n; \\ X_{6j+1,m} &:= \mu_0^m : Q^{6j+1} \rightarrow \Lambda_n \text{ for } m = 1, \dots, n, \text{ if } n > 0, \end{aligned}$$

which is given by

$$\begin{aligned}
 X_{6j+1,0} &: \begin{cases} e_l \otimes e_{l+1} & \mapsto \begin{cases} a_0 & \text{for } l = 0 \\ 0 & \text{for } l = 1, 2 \end{cases} \\ f_r \otimes f_{r+1} & \mapsto \begin{cases} (n+1)b_0 & \text{for } r = 0 \\ 0 & \text{for } r = 1, \end{cases} \end{cases} \\
 X_{6j+1,m} &: \begin{cases} e_l \otimes e_{l+1} & \mapsto \begin{cases} (a_l a_{l+1} a_{l+2})^m a_l & \text{for } l = 0 \\ 0 & \text{for } l = 1, 2 \end{cases} \\ f_r \otimes f_{r+1} & \mapsto 0 & \text{for } r = 0, 1, \end{cases}
 \end{aligned}$$

respectively. Similarly, we denote the basis elements of $\text{HH}^{6j+3}(\Lambda_n)$ given in [F, Lemma 4.1] as follows: for $j \geq 0$

$$\begin{aligned}
 X_{6j+3,0} &:= \phi_0^0 + \phi_1^0 + \phi_2^0 + \psi : Q^{6j+3} \rightarrow \Lambda_n, \text{ if } \text{char } K \mid 3n+2; \\
 X_{6j+3,m} &:= \phi_0^m + \phi_1^m + \phi_2^m : Q^{6j+3} \rightarrow \Lambda_n \\
 &\quad \text{for } m = 1, \dots, n, \text{ if } n > 0 \text{ and } \text{char } K \mid 3n+2; \\
 X_{6j+3,m} &:= \phi_0^m : Q^{6j+3} \rightarrow \Lambda_n \text{ for } m = 1, \dots, n, \text{ if } n > 0 \text{ and } \text{char } K \nmid 3n+2.
 \end{aligned}$$

If $\text{char } K \mid 3n+2$, then $X_{6j+3,m}$ is given by

$$X_{6j+3,m} : \begin{cases} e_l \otimes e_l & \mapsto (a_l a_{l+1} a_{l+2})^m & \text{for } l = 0, 1, 2 \\ f_1 \otimes f_1 & \mapsto \begin{cases} f_1 & \text{if } m = 0 \\ 0 & \text{if } m \geq 1 \end{cases} \\ e_1 \otimes f_1 & \mapsto 0 \\ f_1 \otimes e_1 & \mapsto 0, \end{cases}$$

where $m = 0, 1, \dots, n$. If $\text{char } K \nmid 3n+2$ and $n > 0$, then $X_{6j+3,m}$ is given by

$$X_{6j+3,m} : \begin{cases} e_l \otimes e_l & \mapsto \begin{cases} (a_0 a_1 a_2)^m & \text{for } l = 0 \\ 0 & \text{for } l = 1, 2 \end{cases} \\ f_1 \otimes f_1 & \mapsto 0 \\ e_1 \otimes f_1 & \mapsto 0 \\ f_1 \otimes e_1 & \mapsto 0, \end{cases}$$

where $m = 1, \dots, n$. Moreover, if $n > 0$, we denote the basis elements of $\text{HH}^{6j+5}(\Lambda_n)$ as follows: for $j \geq 0$ and $m = 0, 1, \dots, n-1$

$$X_{6j+5,m} := \theta_0^m : Q^{6j+5} \rightarrow \Lambda_n,$$

which is given by

$$X_{6j+5,m} : \begin{cases} e_l \otimes e_{l+2} & \mapsto \begin{cases} (a_0 a_1 a_2)^m a_0 a_1 & \text{for } l = 0 \\ 0 & \text{for } l = 1, 2 \end{cases} \\ f_r \otimes f_{r+2} & \mapsto 0 \quad \text{for } r = 1, 2. \end{cases}$$

4.1. An initial part of liftings of $X_{1,0}$, $X_{3,0}$, $X_{3,1}$, and $X_{5,0}$

We start by giving an initial part of liftings for $X_{1,0}$, $X_{3,0}$, $X_{3,1}$, and $X_{5,0}$.

Definition 4.1. For $u \geq 0$ and $j = 0, 1$, we define homomorphisms $\sigma_{6u+1,0}^j : Q^{6u+1+j} \rightarrow Q^j$ as Λ_n - Λ_n -bimodules by the following formulas:

$$\begin{cases} \sigma_{6u+1,0}^0 : \begin{cases} e_l \otimes e_{l+1} \mapsto \begin{cases} a_0 \otimes e_1 & \text{for } l = 0 \\ 0 & \text{for } l = 1, 2 \end{cases} \\ f_r \otimes f_{r+1} \mapsto \begin{cases} (n+1)b_0 \otimes f_1 & \text{for } r = 0 \\ 0 & \text{for } r = 1, \end{cases} \end{cases} \\ \sigma_{6u+1,0}^1 : \begin{cases} e_l \otimes e_{l+2} \mapsto \begin{cases} \sum_{k=0}^n (k+1)(a_0 a_1 a_2)^k a_0 \otimes (a_2 a_0 a_1)^{n-k} \\ + \sum_{k=0}^{n-1} (k+1)(a_0 a_1 a_2)^k a_0 a_1 \otimes a_0 a_1 (a_2 a_0 a_1)^{n-k-1} \\ + \sum_{k=1}^n k(a_0 a_1 a_2)^k \otimes a_1 (a_2 a_0 a_1)^{n-k} - (n+1)b_0 \otimes e_2 & \text{for } l = 0 \\ \sum_{k=1}^n k(a_1 a_2 a_0)^k \otimes a_2 (a_0 a_1 a_2)^{n-k} \\ + \sum_{k=1}^n k(a_1 a_2 a_0)^k a_1 \otimes (a_0 a_1 a_2)^{n-k} \\ + \sum_{k=1}^{n-1} k(a_1 a_2 a_0)^k a_1 a_2 \otimes a_1 a_2 (a_0 a_1 a_1)^{n-k-1} & \text{for } l = 1 \\ \sum_{k=0}^{n-1} (k+1)(a_2 a_0 a_1)^k a_2 a_0 \otimes a_2 a_0 (a_1 a_2 a_0)^{n-k-1} \\ + \sum_{k=1}^n k(a_2 a_0 a_1)^k \otimes a_0 (a_1 a_2 a_0)^{n-k} \\ + \sum_{k=1}^n k(a_2 a_0 a_1)^k a_2 \otimes (a_1 a_2 a_0)^{n-k} & \text{for } l = 2 \end{cases} \\ f_r \otimes f_{r+2} \mapsto 0 \quad \text{for } r = 1, 2. \end{cases} \end{cases}$$

By direct computations, we have the following lemma.

Lemma 4.1. *We have*

$$X_{6u+1,0} = \partial^0 \sigma_{6u+1,0}^0 \text{ and } \partial^1 \sigma_{6u+1,0}^1 = \sigma_{6u+1,0}^0 \partial^2$$

for $u \geq 0$. Hence $\sigma_{6u+1,0}^i$ ($i = 0, 1$) is an initial part of a lifting of $X_{6u+1,0}$.

