SUT Journal of Mathematics
Vol. 50, No. 2 (2014), 103–129

Research of submanifolds in symmetric spaces by using the complexification and the infinite dimensional geometry

Naoyuki Koike

(Received July 24, 2014; Revised November 4, 2014)

Abstract. This is a survey for the research of submanifolds in symmetric spaces by using the complexification and the infinite dimensional geometry.

AMS 2010 *Mathematics Subject Classification.* 53C40; 53C35.

Key words and phrases. Equifocal submanifold, complex equifocal submanifold, isoparametric submanifold, symmetric space, Hilbert space.

Introduction

In 1989, C. L. Terng ([62]) introduced the notion of an isoparametric submanifold in the (separable) Hilbert space as the infinite dimensional version of an isoparametric submanifold in the Euclidean space. In the gauge theoretic aspect, for a compact semi-simple Lie group G, so-called the parallel transport map is defined as a Riemannian submersion of the Hilbert space consisting of all L^2-integrable paths in the Lie algebra $\mathfrak{g}$ of G onto G, where we give $\mathfrak{g}$ an $\mathrm{Ad}(G)$-invariant inner product and give G the bi-invariant metric induced from the inner product. Here Ad is the adjoint representaion of G. In 1995, C. L. Terng and G. Thorbergsson [64] introduced the notion of an equifocal submanifold in a (Riemannian) symmetric space. This notion is defined as a compact submanifold with flat section, trivial normal holonomy group and parallel focal structure. Here "with flat section" means that the images of the normal spaces of the submanifold by the normal exponential map are flat totally geodesic submanifolds and the parallelity of the focal structure means that, for any parallel normal vector field v of the submanifold, the focal radii along the normal geodesic γ_{v_x} with $\gamma'_{v_x}(0) = v_x$ are independent of the choice of x (with considering the multiplicities), where $\gamma'_{v_x}(0)$ is the velocity vector

of γ_{v_x} at 0. Note that the focal radii of the submanifold along the normal geodesic γ_{v_x} coincide with the zero points of the real valued function

$$F_{v_x}(s) := \det\left(\cos\left(s\sqrt{R(v_x)}\right) - \frac{\sin\left(s\sqrt{R(v_x)}\right)}{\sqrt{R(v_x)}} \circ A_{v_x}\right)$$

over $\mathbb{R}$ defined in terms of the shape operator A_{v_x} and the normal Jacobi operator $R(v_x)(:= R(\cdot, v_x)v_x)$, where R is the curvature tensor of the ambient symmetric space. In particular, in the case where G/K is a Euclidean space, we have $F_{v_x}(s) = \det(\mathrm{id} - sA_{v_x})$ and hence the focal radii along γ_{v_x} coincide with the inverse numbers of the eigenvalues of A_{v_x} (i.e., the principal curvature radii of direction v_x). Compact isoparametric submanifolds in a Euclidean space and compact isoparametric hypersurfaces in a sphere or a hyperbolic space are equifocal. They ([64]) proved that the research of an equifocal submanifold in a symmetric space G/K of compact type is reduced to that of an isoparametric submanifold in the Hilbert space through the composition of the parallel transport map for G and the natural projection of G onto G/K. In 2002, U. Christ ([8]) proved that a full irreducible equifocal submanifold of codimension greater than one in a symmetric space of compact type is homogeneous. He proved the homogeneity theorem by using the homogeneity theorem (which was proved by E. Heintze and X. Liu ([20])) for an isoparametric submanifold in the Hilbert space.

When a non-compact submanifold M in a symmetric space G/K of non-compact type deforms as its principal curvatures approach to zero, its focal set vanishes beyond the ideal boundary $(G/K)(\infty)$ of G/K (see Figure 1). For example, when an open portion of a totally umbilic sphere in a hyperbolic space of constant curvature $c(< 0)$ deforms as its principal curvatures approach to $\sqrt{-c}$, its focal point approach to $(G/K)(\infty)$ and, when it furthermore deforms as its principal curvatures approach to a positive value smaller than $\sqrt{-c}$, the focal point vanishes beyond $(G/K)(\infty)$. the parallelity of the complex focal structure is an essential condition (even if M is not of C^ω). So, we [31] defined the notion of a complex equifocal submanifold as a (properly embedded) complete submanifold with flat section, trivial normal holonomy group and parallel complex focal structure, where we note that this submanifold should be called an equi-complex focal submanifold but that we called it a complex equifocal submanifold for simplicity. Note that equifocal submanifolds in the symmetric space are complex equifocal. In fact, since they are compact, their principal curvatures are not close to zero and hence the parallelity of their focal structure leads to that of their complex focal structure. On the base of this fact, we recognized that, for a non-compact submanifold in a symmetric space of non-compact type, the parallelity of the focal structure is not an essential condition. So, we ([31]) introduced the notion of a complex focal radius of the

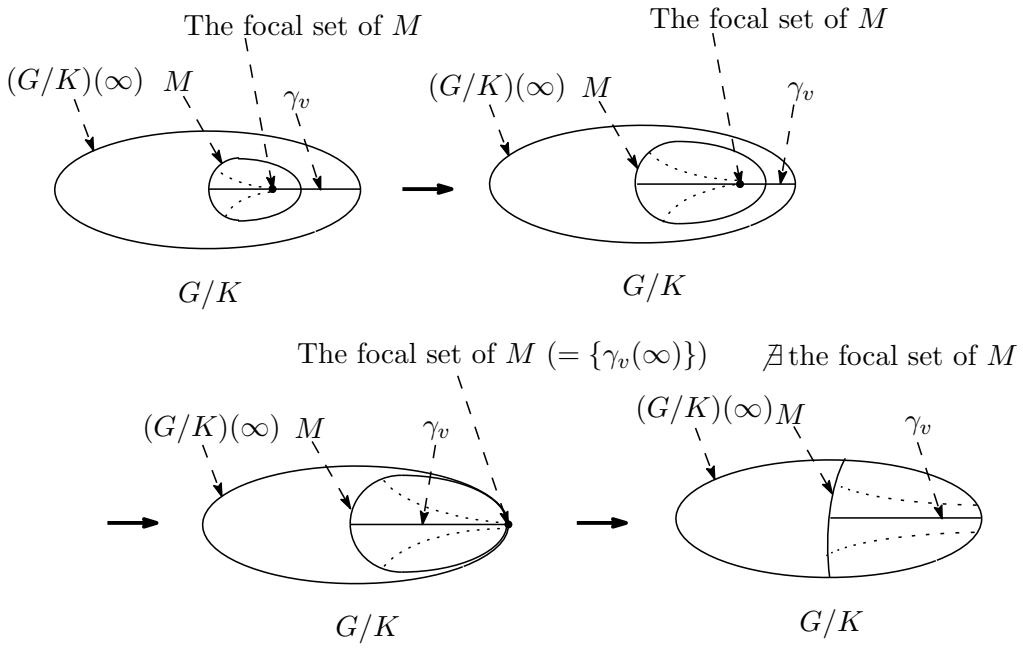


Figure 1.