Definition 4.2. *If $\text{char } K \mid 3n + 2$, for $u \geq 0$ and $j = 0, 1, 2, 3$, we define homomorphisms $\sigma_{6u+3,0}^j : Q^{6u+3+j} \rightarrow Q^j$ as Λ_n - Λ_n -bimodules by the following formulas:*

$$\sigma_{6u+3,0}^0 : \begin{cases} e_l \otimes e_l & \mapsto e_l \otimes e_l \text{ for } l = 0, 1, 2 \\ f_1 \otimes f_1 & \mapsto f_1 \otimes f_1 \\ e_1 \otimes f_1 & \mapsto 0 \\ f_1 \otimes e_1 & \mapsto 0, \end{cases}$$

$$\sigma_{6u+3,0}^1 : \begin{cases} e_l \otimes e_{l+1} & \mapsto \begin{cases} e_l \otimes e_{l+1} & \text{if } n = 0 \text{ and } l = 0, 1, 2 \\ e_0 \otimes (a_1 a_2 a_0)^n \\ \quad + \sum_{k=0}^{3n-2} (k+2) a_0 a_1 a_2 \cdots a_k \otimes a_{k+2} \cdots a_{3n} \\ \quad \quad \quad + (3n+1)(a_0 a_1 a_2)^n \otimes e_1 & \text{if } n \geq 1 \text{ and } l = 0 \end{cases} \\ e_1 \otimes e_{l+1} & \mapsto \begin{cases} e_1 \otimes (a_2 a_0 a_1)^n \\ \quad + \sum_{k=1}^{3n-1} (k+1) a_1 a_2 \cdots a_k \otimes a_{k+2} \cdots a_{3n+1} \\ \quad \quad \quad + (3n+1)(a_1 a_2 a_0)^n \otimes e_2 & \text{if } n \geq 1 \text{ and } l = 1 \end{cases} \\ e_2 \otimes e_{l+1} & \mapsto \begin{cases} e_2 \otimes (a_0 a_1 a_2)^n \\ \quad + \sum_{k=2}^{3n} k a_2 a_3 \cdots a_k \otimes a_{k+2} \cdots a_{3n+2} \\ \quad \quad \quad + (3n+1)(a_2 a_0 a_1)^n \otimes e_0 & \text{if } n \geq 1 \text{ and } l = 2 \end{cases} \\ f_r \otimes f_{r+1} & \mapsto (-1)^r f_r \otimes f_{r+1} \text{ for } r = 0, 1, \end{cases}$$

$$\sigma_{6u+3,0}^2 : \begin{cases} e_l \otimes e_{l+2} & \mapsto e_l \otimes e_{l+2} \text{ for } l = 0, 1, 2 \\ f_r \otimes f_{r+2} & \mapsto \begin{cases} -f_1 \otimes (a_0 a_1 a_2)^n & \text{for } r = 1 \\ -(a_2 a_0 a_1)^n \otimes f_1 & \text{for } r = 2, \end{cases} \end{cases}$$

$$\sigma_{6u+3,0}^3 : \left\{ \begin{array}{l} e_l \otimes e_l \mapsto \begin{cases} e_l \otimes e_l & \text{if } n = 0 \text{ and } l = 0, 1, 2 \\ e_0 \otimes (a_0 a_1 a_2)^n + \\ \sum_{k=0}^{3n-2} (k+2) a_0 a_1 a_2 \cdots a_k \otimes a_{k+1} \cdots a_{3n-1} \\ + (3n+1)(a_0 a_1 a_2)^n \otimes e_0 & \text{if } n \geq 1 \text{ and } l = 0 \\ e_1 \otimes (a_1 a_2 a_0)^n + \\ \sum_{k=1}^{3n-1} (k+1) a_1 a_2 \cdots a_k \otimes a_{k+1} \cdots a_{3n} \\ + (3n+1)(a_1 a_2 a_0)^n \otimes e_1 & \text{if } n \geq 1 \text{ and } l = 1 \\ e_2 \otimes (a_2 a_0 a_1)^n + \\ \sum_{k=2}^{3n} k a_2 a_3 \cdots a_k \otimes a_{k+1} \cdots a_{3n+1} \\ + (3n+1)(a_2 a_0 a_1)^n \otimes e_2 & \text{if } n \geq 1 \text{ and } l = 2 \end{cases} \\ f_1 \otimes f_1 \mapsto \begin{cases} f_1 \otimes f_1 & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases} \\ e_1 \otimes f_1 \mapsto e_1 \otimes f_1 \\ f_1 \otimes e_1 \mapsto -f_1 \otimes e_1. \end{array} \right.$$

Then we have the following lemma.

Lemma 4.2. *If $\text{char } K \mid 3n + 2$, we have that $X_{6u+3,0} = \partial^0 \sigma_{6u+3,0}^0$ and $\partial^i \sigma_{6u+3,0}^i = \sigma_{6u+3,0}^{i-1} \partial^{i+3}$ for all $u \geq 0$ and $i = 1, 2, 3$. Hence $\sigma_{6u+3,0}^i$ ($i = 0, 1, 2, 3$) is an initial part of a lifting of $X_{6u+3,0}$.*

Definition 4.3. *If $\text{char } K \nmid 3n+2$ and $n > 0$, for $u \geq 0$, we define a homomorphism $\sigma_{6u+3,1}^0 : Q^{6u+3} \rightarrow Q^0$ as Λ_n - Λ_n -bimodules by the following formulas:*

$$\sigma_{6u+3,1}^0 : \left\{ \begin{array}{l} e_l \otimes e_l \mapsto \begin{cases} a_0 a_1 a_2 \otimes e_0 & \text{for } l = 0 \\ 0 & \text{for } l = 1, 2 \end{cases} \\ f_1 \otimes f_1 \mapsto 0 \\ e_1 \otimes f_1 \mapsto 0 \\ f_1 \otimes e_1 \mapsto 0, \end{array} \right.$$

Clearly, the following lemma holds.

Lemma 4.3. *If $\text{char } K \nmid 3n + 2$ and $n > 0$, the equation $X_{6u+3,1} = \partial^0 \sigma_{6u+3,1}^0$ holds for all $u \geq 0$. Hence $\sigma_{6u+3,1}^0$ is an initial part of a lifting of $X_{6u+3,1}$.*

Definition 4.4. If $n > 0$, for $u \geq 0$, we define a homomorphism $\sigma_{6u+5,0}^0 : Q^{6u+5} \rightarrow Q^0$ as Λ_n - Λ_n -bimodules by the following formula:

$$\sigma_{6u+5,0}^0 : \begin{cases} e_l \otimes e_{l+2} & \mapsto \begin{cases} a_0 a_1 \otimes e_2 & \text{for } l = 0 \\ 0 & \text{for } l = 1, 2 \end{cases} \\ f_r \otimes f_{r+2} & \mapsto 0 \quad \text{for } r = 1, 2. \end{cases}$$

Lemma 4.4. If $n > 0$, we have that $X_{6u+5,0} = \partial^0 \sigma_{6u+5,0}^0$ for all $u \geq 0$. Hence $\sigma_{6u+5,0}^0$ is an initial part of a lifting of $X_{6u+5,0}$.

4.2. The products in $\text{HH}^{6u}(\Lambda_n) \times \text{HH}^{6v+1}(\Lambda_n)$

First we compute the products of elements in $\text{HH}^{6u}(\Lambda_n)$ and $\text{HH}^{6v+1}(\Lambda_n)$ for $u \geq 0$ and $v \geq 0$.

Lemma 4.5. We have the following products:

- (a) $X_{0,s} \times X_{6u+1,0} = X_{6u+1,s}$ for any integers $u \geq 0$ and s with $0 \leq s \leq n$.
- (b) $X_{6u,0} \times X_{1,0} = X_{6u+1,0}$ for any integer $u \geq 1$.

Therefore for any integers $u_1 \geq 0$, $u_2 \geq 0$, s_1 , and s_2 with $0 \leq s_1, s_2 \leq n$, we have

$$X_{6u_1,s_1} \times X_{6u_2+1,s_2} = \begin{cases} X_{6(u_1+u_2)+1,s_1+s_2} & \text{if } s_1 + s_2 \leq n \\ 0 & \text{if } s_1 + s_2 > n. \end{cases}$$

Proof. Since, for integers u and s with $u \geq 0$ and $0 \leq s \leq n$,

$$X_{0,s} \sigma_{6u+1,0}^0 : \begin{cases} e_l \otimes e_{l+1} \mapsto \begin{cases} (a_0 a_1 a_2)^s a_0 & \text{for } l = 0 \\ 0 & \text{for } l = 1, 2 \end{cases} \\ f_r \otimes f_{r+1} \mapsto \begin{cases} (n+1)b_0 & \text{if } s = r = 0 \\ 0 & \text{otherwise,} \end{cases} \end{cases}$$

we have that $X_{0,s} \sigma_{6u+1,0}^0 = X_{6u+1,s}$. Furthermore, it is clear that $X_{1,0} \sigma_{6u,0}^1 = X_{6u+1,0}$ for all $u > 0$. Thus (a) and (b) are proved. The last equation follows from (a), (b), and Lemma 2.2. □

4.3. The products in $\text{HH}^{6u}(\Lambda_n) \times \text{HH}^{6v+3}(\Lambda_n)$

In the following, we calculate the products of elements in $\text{HH}^{6u}(\Lambda_n)$ and $\text{HH}^{6v+3}(\Lambda_n)$ for $u \geq 0$ and $v \geq 0$.

Lemma 4.6. *Let $\text{char } K \mid 3n + 2$. We have the following products:*

(a) $X_{0,s} \times X_{6u+3,0} = X_{6u+3,s}$ for any integers $u \geq 0$ and s with $0 \leq s \leq n$.

(b) $X_{6u,0} \times X_{3,0} = X_{6u+3,0}$ for any integer $u \geq 1$.

Therefore for any integers $u_1 \geq 0$, $u_2 \geq 0$, s_1 , and s_2 with $0 \leq s_1, s_2 \leq n$, we have

$$X_{6u_1,s_1} \times X_{6u_2+3,s_2} = \begin{cases} X_{6(u_1+u_2)+3,s_1+s_2} & \text{if } s_1 + s_2 \leq n \\ 0 & \text{if } s_1 + s_2 > n. \end{cases}$$

Proof. Since, for integers u and s with $u \geq 0$ and $0 \leq s \leq n$,

$$X_{0,s}\sigma_{6u+3,0}^0 : \begin{cases} e_l \otimes e_l \mapsto (a_l a_{l+1} a_{l+2})^s & \text{for } l = 0, 1, 2 \\ f_1 \otimes f_1 \mapsto \begin{cases} f_1 & \text{if } s = 0 \\ 0 & \text{if } 0 < s \leq n \end{cases} \\ e_1 \otimes f_1 \mapsto 0 \\ f_1 \otimes e_1 \mapsto 0, \end{cases}$$

we have that $X_{0,s}\sigma_{6u+3,0}^0 = X_{6u+3,s}$. Moreover, we have $X_{3,0}\sigma_{6u,0}^3 = X_{6u+3,0}$ for all $u > 0$. So (a) and (b) are proved. The last equation follows from (a), (b), and Lemma 2.2. \square

Lemma 4.7. *Let $\text{char } K \nmid 3n + 2$ and $n > 0$. We have the following products:*

(a) For any integers $u \geq 0$ and s with $0 \leq s \leq n$,

$$X_{0,s} \times X_{6u+3,1} = \begin{cases} X_{6u+3,s+1} & \text{if } 0 \leq s < n \\ 0 & \text{if } s = n. \end{cases}$$

(b) $X_{6u,0} \times X_{3,1} = X_{6u+3,1}$ for any integer $u \geq 1$.

Therefore, for any integers $u_1 \geq 0$, $u_2 \geq 0$, s_1 with $0 \leq s_1 \leq n$, and s_2 with $0 < s_2 \leq n$, we have

$$X_{6u_1,s_1} \times X_{6u_2+3,s_2} = \begin{cases} X_{6(u_1+u_2)+3,s_1+s_2+1} & \text{if } s_1 + s_2 < n \\ 0 & \text{if } s_1 + s_2 \geq n. \end{cases}$$

Proof. Since for integers u and s with $u \geq 0$ and $0 \leq s \leq n$,

$$X_{0,s}\sigma_{6u+3,1}^0 : \begin{cases} e_l \otimes e_l \mapsto \begin{cases} (a_0 a_1 a_2)^{s+1} & \text{for } l = 0 \\ 0 & \text{for } l = 1, 2 \end{cases} \\ f_1 \otimes f_1 \mapsto 0 \\ e_1 \otimes f_1 \mapsto 0 \\ f_1 \otimes e_1 \mapsto 0, \end{cases}$$

we have that $X_{0,s}\sigma_{6u+3,1}^0 = X_{6u+3,s+1}$. Moreover, for all $u > 0$, we have $X_{3,1}\sigma_{6u,0}^3 = X_{6u+3,1}$. Thus (a) and (b) are proved. The last equation follows from (a), (b), and Lemma 2.2. \square

4.4. The products in $\text{HH}^{6u}(\Lambda_n) \times \text{HH}^{6v+5}(\Lambda_n)$

Lemma 4.8. *Let $n > 0$. We have the following products:*

(a) *For any integers $u \geq 0$ and s with $0 \leq s \leq n$,*

$$X_{0,s} \times X_{6u+5,0} = \begin{cases} X_{6u+5,s} & \text{if } 0 \leq s < n \\ 0 & \text{if } s = n. \end{cases}$$

(b) $X_{6u,0} \times X_{5,0} = X_{6u+5,0}$ for any integer $u \geq 1$.

Therefore, for any integers $u_1 \geq 0$, $u_2 \geq 0$, s_1 with $0 \leq s_1 \leq n$, and s_2 with $0 \leq s_2 < n$, we have

$$X_{6u_1,s_1} \times X_{6u_2+5,s_2} = \begin{cases} X_{6(u_1+u_2)+5,s_1+s_2} & \text{if } s_1 + s_2 < n \\ 0 & \text{if } s_1 + s_2 \geq n. \end{cases}$$

Proof. Since, for integers u and s with $u \geq 0$ and $0 \leq s \leq n$,

$$X_{0,s}\sigma_{6u+5,0}^0 : \begin{cases} e_l \otimes e_{l+2} \mapsto \begin{cases} (a_0a_1a_2)^s a_0a_1 & \text{for } l = 0 \\ 0 & \text{for } l = 1, 2 \end{cases} \\ f_r \otimes f_{r+2} \mapsto 0 & \text{for } r = 1, 2, \end{cases}$$

we have that $X_{0,s}\sigma_{6u+5,0}^0 = \theta_0^s = X_{6u+5,s}$ for $0 \leq s < n$ and $X_{0,n}\sigma_{6u+5,0}^0 = \eta \in \text{Im Hom}_{\Lambda_n^e}(\partial^{6u+5}, \Lambda_n)$ by [F, Lemma 4.5 (e)]. Therefore (a) is proved. As for (b), we have $X_{5,0}\sigma_{6u,0}^5 = X_{6u+5,0}$ for all $u > 0$. The last equation follows from (a), (b), and Lemma 2.2. \square

4.5. The products in $\text{HH}^{6u+1}(\Lambda_n) \times \text{HH}^{6v+1}(\Lambda_n)$

Lemma 4.9. *We have $X_{1,0}^2 = 0$. Hence for any integers $u_1 \geq 0$, $u_2 \geq 0$, s_1 , and s_2 with $0 \leq s_1, s_2 \leq n$, we have $X_{6u_1+1,s_1} \times X_{6u_2+1,s_2} = 0$.*

Proof. We have

$$X_{1,0}\sigma_{1,0}^1 : \begin{cases} e_l \otimes e_{l+2} \mapsto \begin{cases} \frac{n(n+1)}{2}(a_0a_1a_2)^n a_0a_1 & \text{for } l = 0 \\ 0 & \text{for } l = 1, 2 \end{cases} \\ f_r \otimes f_{r+2} \mapsto 0 & \text{for } r = 1, 2. \end{cases}$$

Since η is an element in $\text{Im Hom}_{A_n^e}(\partial^2, A_n)$ (see [F, Lemma 4.5 (b)]), we have $X_{1,0}^2 = (n(n+1))/2 \eta = 0$ in $\text{HH}^2(A_n)$. The second equation follows from the first equation and Lemmas 2.2 and 4.5. \square

4.6. The products in $\text{HH}^{6u+1}(A_n) \times \text{HH}^{6v+2}(A_n)$

Next we describe the products in $\text{HH}^{6u+1}(A_n) \times \text{HH}^{6v+4}(A_n)$ for $u \geq 0$ and $v \geq 0$.