submanifold along the normal geodesic γ_{v_x} as the zero points of the complex valued function $F^{\mathbb{C}}_{v_x}$ over $\mathbb{C}$ defined by

$$F^{\mathbb{C}}_{v_x}(z) := \det\left(\cos\left(z\sqrt{R(v_x)^{\mathbb{C}}}\right) - \frac{\sin\left(z\sqrt{R(v_x)^{\mathbb{C}}}\right)}{\sqrt{R(v_x)^{\mathbb{C}}}} \circ A^{\mathbb{C}}_{v_x}\right)$$

over $\mathbb{C}$, where $A^{\mathbb{C}}_{v_x}$ and $R(v_x)^{\mathbb{C}}$ are the complexifications of A_{v_x} and $R(v_x)$, respectively. Here we note that complex focal radii along γ_{v_x} can be directly calculated from datas of A_{v_x} and $R(v_x)$ according to this definition. In the case where M is of class C^ω (i.e., real analytic), we can catch the geometrical essence of complex focal radii as follows. We ([32]) defined the complexification $M^{\mathbb{C}}$ of M as an anti-Kaehler submanifold in the anti-Kaehler symmetric space $G^{\mathbb{C}}/K^{\mathbb{C}}$, where we note that $G^{\mathbb{C}}/K^{\mathbb{C}}$ is a space including both G/K and its compact dual G_κ/K as submanifolds transversal to each other and that it is interpreted as the complexification of both G/K and G_κ/K, where we note that the induced metric on G/K coincides with the original metric of G/K and that the induced metric on G_κ/K is the (-1)-multiple of the metric of G_κ/K. Also, we note that an anti-Kaehler manifold means a manifold M equipped with a pseudo-Riemannian metric g and a complex structure J satisfying $g(JX, JY) = -g(X, Y)$ $(\forall\, X, Y \in TM)$ and $\nabla J = 0$, and an anti-

Kaehler submanifold in the space means a J-invariant submanifold, where ∇ is the Levi-Civita connection of g. We ([32]) showed that z is a complex focal radius of M along γ_{v_x} if and only if $\gamma^{\mathbb{C}}_{v_x}(z)$ is a focal point of $M^{\mathbb{C}}$ along the complexified geodesic $\gamma^{\mathbb{C}}_{v_x}$ (see Figures 2 and 3). Here $\gamma^{\mathbb{C}}_{v_x}$ is defined by $\gamma^{\mathbb{C}}_{v_x}(z) := \gamma_{av_x+bJv_x}(1)$ $(z = a+b\sqrt{-1} \in \mathbb{C})$, where $\gamma_{av_x+bJv_x}$ is the geodesic in $G^{\mathbb{C}}/K^{\mathbb{C}}$ with $\gamma'_{av_x+bJv_x}(0) = av_x + bJv_x$. Thus the complex focal radii of M are the quantities indicating the positions of focal points of $M^{\mathbb{C}}$.

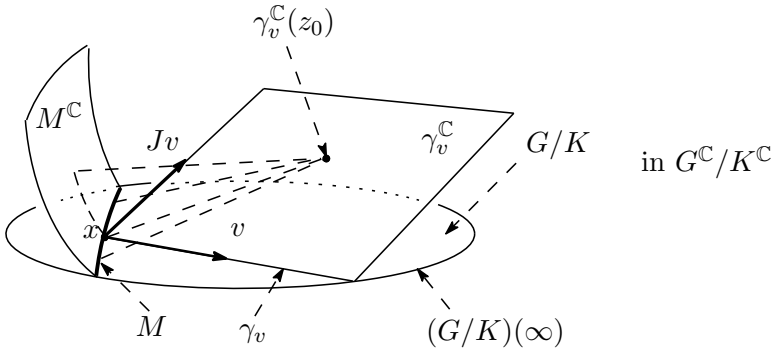
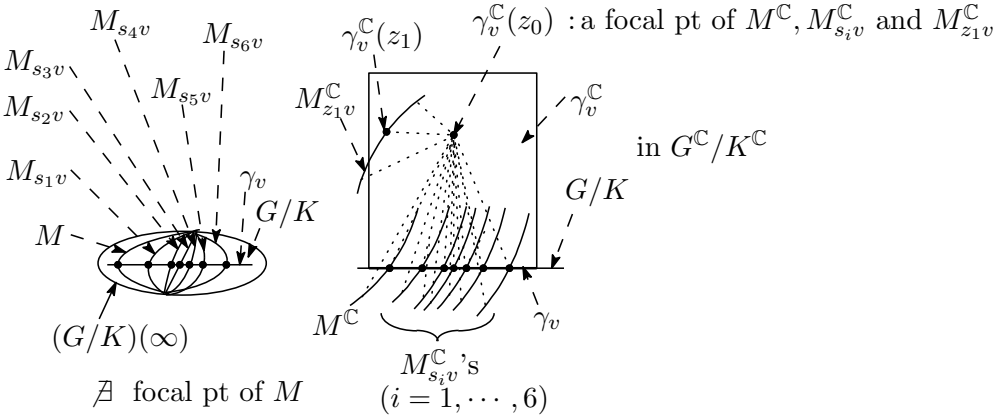


Figure 2.



M_{s_iv} : the parallel submanifold of M through $\gamma_v(s_i)$
$M^{\mathbb{C}}_{s_iv}$: the parallel submanifold of $M^{\mathbb{C}}$ through $\gamma^{\mathbb{C}}_v(s_i)$
$M^{\mathbb{C}}_{z_1v}$: the parallel submanifold of $M^{\mathbb{C}}$ through $\gamma^{\mathbb{C}}_v(z_1)$

Figure 3.

When M variates as above and real analytically, its focal set vanishes beyond $(G/K)(\infty)$ but the focal set of $M^{\mathbb{C}}$ (i.e., the complex focal set of M) does not vanish (see Figure 4). On the base of this fact, for non-compact submanifolds in a symmetric space of non-compact type, we recognize that the parallelity of the complex focal structure is an essential condition (even if M is not of C^{ω}). So, we [31] defined the notion of a complex equifocal submanifold as a (properly embedded) complete submanifold with flat section, trivial normal holonomy group and parallel complex focal structure, where we note that this submanifold should be called an equi-complex focal submanifold but that we called it a complex equifocal submanifold for simplicity. Note that equifocal submanifolds in the symmetric space are complex equifocal. In fact, since they are compact, their principal curvatures are not close to zero and hence the parallelity of their focal structure leads to that of their complex focal structure.