Lemma 4.10. *Let $n > 0$. For any integer $u \geq 0$, we have the following product:*

$$X_{6u+1,0} \times X_{2,0} = \begin{cases} 3^{-1}X_{6u+3,1} & \text{if char } K \mid 3n + 2 \\ X_{6u+3,1} & \text{if char } K \nmid 3n + 2. \end{cases}$$

Hence, for any integers $u_1 \geq 0, u_2 \geq 0, s_1$ with $0 \leq s_1 \leq n$, and s_2 with $0 \leq s_2 < n$, we have

$$\begin{aligned} & X_{6u_1+1,s_1} \times X_{6u_2+2,s_2} \\ &= \begin{cases} 3^{-1}X_{6(u_1+u_2)+3,s_1+s_2+1} & \text{if char } K \mid 3n + 2 \text{ and } s_1 + s_2 < n \\ X_{6(u_1+u_2)+3,s_1+s_2+1} & \text{if char } K \nmid 3n + 2 \text{ and } s_1 + s_2 < n \\ 0 & \text{if } s_1 + s_2 \geq n. \end{cases} \end{aligned}$$

Proof. Since, for any integer $u \geq 0$,

$$X_{6u+1,0}\sigma_{2,0}^1 : \begin{cases} e_l \otimes e_l \mapsto \begin{cases} 0 & \text{for } l = 0, 2 \\ a_1a_2a_0 & \text{for } l = 1 \end{cases} \\ e_1 \otimes f_1 \mapsto 0 \\ f_1 \otimes e_1 \mapsto 0 \\ f_1 \otimes f_1 \mapsto 0, \end{cases}$$

we have

$$\begin{aligned} X_{6u+1,0} \times X_{2,0} &= \phi_1^1 \\ &= \begin{cases} \phi_0^1 + \phi_1^1 + \phi_2^1 + (\phi_0^1 - \phi_2^1) & \text{if char } K \mid 3n + 2 \text{ and char } K = 2 \\ 3^{-1}(\phi_0^1 + \phi_1^1 + \phi_2^1) + 3^{-1}(\phi_0^1 - \phi_2^1) + 3^{-1} \cdot 2(\phi_1^1 - \phi_0^1) & \text{if char } K \mid 3n + 2 \text{ and char } K \neq 2 \\ \phi_0^1 + (\phi_1^1 - \phi_0^1) & \text{if char } K \nmid 3n + 2. \end{cases} \end{aligned}$$

Since $\phi_0^1 - \phi_2^1$ and $\phi_1^1 - \phi_0^1$ are in $\text{Im Hom}_{A_n^e}(\partial^{6u+3}, A_n)$ (see [F, Lemma 4.5 (c)]), the first equation is proved. The second equation follows from the first equation and Lemmas 2.2, 3.3, 4.6, and 4.7. \square

Remark 4.1. Let $\text{char } K \nmid 3n + 2$ and $n > 0$. Then we have $X_{6u+3,s} = X_{6,0}^u X_{0,1}^{s-1} X_{1,0} X_{2,0}$ for $u \geq 0$ and $0 < s \leq n$. Hence $\text{HH}^{6u+3}(\Lambda_n)$ is generated by the products of $X_{6,0}, X_{0,1}, X_{1,0}, X_{2,0}$ for $u \geq 0$.

4.7. The products in $\text{HH}^{6u+1}(\Lambda_n) \times \text{HH}^{6v+3}(\Lambda_n)$

Lemma 4.11. Let $\text{char } K \mid 3n + 2$. For any integer $u \geq 0$, we have the following product:

$$X_{6u+1,0} \times X_{3,0} = \begin{cases} X_{6u+4,n} & \text{if char } K = 2 \text{ and } n \equiv 0 \pmod{4} \\ 0 & \text{otherwise} \end{cases}$$

Hence, for any integers $u_1 \geq 0, u_2 \geq 0, s_1,$ and s_2 with $0 \leq s_1, s_2 \leq n$, we have

$$X_{6u_1+1,s_1} \times X_{6u_2+3,s_2} = \begin{cases} X_{6(u_1+u_2)+4,n} & \text{if char } K = 2, n \equiv 0 \pmod{4} \\ & \text{and } s_1 = s_2 = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since, for an integer $u \geq 0$,

$$X_{6u+1,0}\sigma_{3,0}^1 : \begin{cases} e_l \otimes e_{l+1} \mapsto \begin{cases} \frac{(3n+2)(n+1)}{2} (a_0 a_1 a_2)^n a_0 & \text{for } l = 0 \\ \frac{3n(n+1)}{2} (a_1 a_2 a_0)^n a_1 & \text{for } l = 1 \\ \frac{(3n+1)n}{2} (a_2 a_0 a_1)^n a_2 & \text{for } l = 2 \end{cases} \\ f_r \otimes f_{r+1} \mapsto \begin{cases} (n+1)b_0 & \text{for } r = 1 \\ 0 & \text{for } r = 2 \end{cases} \end{cases}$$

it follows that

$$\begin{aligned} & X_{6u+1,0} \times X_{3,0} \\ &= \frac{(3n+2)(n+1)}{2} \mu_0^n + \frac{3n(n+1)}{2} \mu_1^n + \frac{(3n+1)n}{2} \mu_2^n + (n+1)\nu_0 \\ &= \begin{cases} -(n+1)(\mu_1^n - \mu_2^n - \nu_0) & \text{if } n \text{ odd} \\ \frac{3n+2}{2} (\mu_0^n + \mu_1^n + \mu_2^n) - (n+1)(\mu_1^n - \mu_2^n - \nu_0) & \text{if } n \text{ even} \end{cases} \end{aligned}$$

hold. Note that $\mu_1^n - \mu_2^n - \nu_0 \in \text{Im Hom}_{A_n^e}(\partial^{6u+4}, \Lambda_n)$ by [F, Lemma 4.5 (d)]. Therefore we have that

$$X_{6u+1,0} \times X_{3,0} = \begin{cases} 0 & \text{if } n \text{ odd} \\ \frac{3n+2}{2} X_{6u+4,n} & \text{if } n \text{ even.} \end{cases}$$

If n is even, then $\text{char } K = 2$ or $\text{char } K \mid (3n+2)/2$. Thus we have

$$\frac{3n+2}{2} = \begin{cases} 1 & \text{if char } K = 2 \text{ and } n \equiv 0 \pmod{4} \\ 0 & \text{otherwise.} \end{cases}$$

Hence the first equation is proved. The second equality follows from the first equation and Lemmas 2.2, 3.4, 4.5, and 4.6. \square

Corollary 4.1. *Let $\text{char } K \nmid 3n+2$ and $n > 0$. For any integer $u \geq 0$, we have $X_{6u+1,0} \times X_{3,1} = 0$. Hence, for any integers $u_1 \geq 0$, $u_2 \geq 0$, s_1 with $0 \leq s_1 \leq n$, and s_2 with $0 < s_2 \leq n$, we have $X_{6u_1+1,s_1} \times X_{6u_2+3,s_2} = 0$.*

4.8. The products in $\text{HH}^{6u+1}(\Lambda_n) \times \text{HH}^{6v+4}(\Lambda_n)$

Lemma 4.12. *Suppose that $n > 0$ or $\text{char } K \mid 3n+2$. For any integer $u \geq 0$, we have the following products:*

$$X_{6u+1,0} \times X_{4,0} = \begin{cases} 0 & \text{if char } K = 2 \text{ and } n = 0 \\ X_{6u+5,0} & \text{otherwise.} \end{cases}$$

Hence for any integers $u_1 \geq 0$, $u_2 \geq 0$, s_1 with $0 \leq s_1 \leq n$, and s_2 with $0 \leq s_2 \leq n$ (if $\text{char } K \mid 3n+2$) or $0 \leq s_2 < n$ (if $\text{char } K \nmid 3n+2$), we have

$$X_{6u_1+1,s_1} \times X_{6u_2+4,s_2} = \begin{cases} X_{6(u_1+u_2)+5,s_1+s_2} & \text{if } n > 0 \text{ and } s_1 + s_2 < n \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since, for any integer $u \geq 0$,

$$X_{6u+1,0} \sigma_{4,0}^1 : \begin{cases} e_l \otimes e_{l+2} \mapsto \begin{cases} 0 & \text{for } l = 0, 1 \\ a_2 a_0 & \text{for } l = 2 \end{cases} \\ f_r \otimes f_{r+2} \mapsto 0 & \text{for } r = 1, 2, \end{cases}$$

we have that

$$\begin{aligned} X_{6u+1,0} \times X_{4,0} &= \begin{cases} 0 & \text{if char } K \mid 3n+2 \text{ and } n = 0 \\ \theta_2^0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 0 & \text{if char } K = 2 \text{ and } n = 0 \\ \theta_0^0 - (\theta_0^0 - \theta_2^0) & \text{otherwise.} \end{cases} \end{aligned}$$

Since $\theta_0^0 - \theta_2^0$ is in $\text{Im Hom}_{A_n^e}(\partial^{6u+5}, \Lambda_n)$ (see [F, Lemma 4.5 (e)]), the first equation is proved. The second equation follows from the first equation and Lemmas 2.2, 3.4, 4.5, and 4.8. \square

Remark 4.2. *If $n > 0$, then by Lemma 4.8 and 4.12 we have $X_{6u+5,s} = X_{6,0}^u X_{0,1}^s X_{1,0} X_{4,0}$ holds for $u \geq 0$ and $0 \leq s < n$. Thus $\text{HH}^{6u+5}(\Lambda_n)$ is generated by the products of $X_{6,0}, X_{1,0}, X_{4,0}, X_{0,1}$ for $u > 0$. Note that the equation $X_{0,1}^n X_{1,0} X_{4,0} = 0$ holds.*

4.9. The products in $\text{HH}^{6u+1}(\Lambda_n) \times \text{HH}^{6v+5}(\Lambda_n)$

By the previous sections, we have the following corollary:

Corollary 4.2. *Let $n > 0$. For any integer $u \geq 0$, we have $X_{6u+1,0} \times X_{5,0} = 0$. Hence, for any integers $u_1 \geq 0, u_2 \geq 0, s_1$ with $0 \leq s_1 \leq n$, and s_2 with $0 \leq s_2 < n$, we have $X_{6u_1+1,s_1} \times X_{6u_2+5,s_2} = 0$.*

4.10. The products in $\text{HH}^{6u+2}(\Lambda_n) \times \text{HH}^{6v+3}(\Lambda_n)$

Lemma 4.13. *Let $\text{char } K \mid 3n + 2$ and $n > 0$ (hence $\text{char } K \neq 3$). For any integer $u \geq 0$, we have the following product:*

$$X_{6u+2,0} \times X_{3,0} = 3X_{6u+5,0}$$

Hence, for any integers $u_1 \geq 0, u_2 \geq 0, s_1$ with $0 \leq s_1 < n$, and s_2 with $0 \leq s_2 \leq n$, we have

$$X_{6u_1+2,s_1} \times X_{6u_2+3,s_2} = \begin{cases} 3X_{6(u_1+u_2)+5,s_1+s_2} & \text{if } s_1 + s_2 \leq n \\ 0 & \text{if } s_1 + s_2 > n. \end{cases}$$

Proof. Since

$$X_{6u+2,0} \sigma_{3,0}^2 = \begin{cases} e_l \otimes e_{l+2} \mapsto a_l a_{l+1} & \text{for } l = 0, 1, 2 \\ f_r \otimes f_{r+2} \mapsto 0 & \text{for } r = 1, 2, \end{cases}$$

it follows that

$$\begin{aligned} X_{6u+2,0} \times X_{3,0} &= \theta_0^0 + \theta_1^0 + \theta_2^0 \\ &= (\theta_1^0 - \theta_0^0) - (\theta_0^0 - \theta_2^0) + 3\theta_0^0 \\ &= 3\theta_0^0. \end{aligned}$$

holds in $\text{HH}^{6u+5}(\Lambda_n)$. Notice that $\theta_1^0 - \theta_0^0$ and $\theta_0^0 - \theta_2^0$ are in $\text{Im Hom}_{A_n^e}(\partial^{6u+5}, \Lambda_n)$ (see [F, Lemma 4.5 (e)]). The second equality follows from the first equation and Lemmas 2.2, 3.3, 4.6, and 4.8. \square

Corollary 4.3. *Let $\text{char } K \nmid 3n + 2$ and $n > 0$. For any integer $u \geq 0$, we have*

$$X_{6u+2,0} \times X_{3,1} = \begin{cases} 0 & \text{if } n = 1 \\ X_{6u+5,1} & \text{if } n \geq 2. \end{cases}$$

Therefore, for any integers $u_1 \geq 0$, $u_2 \geq 0$, s_1 with $0 \leq s_1 < n$, and s_2 with $0 < s_2 \leq n$, we have

$$X_{6u_1+2,s_1} \times X_{6u_2+3,s_2} = \begin{cases} X_{6(u_1+u_2)+5,s_1+s_2} & \text{if } s_1 + s_2 < n \\ 0 & \text{if } s_1 + s_2 \geq n. \end{cases}$$

4.11. The products in $\text{HH}^{6u+2}(\Lambda_n)$ and $\text{HH}^{6v+5}(\Lambda_n)$

By the previous sections we get the following corollary:

Corollary 4.4. *Let $n > 0$. For any integer $u \geq 0$, we have $X_{6u+2,0} \times X_{5,0} = X_{6u+7,1}$. Hence, for any integers $u_1 \geq 0$, $u_2 \geq 0$, s_1 , and s_2 with $0 \leq s_1, s_2 \leq n$, we have*

$$X_{6u_1+2,s_1} \times X_{6u_2+5,s_2} = \begin{cases} X_{6(u_1+u_2+1)+1,s_1+s_2+1} & \text{if } s_1 + s_2 < n \\ 0 & \text{if } s_1 + s_2 \geq n. \end{cases}$$

4.12. The products in $\text{HH}^{6u+3}(\Lambda_n) \times \text{HH}^{6v+3}(\Lambda_n)$

We next describe the products in $\text{HH}^{6u+3}(\Lambda_n) \times \text{HH}^{6v+3}(\Lambda_n)$ for $u, v \geq 0$. First we consider the case where $\text{char } K \mid 3n + 2$.

Lemma 4.14. *Let $\text{char } K \mid 3n + 2$. For any integer $u \geq 0$, we have the following product:*

$$X_{3,0}^2 = \begin{cases} X_{6,n} & \text{if } \text{char } K = 2 \text{ and } n \equiv 0 \pmod{4} \\ 0 & \text{otherwise.} \end{cases}$$

Hence, for any integers $u_1 \geq 0$, $u_2 \geq 0$, s_1 and s_2 with $0 \leq s_1, s_2 \leq n$, we have

$$X_{6u_1+3,s_1} \times X_{6u_2+3,s_2} = \begin{cases} X_{6(u_1+u_2+1),n} & \text{if } \text{char } K = 2, n \equiv 0 \pmod{4} \\ & \text{and } s_1 = s_2 = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since

$$X_{3,0}\sigma_{3,0}^3 : \begin{cases} e_l \otimes e_l \mapsto \frac{(3n+1)(3n+2)}{2} (a_l a_{l+1} a_{l+2})^n & \text{for } l = 0, 1, 2 \\ f_1 \otimes f_1 \mapsto \begin{cases} f_1 & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases} \\ e_1 \otimes f_1 \mapsto 0 \\ f_1 \otimes e_1 \mapsto 0, \end{cases}$$

we have

$$\begin{aligned} X_{3,0}^2 &= \begin{cases} \phi_0^0 + \phi_1^0 + \phi_2^0 + \psi & \text{if } n = 0 \\ \frac{(3n+1)(3n+2)}{2} (\phi_0^0 + \phi_1^0 + \phi_2^0) & \text{if } n \geq 1 \end{cases} \\ &= \frac{(3n+1)(3n+2)}{2} X_{6,n}. \end{aligned}$$

In the case n odd, we have $X_{3,0}^2 = 0$. If n is even, then $\text{char } K = 2$ or $\text{char } K \mid (3n+2)/2$. Thus we have that

$$\begin{aligned} \frac{(3n+1)(3n+2)}{2} &= -\frac{3n+2}{2} = \frac{3n+2}{2} \\ &= \begin{cases} 1 & \text{if } \text{char } K = 2 \text{ and } n \equiv 0 \pmod{4} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore the first equation is proved. The second equation follows from the first equation and Lemmas 2.2 and 4.6. \square

Similarly, by direct computations and the previous sections, we also have the following:

Corollary 4.5. *Let $\text{char } K \nmid 3n+2$ and $n > 0$. We have $X_{3,1}^2 = 0$. Therefore, for any integers $u_1 \geq 0$, $u_2 \geq 0$, s_1 , and s_2 with $0 < s_1, s_2 \leq n$, we have $X_{6u_1+3, s_1} \times X_{6u_2+3, s_2} = 0$.*