In 2004, the author ([31]) introduced the notion of a complex isoparametric submanifold in the pseudo-Hilbert space as the infinite dimensional version of an isoparametric submanifold in the pseudo-Euclidean space, and furthermore, he defined the parallel transport map for a semi-simple Lie group G as a pseudo-Riemannian submersion of the pseudo-Hilbert space consisting of certain kind of paths in the Lie algebra $\mathfrak{g}$ of G onto G. Also, in 2005, the author ([32]) introduced the notion of an anti-Kaehler isoparametric submanifold in the infinite dimensional anti-Kaehler space, and furthermore, he defined the parallel transport map for $G^{\mathbb{C}}$ as an anti-Kaehler submersion of the infinite dimensional anti-Kaehler space consisting of certain kind of paths in the Lie algebra $\mathfrak{g}^{\mathbb{C}}$ of $G^{\mathbb{C}}$ onto $G^{\mathbb{C}}$. He ([31]) proved that the research of a complex equifocal submanifold in a symmetric space G/K of non-compact type is reduced to that of a complex isoparametric submanifold in the pseudo-Hilbert space through the composition of the parallel transport map for G and the natural projection of G onto G/K. Also, he ([32]) proved that the research of a real analytic complex equifocal submanifold in a symmetric space G/K of non-compact type is reduced to that of an anti-Kaehler isoparametric submanifold in the infinite dimensional anti-Kaehler space through the composition of the parallel transport map for $G^{\mathbb{C}}$ and the natural projection of $G^{\mathbb{C}}$ onto $G^{\mathbb{C}}/K^{\mathbb{C}}$. Recently, he ([46]) proved that a certain kind of full irreducible complex equifocal submanifold of codimension greater than one in a symmetric space of non-compact type is homogeneous. He ([46]) proved the homogeneity theorem by using the homogeneity theorem (which also was proved by him ([41,45])) for a certain kind of anti-Kaehler isoparametric submanifold in the infinite dimensional anti-Kaehler space.

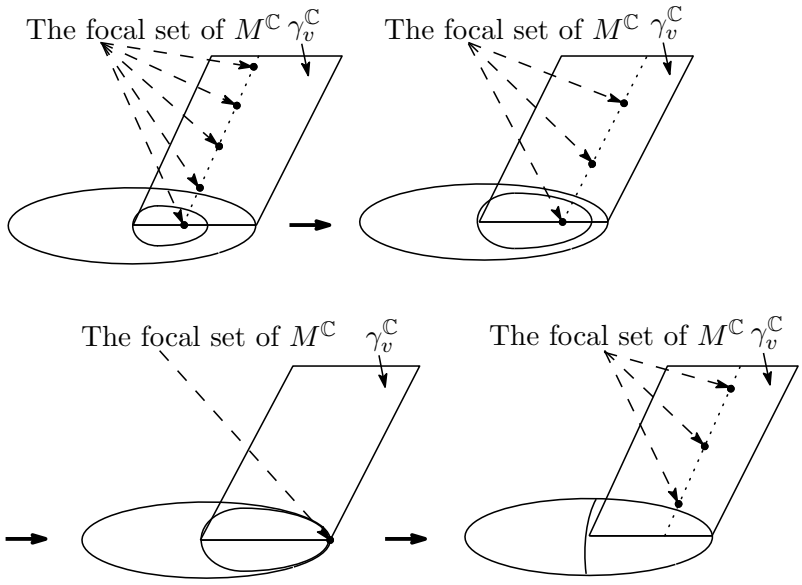


Figure 4.

§1. Isoparametric submanifolds, complex isoparametric submanifolds and anti-Kaehler isoparametric submanifolds

In 1989, Terng [62] introduced the notion of an isoparametric submanifold in a (separable) Hilbert space. This notion is defined as a (proper) Fredholm submanifold with trivial normal holonomy group and constant principal curvatures, where a (proper) Fredholm submanifold means a (properly embedded) submanifold of finite codimension such that the normal exponential map $\exp^{\perp}$ of the submanifold is a Fredholm map (i.e., the differential of $\exp^{\perp}$ at each point is a Fredholm operator) and that the restriction of $\exp^{\perp}$ to unit ball normal bundle of M is proper. Note that the shape operators of this submanifold are compact operators and that they are simultaneously diagonalizable with respect to an orthonormal base. Also she [62] introduced the notion of the parallel transport map for a compact semi-simple Lie group G. This map is defined as a Riemannian submersion of a (separable) Hilbert space $H^0([0,1],\mathfrak{g})$ onto G, where $H^0([0,1],\mathfrak{g})$ is the space of all L^2-integrable paths in the Lie algebra $\mathfrak{g}$ of G. Let G/K be a symmetric space of compact type, π the natural projection of G onto G/K and ϕ the parallel transport map for G. Let M be a submanifold in G/K and $\widetilde{M}$ a component of the lifted submanifold $(\pi\circ\phi)^{-1}(M)$. The relation between the focal structures of M and $\widetilde{M}$ is as in Figure 5. In 1995, Terng-Thorbergsson [64] showed that M is equifocal if

and only if $\widetilde{M}$ is isoparametric. Thus the research of an equifocal submanifold in a symmetric space of compact type is reduced to that of an isoparametric submanifold in a (separable) Hilbert space. An advantage of this reducement of the research is as follows. The symmetric space is of non-trivial holonomy group but the Hilbert space is a linear space, that is, it is of trivial holonomy group and is identified with its tangent space at each point. By using this reducement of the research, they proved some facts for an equifocal submanifold in the symmetric space (see [64]). In [64], they proposed the following problem:

Problem. *Is there a similar method of research for equifocal submanifolds in symmetric spaces of non-compact type?*

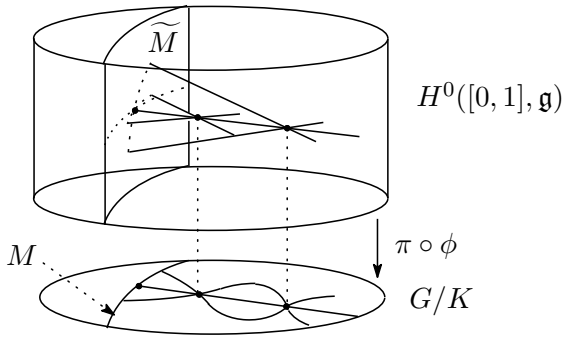


Figure 5.