4.13. The products in $\text{HH}^{6u+3}(\Lambda_n) \times \text{HH}^{6v+4}(\Lambda_n)$

Lemma 4.15. *Let $\text{char } K \mid 3n+2$. For any integer $u \geq 0$, we have $X_{6u+4,0} \times X_{3,0} = 3X_{6u+7,0}$. Hence, for any integers $u_1 \geq 0$, $u_2 \geq 0$, s_1 and s_2 with $0 \leq s_1, s_2 \leq n$, we have*

$$X_{6u_1+3, s_1} \times X_{6u_2+4, s_2} = \begin{cases} X_{6(u_1+u_2)+7, s_1+s_2} & \text{if } s_1 + s_2 \leq n \\ 0 & \text{if } s_1 + s_2 > n. \end{cases}$$

Proof. For integer $u \geq 0$, we have the following:

$$X_{6u+3,0}\sigma_{4,0}^3 : \begin{cases} e_l \otimes e_{l+1} \mapsto a_l & \text{for } l = 0, 1, 2 \\ f_r \otimes f_{r+1} \mapsto 0 & \text{for } r = 0, 1. \end{cases}$$

Therefore we have that

$$\begin{aligned} X_{6u+4,0} \times X_{3,0} &= \mu_0^0 + \mu_1^0 + \mu_2^0 \\ &= 3(\mu_0^0 + (n+1)\nu_0) + (\mu_1^0 - \mu_0^0) - (\mu_0^0 - \mu_2^0 + \nu_0). \end{aligned}$$

Then we get $X_{6u+4,0} \times X_{3,0} = 3X_{6u+7,0}$, since $\mu_1^0 - \mu_0^0$ and $\mu_0^0 - \mu_2^0 + \nu_0$ are in $\text{Im Hom}_{\Lambda_n^e}(\partial^{6u+7}, \Lambda_n)$ (see [F, Lemma 4.5 (a)]). Therefore the first equation is proved. The second equation follows from the first equation and Lemmas 2.2, 3.4, 4.5, and 4.6. \square

Similarly, by direct computations and the previous sections, we also have the following:

Corollary 4.6. *Let $\text{char } K \nmid 3n + 2$ and $n > 0$. For any integer $u \geq 0$, we have $X_{6u+4,0} \times X_{3,1} = X_{6u+7,1}$. Therefore, for any integers $u_1 \geq 0$, $u_2 \geq 0$, s_1 , and s_2 with $0 < s_1 \leq n$, $0 \leq s_2 < n$, we have*

$$X_{6u_1+3,s_1} \times X_{6u_2+4,s_2} = \begin{cases} X_{6(u_1+u_2)+7,s_1+s_2} & \text{if } s_1 + s_2 \leq n \\ 0 & \text{if } s_1 + s_2 > n. \end{cases}$$

4.14. The products in $\text{HH}^{6u+i}(\Lambda_n) \times \text{HH}^{6v+5}$ for $i = 3, 4, 5$

By the previous sections we get the following corollaries:

Corollary 4.7. *Let $n \geq 0$. Then the following statements hold.*

(a) *Let $\text{char } K \mid 3n + 2$ (hence $\text{char } K \neq 3$). Then, for any integer $u \geq 0$,*

$$X_{6u+3,0} \times X_{5,0} = 0 \quad \text{and} \quad X_{6u+4,0} \times X_{5,0} = 3^{-1}X_{6u+9,1}.$$

Hence, for any integers $u_1 \geq 0$, $u_2 \geq 0$, s_1 and s_2 with $0 \leq s_1, s_2 < n$,

$$\begin{aligned} X_{6u_1+3,s_1} \times X_{6u_2+5,s_2} &= 0 \quad \text{and} \\ X_{6u_1+4,s_1} \times X_{6u_2+5,s_2} &= \begin{cases} 3^{-1}X_{6(u_1+u_2)+9,s_1+s_2+1} & \text{if } s_1 + s_2 < n \\ 0 & \text{if } s_1 + s_2 \geq n. \end{cases} \end{aligned}$$

(b) Let $\text{char } K \nmid 3n + 2$. Then, for any integer $u \geq 0$,

$$X_{6u+3,1} \times X_{5,0} = 0 \quad \text{and} \quad X_{6u+4,0} \times X_{5,0} = X_{6u+9,1}.$$

Hence, for any integers $u_1 \geq 0, u_2 \geq 0$,

$$\begin{aligned} X_{6u_1+3,s_1} \times X_{6u_2+5,s_2} &= 0 \\ &\text{for } s_1 \text{ with } 0 < s_1 \leq n, \text{ and } s_2 \text{ with } 0 \leq s_2 < n, \text{ and} \\ X_{6u_1+4,s_1} \times X_{6u_2+5,s_2} &= \begin{cases} X_{6(u_1+u_2)+9,s_1+s_2+1} & \text{if } s_1 + s_2 < n \\ 0 & \text{if } s_1 + s_2 \geq n \end{cases} \\ &\text{for } s_1 \text{ and } s_2 \text{ with } 0 \leq s_1, s_2 < n. \end{aligned}$$

(c) $X_{5,0}^2 = 0$. Hence for any integers $u_1 \geq 0, u_2 \geq 0, s_1$ and s_2 with $0 \leq s_1, s_2 < n$.

4.15. Generators and relations of $\text{HH}^*(\Lambda_n)$

By summarizing Sections 2 and 3 and Sections 4.1 through 4.16, we have the following theorem.

Theorem 2. *The Hochschild cohomology ring of Λ_n is commutative, and it is given as follows:*

- (a) *The case $\text{char } K \mid 3n + 2$:*
 - (1) *If $\text{char } K = 2$ and $n = 0$, then*

$$\text{HH}^*(\Lambda_0) \simeq K[y_1, y_3]/(y_1^2),$$

where $\deg y_i = i$ ($i = 1, 3$).

- (2) *If $\text{char } K = 2, n \equiv 0 \pmod{4}$ and $n \neq 0$, then*

$$\begin{aligned} \text{HH}^*(\Lambda_n) &\simeq K[y_0, y_1, y_2, y_3, y_4, y_6] \\ &/(y_0^{n+1}, y_1^2, y_0^n y_2, y_1 y_2 - y_0 y_3, y_1 y_3 - y_0^n y_4, \\ &\quad y_2^2 - y_0 y_4, y_1 y_4 - y_2 y_3, y_2 y_4 - y_0 y_6, y_3^2 - y_0^n y_6, \\ &\quad y_1 y_6 - y_3 y_4, y_4^2 - y_2 y_6), \end{aligned}$$

where $\deg y_i = i$ ($i = 0, 1, 2, 3, 4, 6$).

- (3) *If $\text{char } K = 2, n \not\equiv 0 \pmod{4}$, or if $\text{char } K \neq 2$, then*

$$\begin{aligned} \text{HH}^*(\Lambda_n) &\simeq K[y_0, y_1, y_2, y_3, y_4, y_6] \\ &/(y_0^{n+1}, y_1^2, y_0^n y_2, 3y_1 y_2 - y_0 y_3, y_1 y_3, \\ &\quad y_2^2 - y_0 y_4, 3y_1 y_4 - y_2 y_3, y_2 y_4 - y_0 y_6, y_3^2, \\ &\quad 3y_1 y_6 - y_3 y_4, y_4^2 - y_2 y_6), \end{aligned}$$

where $\deg y_i = i$ ($i = 0, 1, 2, 3, 4, 6$).

(b) *The case char $K \nmid 3n + 2$:*

(1) *If $n = 0$, then*

$$\mathrm{HH}^*(\Lambda_0) \simeq K[y_1, y_6]/(y_1^2),$$

where $\deg y_i = i$ ($i = 1, 6$).

(2) *If $n = 1$, then*

$$\begin{aligned} \mathrm{HH}^*(\Lambda_1) \simeq K[y_0, y_1, y_2, y_4, y_6] \\ / (y_0^2, y_1^2, y_0y_2, y_0y_4, y_2^2, y_2y_4 - y_0y_6, y_4^2 - y_2y_6), \end{aligned}$$

where $\deg y_i = i$ ($i = 0, 1, 2, 4, 6$).

(3) *If $n > 1$, then*

$$\begin{aligned} \mathrm{HH}^*(\Lambda_n) \simeq K[y_0, y_1, y_2, y_4, y_6] / (y_0^{n+1}, y_1^2, y_0^n y_2, y_0^n y_4, \\ y_2^2 - y_0y_4, y_2y_4 - y_0y_6, y_4^2 - y_2y_6), \end{aligned}$$

where $\deg y_i = i$ ($i = 0, 1, 2, 4, 6$).

Proof. (a) Suppose $\mathrm{char} K \mid 3n + 2$. We put

$$y_0 := X_{0,1}, \quad y_1 := X_{1,0}, \quad y_2 := X_{2,0}, \quad y_3 := X_{3,0}, \quad y_4 := X_{4,0}, \quad y_6 := X_{6,0}.$$

(1): If $n = 0$ (hence $\mathrm{char} K = 2$), then note that $\mathrm{HH}^{3u+s}(\Lambda_0) = K$ and $\mathrm{HH}^{3u+2}(\Lambda_0) = 0$ hold for all $u \geq 0$ and $s = 0, 1$. Since $X_{4,0} = y_1y_3$ and $y_6 = y_3^2$ hold, we have $X_{3u,0} = y_3^u$ and $X_{3u+1,0} = y_3^u y_1$ hold for all $u \geq 0$. Thus $\mathrm{HH}^*(\Lambda_0)$ is multiplicatively generated by y_1 and y_3 , and the equation $y_1^2 = 0$ holds. Therefore we have the desired isomorphism.

(2) and (3): If $n > 0$, from Sections 2 and 3 and Sections 4.1 through 4.16 we have the following equations:

$$\begin{aligned} y_0^{n+1} = 0, \quad y_1^2 = 0, \quad y_0^n y_2 = 0, \quad 3y_1y_2 = y_0y_3, \\ y_1y_3 = \begin{cases} y_0^n y_4 & \text{if } \mathrm{char} K = 2, \quad n \equiv 0 \pmod{4} \\ 0 & \text{otherwise,} \end{cases} \\ y_2^2 = y_0y_4, \quad 3y_1y_4 = y_2y_3, \quad y_2y_4 = y_0y_6, \\ y_3^2 = \begin{cases} y_0^n y_6 & \text{if } \mathrm{char} K = 2, \quad n \equiv 0 \pmod{4} \\ 0 & \text{otherwise,} \end{cases} \\ 3y_1y_6 = y_3y_4, \quad y_4^2 = y_2y_6. \end{aligned}$$

Note that the equations $y_0^n y_2 = 0$ and $3y_1y_4 = y_2y_3$ yield the equation $y_0^n y_1y_4 = 0$. Moreover we have that

$$X_{6u,s} = y_6^u y_0^s \quad \text{for } u \geq 0 \text{ and } 0 \leq s \leq n,$$

$$\begin{aligned} X_{6u+1,s} &= y_6^u y_0^s y_1 && \text{for } u \geq 0 \text{ and } 0 \leq s \leq n, \\ X_{6u+2,s} &= y_6^u y_0^s y_2 && \text{for } u \geq 0 \text{ and } 0 \leq s < n, \\ X_{6u+3,s} &= y_6^u y_0^s y_3 && \text{for } u \geq 0 \text{ and } 0 \leq s \leq n, \\ X_{6u+4,s} &= y_6^u y_0^s y_4 && \text{for } u \geq 0 \text{ and } 0 \leq s \leq n, \\ X_{6u+5,s} &= y_6^u y_0^s y_1 y_4 && \text{for } u \geq 0 \text{ and } 0 \leq s < n. \end{aligned}$$

Thus it is shown that the relations are enough, and therefore we can take $\{y_0, y_1, y_2, y_3, y_4, y_6\}$ as algebra generators of $\text{HH}^*(A_n)$. Note that $\text{HH}^*(A_n)$ is a commutative algebra, since $\text{HH}^*(A_n)$ is graded commutative and $y_1^2, y_1 y_3, y_3^2$ are zero if $\text{char } K \neq 2$. Hence we have the results.

(b) Suppose $\text{char } K \nmid 3n + 2$. We put

$$y_0 := X_{0,1}, \quad y_1 := X_{1,0}, \quad y_2 := X_{2,0}, \quad y_4 := X_{4,0}, \quad y_6 := X_{6,0}.$$

(1): If $n = 0$ then note that $\text{HH}^{6u+s}(A_0) = K$ and $\text{HH}^{6u+t}(A_0) = 0$ hold for all $u \geq 0, s = 0, 1$, and $t = 2, 3, 4, 5$. Then we have $y_1^2 = 0$, and the equations $X_{6u,0} = y_6^u$ and $X_{6u+1,0} = y_6^u y_1$ hold for $u \geq 0$. Hence $\text{HH}^*(A_0)$ is multiplicatively generated by y_1, y_6 , and we have the result.

(2) and (3): If $n > 0$, from Sections 2 and 3 and Sections 4.1 through 4.16 we have the following equations:

$$\begin{aligned} y_0^{n+1} = 0, \quad y_1^2 = 0, \quad y_0^n y_2 = 0, \quad y_0^n y_4 = 0, \quad y_2^2 &= \begin{cases} 0 & \text{for } n = 1 \\ y_0 y_4 & \text{for } n > 1, \end{cases} \\ y_2 y_4 = y_0 y_6, \quad y_4^2 = y_2 y_6. \end{aligned}$$

Furthermore we have

$$\begin{aligned} X_{6u,s} &= y_6^u y_0^s && \text{for } u \geq 0 \text{ and } 0 \leq s \leq n, \\ X_{6u+1,s} &= y_6^u y_0^s y_1 && \text{for } u \geq 0 \text{ and } 0 \leq s \leq n, \\ X_{6u+2,s} &= y_6^u y_0^s y_2 && \text{for } u \geq 0 \text{ and } 0 \leq s < n, \\ X_{6u+3,s} &= y_6^u y_0^{s-1} y_1 y_2 && \text{for } u \geq 0 \text{ and } 0 < s \leq n, \\ X_{6u+4,s} &= y_6^u y_0^s y_4 && \text{for } u \geq 0 \text{ and } 0 \leq s < n, \\ X_{6u+5,s} &= y_6^u y_0^s y_1 y_4 && \text{for } u \geq 0 \text{ and } 0 \leq s < n. \end{aligned}$$

Hence it is shown that the relations are enough, and therefore we can take $\{y_0, y_1, y_2, y_4, y_6\}$ as algebra generators of $\text{HH}^*(A_n)$. Moreover $\text{HH}^*(A_n)$ is commutative, since $\text{HH}^*(A_n)$ is graded commutative and $y_1^2 = 0$. \square

Finally, by Theorem 2, we have the following structure of the Hochschild cohomology ring modulo nilpotence of A_n :

Corollary 4.8. *There is the following of isomorphism of commutative graded algebras:*

$$\mathrm{HH}^*(\Lambda_n)/\mathcal{N} \simeq K[x], \quad \text{where } \deg x = \begin{cases} 3 & \text{if } n = 0 \text{ and } \mathrm{char} K = 2 \\ 6 & \text{otherwise.} \end{cases}$$

Hence for $n \geq 0$, $\mathrm{HH}^*(\Lambda_n)/\mathcal{N}$ is finitely generated as an algebra.

Proof. If $n = 0$, the statement is clear. Also, if $\mathrm{char} K \mid 3n + 2$ and $n > 0$, then y_0, y_1, y_2, y_3 , and y_4 are nilpotent elements, and moreover if $\mathrm{char} K \nmid 3n + 2$ and $n > 0$, then y_0, y_1, y_2 , and y_4 are nilpotent elements. This completes the proof. \square

§5. Applications

Throughout this section we suppose that $n = 0$, that is, we only deal with the cluster-tilted algebra Λ_0 of type \mathbb{D}_4 , so that denote Λ_0 by A , for simplicity. Also we keep the notation from the previous sections.

In this section, as an application, we show that A satisfies the finiteness conditions **(Fg1)** and **(Fg2)**, and describe the Hochschild cohomology rings modulo nilpotence for all cluster-tilted algebras of type \mathbb{D}_4 .