By private discussion with Thorbergsson at Nagoya University in 2002, I knew that this problem is important and began to tackle to this problem. In 2004-2005, we [31,32] constructed a similar method of research for complex equifocal submanifolds in symmetric spaces of non-compact type in more general. We shall explain this method of research. First we shall recall the notions of an isoparametric submanifold, a real isoparametric submanifold, a complex isoparametric submanifold and a proper complex isoparametric submanifold in a (finite dimensional) pseudo-Euclidean space. Let M be a (properly embedded) complete submanifold in a pseudo-Euclidean space. Denote by A the shape tensor of M. Assume that the normal holonomy group of M is trivial. Let v be a parallel normal vector field of M. For each $x \in M$, the shape operator A_{v_x} is expressed as in $(*)$ with respect to a pseudo-orthonormal base of the tangent space (see [57] in detail), where 0 is entried in blank components in each matrix in $(*)$. If $S_i^x = \emptyset$ $(i \geq 2)$ and $S'^x_i = \emptyset$ $(i \geq 2)$, that is, the complexification $A^{\mathbb{C}}_{v_x}$ of A_{v_x} is diagonalizable with respect to a pseudo-orthonormal base, then A_{v_x} is called be *proper* (see [29]). If, for each parallel normal vector field v of M, the set $\{\lambda^x_{ij} \,|\, 1 \leq i \leq n,\ j \in S^x_i\}$

of all real eigenvalues of A_{v_x} is independent of the choice of $x \in M$ (with considering the multiplicities), then M is called a *real isoparametric submanifold*. Also, if, for each parallel normal vector field v of M, the set $\{\lambda^x_{ij} \,|\, 1 \le i \le n,\ j \in S^x_i\} \cup \{\alpha^x_{ij} + \sqrt{-1}\beta^x_{ij} \,|\, 1 \le i \le [\frac{n}{2}],\ j \in S'^x_i\}$

$$\left(\begin{array}{c} \\ \bigoplus\limits_{i=1}^{n} \bigoplus\limits_{j \in S^x_i} \end{array} \left(\begin{array}{cccc} \lambda^x_{ij} & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda^x_{ij} \end{array} \right) \right)$$

$$(i,i)-\text{type}$$

$$(*) \qquad \oplus \left(\begin{array}{c} \\ \bigoplus\limits_{i=1}^{[\frac{n}{2}]} \bigoplus\limits_{j \in S'^x_i} \end{array} \left(\begin{array}{cccccc} \alpha^x_{ij} & -\beta^x_{ij} & 1 & 0 & & \\ \beta^x_{ij} & \alpha^x_{ij} & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & 1 \quad 0 \\ & & & & \ddots & 0 \quad 1 \\ & & & & & \alpha^x_{ij} \ -\beta^x_{ij} \\ & & & & & \beta^x_{ij} \ \alpha^x_{ij} \end{array} \right) \right)$$

$$(2i,2i)-\text{type}$$

of all complex eigenvalues of A_{v_x} is independent of the choice of $x \in M$ (with considering the multiplicities), then M is called a *complex isoparametric submanifold*. In particular, if M is complex isoparametric and each shape operator of M is proper, then M is called a *proper complex isoparametric submanifold*. Also, if, for any parallel normal vector field v of M, the characterisitic polynomials of A_{v_x} are independent of the choice of $x \in M$, then M is called an *isoparametric submanifold* (see [15,16,27,49] for example). Clearly we have

$$\begin{array}{l} M : \text{proper complex isoparametric} \ \Rightarrow \ M : \text{isoparametric} \\ \Rightarrow \ M : \text{complex isoparametric} \ \Rightarrow \ M : \text{real isoparametric.} \end{array}$$

In 2004, we [31] defined the notions of a real isoparametric submanifold, a complex isoparametric submanifold and a proper complex isoparametric submanifold in a pseudo-Hilbert space as Fredholm submanifolds satisfying the similar conditions, where a pseudo-Hilbert space means a topological vector space equipped with a (weak-sense) non-degenerate continuous symmetric bilinear form which is Hilbertable. See [31] about the meaning of the Hilbertability and the definition of a Fredholm submanifold in a pseudo-Hilbert space. Also, we [31] introduced the notion of the parallel transport map for a (not necessarily compact) semi-simple Lie group G. This map is defined as a pseudo-Riemannian submersion of a pseudo-Hilbert space $H^0([0,1],\mathfrak{g})$ onto G, where

$H^0([0,1],\mathfrak{g})$ is the space of all paths in the Lie algebra $\mathfrak{g}$ of G which are L^2-integrable with respect to the positive definite inner product associated with the $\mathrm{Ad}(G)$-invariant non-degenerate inner product of $\mathfrak{g}$. Let G/K be a symmetric space of non-compact type, π the natural projection of G onto G/K and ϕ the parallel transport map for G. Also, let M be a (properly embedded) complete submanifold in G/K and $\widetilde{M}$ a component of the lifted submanifold $(\pi\circ\phi)^{-1}(M)$. We [31] showed that M is complex equifocal if and only if $\widetilde{M}$ is complex isoparametric. Thus the research of complex equifocal submanifolds in symmetric spaces of non-compact type is reduced to that of complex isoparametric submanifolds in pseudo-Hilbert spaces. If $\widetilde{M}$ is proper complex isoparametric, then we ([33]) called M a *proper complex equifocal submanifold*. Since the shape operators of a proper complex isoparametric submanifold is simultaneously diagonalizeble with respect to a pseudo-orthonormal base, the complex focal set of the submanifold at any point u consists of infinitely many complex hyperplanes in the complexified normal space at u and the group generated by the complex reflections of order two with respect to the complex hyperplanes is discrete. From this fact, it follows that the complex focal set of a proper complex equifocal submanifold at any point x consists of infinitely many totally geodesic complex hypersurfaces in the complexified flat section through x and the group generated by the complex reflections of order two with respect to the totally geodesic complex hypersurfaces is discrete. In 2005, we [32] introduced the notions of an anti-Kaehler isoparametric submanifold and a proper anti-Kaehler isoparametric submanifold in an infinite dimensional anti-Kaehler space, where an infinite dimensional anti-Kaehler space means a topological complex vector space (V,J) equipped with a non-degenerate continuous symmetric bilinear form $\langle\ ,\ \rangle$ such that $\langle JX,JY\rangle=-\langle X,Y\rangle$ for any $X,Y\in V$ and that $(V,\langle\ ,\ \rangle)$ is Hilbertable. See [32] about the definitions of these notions. Let $\pi^{\mathbb{C}}$ the natural projection of $G^{\mathbb{C}}$ onto $G^{\mathbb{C}}/K^{\mathbb{C}}$ and $\phi^{\mathbb{C}}$ the parallel transport map for $G^{\mathbb{C}}$. Assume that M is of class C^ω. Let $\widetilde{M^{\mathbb{C}}}$ be a component of the lifted submanifold $(\pi^{\mathbb{C}}\circ\phi^{\mathbb{C}})^{-1}(M^{\mathbb{C}})$ of the complexification $M^{\mathbb{C}}$ of M. We [32] showed that M is complex equifocal (resp. proper complex equifocal) if and only if $\widetilde{M^{\mathbb{C}}}$ is anti-Kaehler isoparametric (resp. proper anti-Kaehler isoparametric) in the infinite dimensional anti-Kaehleian space $H^0([0,1],\mathfrak{g}^{\mathbb{C}})$. Thus, in the case where M is of class C^ω, the research of complex equifocal (resp. proper complex equifocal) submanifolds is reduced to that of anti-Kaehler isoparametric (resp. proper anti-Kaehler isoparametric) submanifolds.

§2. Hyperpolar actions

Let H be a closed subgroup of G. The H-action on G/K is called a *polar action* if H is compact and if, for each $x \in G/K$, there exists a complete embedded submanifold Σ_x through x meeting all principal H-orbits orthogonally. This

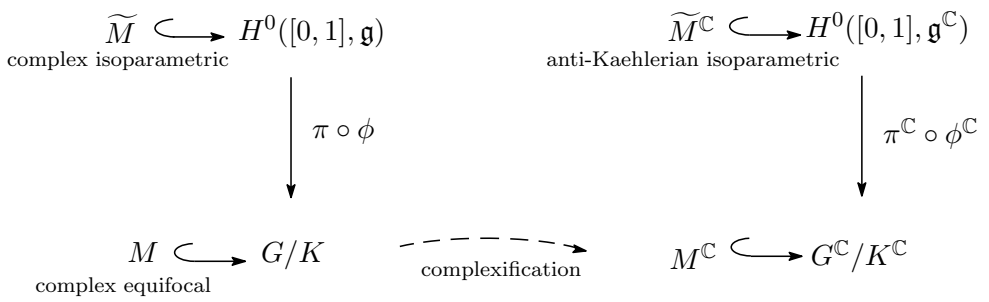


Figure 6.

submanifold Σ_x is called a *section* of this action through x. Furthermore, if the induced metric on Σ_x is flat, then the H-action is called a *hyperpolar action*. Here we illustrate that the assumption of the compactness of H is indispensable in these definitions. Consider the circle $S^1 := \{z \in \mathbb{C} \,|\, |z| = 1\}$ on $\mathbb{R}^2(= \mathbb{C})$ by the multiplication in $\mathbb{C}$. This action $S^1 \curvearrowright \mathbb{R}^2(= \mathbb{C})$ is a compact group action with flat section, that is, a hyperpolar action, and the orbits and the sections of this action give the images of parameter curves of the polar coordinate of $\mathbb{R}^2$ (see Figure 7). This action has the only fixed point (i.e., pole) $(0, 0)$. Define the S^1-action on the unit sphere $S^2 := \{(x, w) \in \mathbb{R} \times \mathbb{C} \,|\, x^2 + |w|^2 = 1\}$ by $z \cdot (x, w) = (x,\, zw)$ $((z \in S^1,\ (x, w) \in S^2)$. This action also is hyperpolar and has two fixed points (i.e., poles) $(1, 0)$ and $(-1, 0)$. On the other hand, the group action $\mathbb{R} \curvearrowright \mathbb{R}^2$ defined by $t \cdot (x, y) := (x + t, y)$ $(t \in \mathbb{R},\ (x, y) \in \mathbb{R}^2)$ is a non-compact group action with flat section. This action has no fixed point (i.e., pole) and the orbits and the sections of this action give the images of the parameter curves of the Euclidean coordinate (i.e., non-polar coordinate) of $\mathbb{R}^2$ (see Figure 7). Thus the assumption of the compactness of the group is indispensable in the definition of a polar (or hyperpolar) action. It is known that principal orbits of a hyperpolar action are equifocal. On the other hand, in 1995, E. Heintze, R.S. Palais, C.L. Terng and G. Thorbergsson ([24]) proved that any homogeneous equifocal submanifold in a simply connected symmetric space of compact type occurs as a principal orbit of a hyperpolar action.

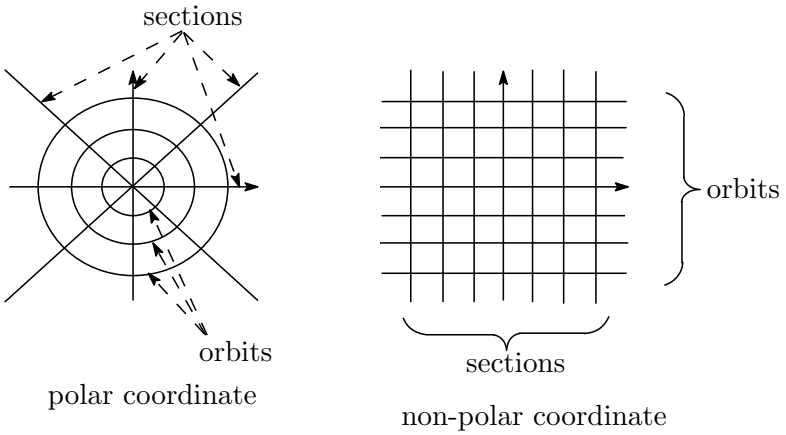


Figure 7.

If there exists an involution σ of G with $(\mathrm{Fix}\,\sigma)_0 \subset H \subset \mathrm{Fix}\,\sigma$, then the H-action on G/K is called a *Hermann action*, where $\mathrm{Fix}\,\sigma$ is the fixed point group of σ and $(\mathrm{Fix}\,\sigma)_0$ is the identity component of $\mathrm{Fix}\,\sigma$. It is easy to show that Hermann actions are hyperpolar. In 2001, A. Kollross ([47]) proved that hyperpolar actions of cohomogeneity greater than one on an irreducible simply connected symmetric space of compact type are orbit equivalent to Hermann actions. where the curvature-adaptedness means that, If, for any $x \in M$ and any unit normal vector v of M at x, the normal Jacobi operator $R(v)$ preserves the tangent space T_xM and $R(v)|_{T_xM}$ and the shape operator A_v of M commute, then it is said to be *curvature-adapted*. In 2007, O. Goertsches and G. Thorbergsson proved the following fact.

Proposition 2.1([13]). *Principal orbits of a Hermann action are curvature-adapted.*

This proposition together with above facts derives the following fact.

Proposition 2.2. *All homogeneous equifocal submanifolds of codimension greater than one in an irreducible simply connected symmetric space of compact type are curvature-adapted.*

§3. Homogeneity of equifocal submanifolds

In this section, we shall state a homogeneity theorem for an equifocal submanifold in a symmetric space of compact type. In 1999, E. Heintze and X. Liu proved the following homogeneity theorem for an isoparametric submanifold in a Hilbert space.

Theorem 3.1([19]). *All irreducible isoparametric submanifolds of codimension greater than one in a Hilbert space are homogeneous.*

This result is the infinite dimensional version of the homogeneity theorem for isoparametric submanifolds in a (finite dimensional) Euclidean space by G. Thorbergsson ([65]), which states that all irreducible isoparametric submanifolds of codimension greater than two in a Euclidean space are homogeneous. G. Thorbergsson proved this fact by using the building theory. On the other hand, E. Heintze and X. Liu proved the above homogeneity theorem by constructing an isometry of the ambient Hilbert space mapping x to y and preserving the submanifold invariantly for any two points x and y of the submanifold connected by a certain kind of curve. In both proofs, is used the fact that the ambient space is a linear space. In 2002, by using the result of Heintze-Liu, U. Christ [8] proved the following homogeneity theorem for an equifocal submanifold in a simply connected symmmetric space of compact type.

Theorem 3.2([8]). *All irreducible equifocal submanifolds of codimension greater than one in a simply connected symmetric space of compact type are homogeneous.*

Here we note that C. Gorodski and E. Heintze ([14]) closed a gap in his proof. From this homogeneity theorem and Proposition 2.1, we have the following fact.