5.1. (Fg1) and (Fg2)

We start by recalling the finiteness conditions **(Fg1)** and **(Fg2)** of [EHSST]. Let A be a finite-dimensional algebra, and let $E(A)$ denote the Ext algebra of A

$$E(A) := \mathrm{Ext}_A^*(A/\mathfrak{r}_A, A/\mathfrak{r}_A) = \bigoplus_{i \geq 0} \mathrm{Ext}_A^i(A/\mathfrak{r}_A, A/\mathfrak{r}_A),$$

where \mathfrak{r}_A is the Jacobson radical of A . We then see that the functor $A/\mathfrak{r}_A \otimes_A -$ naturally induces a homomorphism $\phi_A : \mathrm{HH}^*(A) \rightarrow E(A)$ of graded algebras. For a graded subalgebra S of $\mathrm{HH}^*(A)$ we will consider $E(A)$ as a S -module by using ϕ_A . Then **(Fg1)** and **(Fg2)** are as follows:

(Fg1) There is a graded subalgebra H of $\mathrm{HH}^*(A)$ such that:

- (i) H is a commutative noetherian ring.
- (ii) $H^0 = \mathrm{HH}^0(A) = Z(A)$.

(Fg2) $E(A)$ is finitely generated as a H -module.

Recall that the graded centre $Z_{\text{gr}}(E(A))$ of $E(A)$ is the subring

$$Z_{\text{gr}}(E(A)) := \left(x \in \text{Ext}_A^i(A/\mathfrak{r}_A, A/\mathfrak{r}_A) \right. \\ \left. \mid i \geq 0; \text{ and } xy = (-1)^{ij}yx \text{ for all } \in \text{Ext}_A^j(A/\mathfrak{r}_A, A/\mathfrak{r}_A) (\forall j \geq 0) \right).$$

We first show the following lemma.

Lemma 5.1. *The following statements hold:*

- (a) $E(\Lambda) = K\mathcal{Q}/(a_0a_1 + b_0b_1)$.
- (b) *The element $w := \left(\sum_{i=0}^2 (a_i a_{i+1} a_{i+2})^2 \right) + (b_1 b_2 b_0)^2 \in \text{Ext}_\Lambda^6(\Lambda/\mathfrak{r}_\Lambda, \Lambda/\mathfrak{r}_\Lambda)$ belongs to $Z_{\text{gr}}(E(\Lambda))$.*
- (c) $\phi_\Lambda(y_6) (= \phi_\Lambda(X_{6,0})) = w$.

Proof. (a) By [F] Λ is a Koszul algebra. Hence it follows by [GM, Theorem 2.2] (see also [So]) that $E(\Lambda) = K\mathcal{Q}/I^\perp$, where $I^\perp := (a_0a_1 + b_0b_1)$.

(b) It is straightforward to check that w commutes with all arrows a_i, b_i and trivial paths e_i, f_i , and therefore $w \in Z_{\text{gr}}(E(\Lambda))$.

(c) This easily follows from the definition of ϕ_Λ . □

Note that since Λ is a Koszul algebra, the image of ϕ_Λ is exactly $Z_{\text{gr}}(E(\Lambda))$ by [BGSS, Theorem 4.1].

Now we can prove the main result in this section.

Theorem 3. *$E(\Lambda)$ is finitely generated as a $\text{HH}^{6*}(\Lambda)$ -module. Accordingly Λ satisfies (Fg1) and (Fg2).*

Proof. We verify that $E(\Lambda)$ is a $\text{HH}^{6*}(\Lambda)$ -module generated by the set

$$U = \{ e_i, f_1, a_i, b_j, a_i a_{i+1}, b_j b_{j+1}, a_i a_{i+1} a_{i+2}, a_1 a_2 b_0, b_1 b_2 a_0, b_1 b_2 b_0, \\ a_i a_{i+1} a_{i+2} a_i, b_j b_{j+1} b_{j+2} b_j, a_i a_{i+1} a_{i+2} a_i a_{i+1}, b_j b_{j+1} b_{j+2} b_j b_{j+1}, \\ (a_i a_{i+1} a_{i+2})^2, a_1 a_2 a_0 a_1 a_2 b_0, b_1 b_2 b_0 b_1 b_2 a_0, (b_j b_{j+1} b_{j+2})^2 \\ \mid i = 0, 1, 2; j = 0, 1 \}.$$

Noting that $a_0a_1 = -b_0b_1$ in $E(\Lambda)$, it can be seen that U gives a K -basis of $\bigoplus_{l=0}^6 \text{Ext}_\Lambda^l(\Lambda/\mathfrak{r}_\Lambda, \Lambda/\mathfrak{r}_\Lambda)$, and moreover the set $\{a_i w, b_j w \mid i = 0, 1, 2; j = 0, 1\}$ gives a K -basis of $\text{Ext}_\Lambda^7(\Lambda/\mathfrak{r}_\Lambda, \Lambda/\mathfrak{r}_\Lambda)$. Then it is straightforward to check that all homogeneous elements in $E(\Lambda)$ can be written in the form $\sum_{p \in U} k_p p w^t$ for some $k_p \in K$ ($p \in U$) and $t \geq 0$, and so $E(\Lambda)$ is finitely generated as a right $\text{HH}^{6*}(\Lambda)$ -module.

Also it follows by Proposition 2.1 that $\text{HH}^{6*}(\Lambda)$ is isomorphic to the polynomial ring $K[y_6]$ and hence is a commutative noetherian ring. Therefore Λ satisfies (Fg1) and (Fg2). □

It is well-known that there are 12 isomorphism classes of indecomposable right modules for the path algebra of a Dynkin quiver of type \mathbb{D}_4 (see, for example, [ASS, Chapter VII, Theorem 5.10]). Hence, by [BMR], Λ has 12 isomorphism classes of indecomposable right Λ -modules. In fact, there are precisely the following indecomposable right Λ -modules up to isomorphism:

$$e_i\Lambda/e_i\mathfrak{r}_\Lambda \quad (i = 0, 1, 2), \quad f_1\Lambda/f_1\mathfrak{r}_\Lambda, \quad e_0\Lambda/a_0\mathfrak{r}_\Lambda, \quad f_0\Lambda/b_0\mathfrak{r}_\Lambda, \\ e_j\Lambda \quad (j = 1, 2), \quad f_1\Lambda, \quad e_0\Lambda/e_0\mathfrak{r}_\Lambda^2, \quad e_0\mathfrak{r}_\Lambda, \quad e_0\Lambda.$$

Then we directly see that an indecomposable right Λ -module has finite projective dimension if and only if it is an injective module or a projective module. On the other hand, since Λ satisfies **(Fg1)** and **(Fg2)**, by [EHSST, Theorem 2.5] a right Λ -module has finite projective dimension if and only if it has trivial variety. Therefore we have got the following corollary.

Corollary 5.1. *For an indecomposable right Λ -module M , the following are equivalent:*

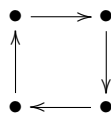
- (a) *The support variety of M is trivial.*
- (b) *M is a projective module or an injective module.*

5.2. The Hochschild cohomology rings modulo nilpotence for cluster-tilted algebras of type \mathbb{D}_4

We end this paper by determining the Hochschild cohomology rings modulo nilpotence for all cluster-tilted algebras of type \mathbb{D}_4 .

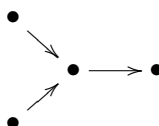
We know from [BHL] that there are three derived equivalence classes of cluster-tilted algebras of type \mathbb{D}_4 , and moreover, as their representatives, we can take Λ and the following algebras:

- (a) The selfinjective algebra $A_1 = K\Gamma_1/I_1$ of finite representation type, where Γ_1 is the cyclic quiver



and I_1 is the ideal generated by all paths of length 3.

- (b) The hereditary algebra $A_2 = K\Gamma_2$, where Γ_2 is the Dynkin quiver



of type \mathbb{D}_4 .

Then by [GSS1] we get $\mathrm{HH}^*(A_1)/\mathcal{N} \simeq K[x]$ whereas, by [H], $\mathrm{HH}^*(A_2)/\mathcal{N} \simeq \mathrm{HH}^*(A_2) \simeq K$. (Note that the structure of $\mathrm{HH}^*(A_1)$ is described in [BLM, EH].) Hence by Corollary 4.8 we have the following:

Corollary 5.2. *The Hochschild cohomology rings modulo nilpotence for all cluster-tilted algebras of type \mathbb{D}_4 are finitely generated as algebras, and are isomorphic to K or $K[x]$.*

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