Theorem 3.3. *All equifocal submanifolds of codimension greater than one in an irreducible simply connected symmetric space of compact type occur as principal orbits of Hermann actions.*

Since the principal orbits of a Hermann action is curvature-adapted, we have the following fact.

Corollary 3.4. *All equifocal submanifolds of codimension greater than one in an irreducible simply connected symmetric space of compact type are curvature-adapted.*

Conversely we have recently proved the following fact.

Theorem 3.5([44]). *Let M be a compact curvature-adapted submanifold with maximal flat section and trivial normal holonomy group in a symmetric space G/K of compact type, where "with maximal flat section" means that it has flat section and* $\operatorname{codim} M = \operatorname{rank} G/K$. *Also, let $TM = \oplus_{i\in I_R} D_i^R$ be the common eigenspace decomposition of the normal Jacobi operator $R(v)$'s ($v \in T^\perp M$) and $TM = \oplus_{i\in I_A} D_i^A$ that of the shape operators A_v's ($v \in T^\perp M$). Assume that, for each $i \in I_R$,* $\dim D_i^R \geq 2$ *and there exists $j \in I_A$ such that*

$D_i^R \subset D_j^A$. *Then M is equifocal.*

§4. Complex hyperpolar actions

Let G/K be a symmetric space of non-compact type and H be a closed subgroup of G. We ([32]) called the H-action on G/K a *complex polar action* if, for each $x \in G/K$, there exists a complete embedded submanifold Σ_x through x meeting all principal H-orbits orthogonally. Furthermore, if the induced metric on Σ_x is flat, then we ([32]) called H-action a *complex hyperpolar action*, where we note that this action should be called a hyper-complex polar action but that we called it a complex hyperpolar action for simplicity. We illustrate why we named this action thus. Define the $\mathbb{R}$-action on the hyperbolic space $H^2(= SO(1,2)/SO(2) = \{(x_1,x_2,x_3) \mid -x_1^2+x_2^2+x_3^2 = -1\}(\subset \mathbb{R}_1^3))$ by $\theta\cdot(x_1,x_2,x_3) = (x_1\cosh\theta + x_2\sinh\theta,\ x_1\sinh\theta + x_2\cosh\theta, x_3)$ $((\theta \in \mathbb{R},\ (x_1,x_2,x_3) \in H^2)$, where $\mathbb{R}_1^3$ is the Lorentzian space equipped with the Lorentzian inner product $-dx_1^2+dx_2^2+dx_3^2$. This action is a complex hyperpolar action. By the way, this action has no fixed point (i.e., pole) but the complexified action $\mathbb{C}$ on the anti-Kaehler symmetric space $SO(3,\mathbb{C})/SO(2,\mathbb{C})$ (which is the complexification of H^2) has fixed points (i.e. poles). These fixed points should be called complex poles of the original action. In this sense, we named the above action a complex (hyper)polar action. See also Figure 8.

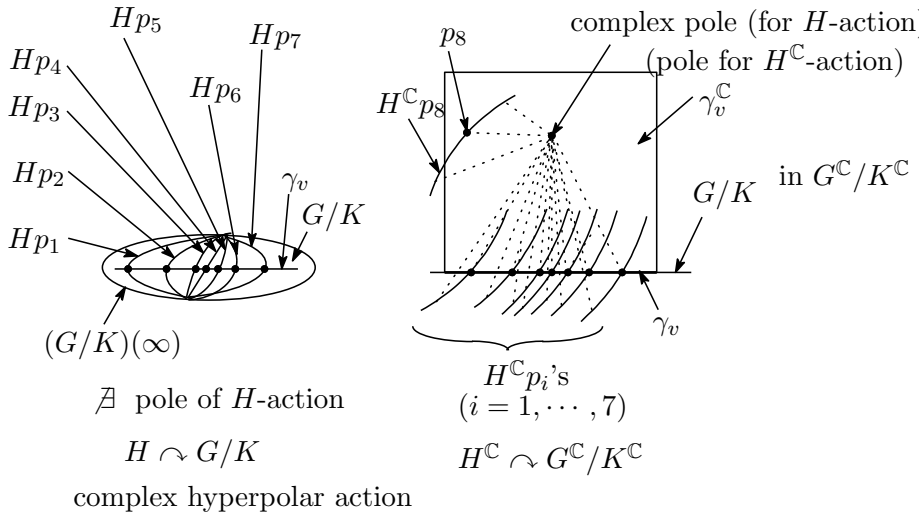


Figure 8.

We proved the following facts for a complex hyperpolar action.

Proposition 4.1([32,35]). (i) *Principal orbits of a complex hyperpolar ac-*

tion are complex equifocal.

(ii) *Any homogeneous submanifolds with flat section in a symmetric space of non-compact type occur as principal orbits of complex hyperpolar actions.*

If there exists an involution σ of G with $(\mathrm{Fix}\,\sigma)_0 \subset H \subset \mathrm{Fix}\,\sigma$, then we called the H-action on G/K a *Hermann type action*. We proved the following fact for a Hermann type action.

Proposition 4.2([33]). *Principal orbits of a Hermann type action are proper complex equifocal and curvature-adapted.*

Also, we proved the following fact for a complex hyperpolar action.

Proposition 4.3([35]). *Complex hyperpolar actions with a reflective orbit are orbit equivalent to Hermann type actions.*

From these facts, we can derive the following facts.

Proposition 4.4. *Let M be a homogeneous submanifold with flat section in a symmetric space of non-compact type. If it admits a reflective focal submanifold, then it is a principal orbit of a Hermann type action. Hence it is proper complex equifocal and curvature-adapted.*

§5. Homogeneity of proper complex equifocal submanifolds

In this section, we shall state a homogeneity theorem for proper complex equifocal submanifolds. In [41], we first proved the following homogeneity theorem for a proper anti-Kaehler isoparametric submanifold in an infinite dimensional anti-Kaehler space.

Theorem 5.1([41]). *All irreducible proper anti-Kaehler isoparametric submanifolds of codimension greater than one in the infinite dimensional anti-Kaehler space are homogeneous.*

Denote by $I(V)$ the group of all isometries of V and $I_b(V)$ the (Banach Lie) group of all isometries of V whose associated Killing field is defined over the whole of V. The homogeneity in the above theorem means that the submanifold is an orbit of a subgroup action of $I(V)$. We ([45]) have recently proved that the submanifold is an orbit of a subgroup action of $I_b(V)$. Furthermore, by using this improved homogeneity theorem, we proved the following homogeneity theorem for a proper complex equifocal C^ω-submanifold in a symmetric

space of non-compact type.

Theorem 5.2([46]). *All irreducible curvature-adpated proper complex equifocal C^ω-submanifolds of codimension greater than one in a symmetric space of non-compact type are homogeneous. Furthermore they are principal orbits of Hermann type actions.*

Remark 1. In this theorem, we cannot replace "proper complex equifocal" to "complex equifocal". In fact, principal orbits of the N-action on an irreducible symmetric space G/K of non-compact type and rank greater than one are irreducible curvature-adapted complex equifocal submanifolds of codimension greater than one but they do not occur as principal orbits of a Hermann type action, where N is the nilpotent part in the Iwasawa's decomposition $G = KAN$ of G.

On the other hand, we ([42]) proved the following fact for a curvature-adapted proper complex equifocal C^ω-hypersurface.

Theorem 5.3([42]). *All curvature-adapted proper complex equifocal C^ω-hypersurfaces in a symmetric space of non-compact type occur as principal orbits of Hermann type actions.*

The proof of this theorem is performed by deriving the Cartan type identity (which is a relation among the prinipal curvatures and the eigenvalues of the normal Jacobi operators) for the hypersurface and showing the existenceness of a totally geodesic focal submanifold in terms of the identity.

Also, we have recently proved the following fact.

Theorem 5.5([44]). *Let M be a complete curvature-adapted submanifold with maximal flat section and trivial normal holonomy group in a symmetric space G/K of non-compact type. Also, let $TM = \oplus_{i\in I_R} D_i^R$ be the common eigenspace decomposition of $R(v)$'s ($v \in T^\perp M$) and $TM = \oplus_{i\in I_A} D_i^A$ that of A_v's ($v \in T^\perp M$). Assume that, for each $i \in I_R$, $\dim D_i^R \geq 2$ and there exists $j \in I_A$ such that $D_i^R \subset D_j^A$. Then M is complex equifocal.*

§6. Isoparametric submanifolds with flat section in the sense of Heintze-Liu-Olmos

In 2006, Heintze-Liu-Olmos [21] defined the notion of isoparametric submanifold with flat section in a general Riemannian manifold as a (properly embedded) complete submanifold with flat section and trivial normal holonomy

group whose sufficiently close parallel submanifolds have constant mean curvature with respect to the radial direction. For a compact submanifold with trivial holonomy group and flat section in a symmetric space of compact type, they [21] showed that it is equifocal if and only if, for each parallel normal vector field v, F_{v_x} is independent of the choice of a point x of the submanifold, where F_{v_x} is the function defined in Page 1. Thus, if it is an isoparametric submanifold with flat section, then it is equifocal. Furthermore, for a compact submanifold in a symmetric space of compact type, they proved te following fact.

Theorem 6.1([21]). *Let M be a compact submanifold in a symmetric space of compact type. Then M is equifocal if and only if it is an isoparametric submanifold with flat section.*

The proof of this fact is performed by investigating the lift $(\pi \circ \phi)^{-1}(M)$ of M to the Hilbert space $H^0([0,1], \mathfrak{g})$.

On the other hand, we [32] showed that, for a (properly embedded) complete submanifold with trivial normal holonomy group and flat section in a symmetric space of non-compact type, it is an isoparametric submanifold with flat section if and only if, for each parallel normal vector field v, $F^{\mathbb{C}}_{v_x}$ is independent of the choice of a point x of the submanifold, where $F^{\mathbb{C}}_{v_x}$ is the function defined in Page 2. Thus if it is an isoparametric submanifold with flat section, then it is complex equifocal. Conversely, we proved the following fact.

Theorem 6.2([32]). *All curvature-adapted complex equifocal submanifolds in a symmetric space of non-compact type are isoparametric submanifolds with flat section.*

For a submanifold M in a Hadamard manifold N, we ([39]) defined the notion of a focal point of non-Euclidean type on the ideal boundary $N(\infty)$ as follows. Denote by $\widetilde{\nabla}$ the Levi-Civita connection of N and A the shape tensor of M. Let $\gamma_v : [0,\infty) \to N$ be the normal geodesic of M of direction $v \in T^{\perp}_x M$. If there exists a M-Jacobi field (resp. strongly M-Jacobi field) Y along γ_v satisfying $\lim_{t\to\infty} \frac{||Y_t||}{t} = 0$, then we call $\gamma_v(\infty)\,(\in N(\infty))$ a *focal point* (resp. *strongly focal point*) *on the ideal boundary $N(\infty)$ of M along γ_v* (see Figure 9), where $\gamma_v(\infty)$ is the asymptotic class of γ_v. Also, if there exists a M-Jacobi field Y along γ_v satisfying $\lim_{t\to\infty} \dfrac{||Y_t||}{t} = 0$ and $\mathrm{Sec}(v, Y(0)) < 0$, then we call $\gamma_v(\infty)$ a *focal point of non-Euclidean type on $N(\infty)$ of M along γ_v*, where $\mathrm{Sec}(v, Y(0))$ is the sectional curvature for the 2-plane spanned by v and $Y(0)$.

We proved the following fact.

Theorem 6.3([39]). *Let M be a curvature-adapted submanifold in a symmetric space $N := G/K$ of non-compact type. Then M is proper complex equifocal if and only if it is an isoparametric submanifold with flat section which admits no focal point of non-Euclidean type on the ideal boundary $N(\infty)$ of N.*

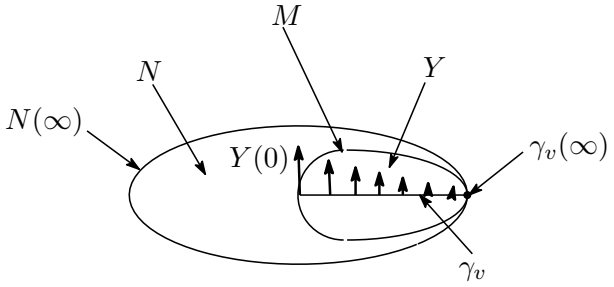


Figure 9.

At the end of this section, we propose the following question.

Question. *Let M be a (properly embedded) complete submanifold in a symmetric space $N = G/K$ of non-compact type. Is M a proper complex equifocal submanifold if and only if it is an isoparametric submanifold with flat section which admits no focal point of non-Euclidean type on $N(\infty)$?*

§7. Duality

In this section, we explain the duality of Hermann actions on symmetric spaces of compact type and Hermann type actions on symmetric spaces of non-compact type. Let G/K be a symmetric space of non-compact type and G_κ/K the compact dual of G/K. Also, let θ be the Cartan involution of G with $(\mathrm{Fix}\,\theta)_0 \subset K \subset \mathrm{Fix}\,\theta$, where $\mathrm{Fix}\,\theta$ is the fixed point group of θ and $(\mathrm{Fix}\,\theta)_0$ is the identity component of $\mathrm{Fix}\,\theta$. If H is a symmetric subgroup of G (i.e., $(\mathrm{Fix}\,\sigma)_0 \subset H \subset \mathrm{Fix}\,\sigma$ for some involution σ of G), then the H-action on G/K is called a *Hermann type action*. Here we explain the duality between Hermann actions on G_κ/K and Hermann type actions on G/K. We may assume that $\theta \circ \sigma = \sigma \circ \theta$ by replacing H to a its suitable conjugate group if necessary. Then we obtain the involution $\hat{\sigma}$ of G_κ with $\theta \circ \hat{\sigma} = \hat{\sigma} \circ \theta$ from σ. Set $\hat{H} := (\mathrm{Fix}\,\hat{\sigma})_0$. Thus we obtain a Hermann action $\hat{H} \curvearrowright G_\kappa/K$. Conversely, we may assume that $\theta \circ \tau = \tau \circ \theta$ by replacing H' to a its suitable conjugate group if necessary except for three exceptional ones. Then we obtain the involution $\hat{\tau}$ of G with $\theta \circ \hat{\tau} = \hat{\tau} \circ \theta$ from τ. Set $\hat{H}' := (\mathrm{Fix}\,\hat{\tau})_0$. Thus we obtain a Hermann type action $\hat{H}' \curvearrowright G_\kappa/K$. Thus Hermann type actions on G/K

correspond almost one-to-one to Hermann actions on G_κ/K.

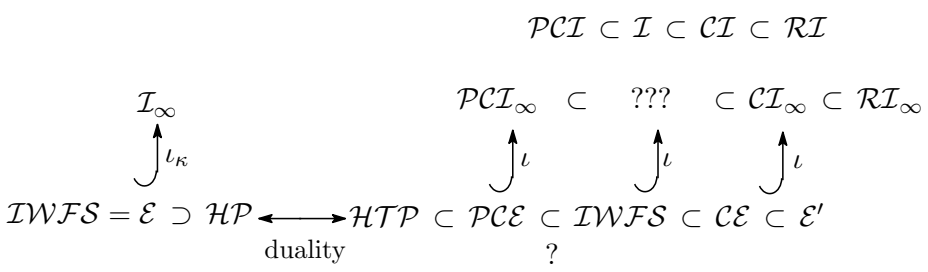


Figure 10.

Notations in Figure 10 are as follows.

$\mathcal{E}$: the set of all congruence classes of equifocal submanifolds in G_κ/K

$\mathcal{HP}$: the set of all congruence classes of principal orbits of a Hermann actions on G_κ/K

$\mathcal{I}_\infty$: the set of all congruence classes of isoparametric submanifolds in $H^0([0,1],\mathfrak{g}_\kappa)$

$\mathcal{E}'$: the set of all congruence classes of equifocal submanifolds in G/K, where they may not be compact

$\mathcal{CE}$: the set of all congruence classes of complex equifocal submanifolds in G/K

$\mathcal{IWFS}$: the set of all congruence classes of isoparametric submanifolds with flat section in G/K

$\mathcal{PCE}$: the set of all congruence classes of proper complex equifocal submanifolds in G/K

$\mathcal{HTP}$: the set of all congruence classes of principal orbits of Hermann type actions on G/K

$\mathcal{RI}_\infty$: the set of all congruence classes of real isoparametric submanifolds in $H^0([0,1],\mathfrak{g})$

$\mathcal{CI}_\infty$: the set of all congruence classes of complex isoparametric submanifolds in $H^0([0,1],\mathfrak{g})$

$\mathcal{PCI}_\infty$: the set of all congruence classes of proper complex isoparametric submanifolds in $H^0([0,1],\mathfrak{g})$

$\mathcal{RI}$: the set of all congruence classes of real isoparametric submanifolds in $\mathbb{R}^m_\nu$

$\mathcal{CI}$: the set of all congruence classes of complex isoparametric submanifolds in $\mathbb{R}^m_\nu$

$\mathcal{I}$: the set of all congruence classes of isoparametric submanifolds in $\mathbb{R}^m_\nu$

$\mathcal{PCI}$: the set of all congruence classes of proper complex isoparametric submanifolds in $\mathbb{R}^m_\nu$

$\iota : \mathcal{CE} \to \mathcal{CI}_\infty \underset{\text{def}}{\Longleftrightarrow} \iota([M]) := [(\pi \circ \phi)^{-1}(M)]$

$\iota_\kappa : \mathcal{E} \to \mathcal{I}_\infty \underset{\text{def}}{\Longleftrightarrow} \iota_\kappa([M]) := [(\pi \circ \phi)^{-1}(M)]$

The congruence classes of the orbits of the action of the nilpotent group N on G/K belong to $\mathcal{IWFS} \setminus \mathcal{PCE}$, where N is the nilpotent part in the Iwasawa's decomposition $G = KAN$ of G. See [37] about examples other than these classes belonging to $\mathcal{IWFS} \setminus \mathcal{PCE}$. Also, for almost all complete submanifolds all of whose principal curvatures are sufficiently close to zero in G/K, the ε-tubes over them belong to $\mathcal{E}' \setminus \mathcal{CE}$, where ε is any positive constant. Thus

$\mathcal{E}'$ is a very big class.

§8. The mean curvature flows

In this section, we state the results for the mean curvature flows starting from an equifocal submanifold and a proper complex equifocal submanifold. In 2009, X. Liu and C.L. Terng proved the following result for the mean curvature flow starting from a compact isoparametric submanifold in a (finite dimensional) Euclidean space.

Theorem 8.1([48]). *Let M be a compact isoparametric submanifold in a Euclidean space. Then the following statements* (i) *and* (ii) *hold:*

(i) *The mean curvature flow M_t starting from M collapses to a focal submanifold of M in a finite time T. If a focal map of M onto F is spherical, then the mean curvature flow M_t has type I singularity, that is,*

$$\lim_{t\to T-0}\max_{v\in S^{\perp}M_t}||A^t_v||^2_{\infty}(T-t)<\infty,$$

where A^t_v is the shape operator of M_t for v, $||A^t_v||_{\infty}$ is the sup norm of A^t_v and $S^{\perp}M_t$ is the unit normal bundle of M_t.

(ii) *For any focal submanifold F of M, there exists a parallel submanifold of M collapsing to F along the mean curvature flow and the set of all parallel submanifolds collapsing to F along the mean curvature flow is a one-parameter C^{∞}-family.*

In 2011, we proved the following result for the mean curvature flow starting from an equifocal submanifold in a symmetric space of compact type.

Theorem 8.2([40]). *Let M be an equifocal submanifold in a symmetric space G/K of compact type. Then the following statements* (i) *and* (ii) *hold:*

(i) *If M is not minimal, then the mean curvature flow M_t starting from M collapses to a focal submanifold F of M in a finite time T. Furtheremore, if M is irreducible, the codimension of M is greater than one and if the fibration of M onto F is spherical, then the flow M_t has type I singularity.*

(ii) *For any focal submanifold F of M, there exists a parallel submanifold of M collapsing to F along the mean curvature flow and the set of all parallel submanifolds collapsing to F along the mean curvature flow is a one-parameter C^{∞}-family.*

The proof of this theorem was performed by reducing to the investigation of the mean curvature flow starting from the lift of the submanifold to the Hilbert space $H^0([0,1],\mathfrak{g})$.

Recently we have recently proved the similar result for the mean curvature flow starting from a certain kind of curvature-adapted and proper complex equifocal submanifold with maximal flat section in a symmetric space G/K of non-compact type (see [43, Theorem A]) . The proof of this theorem was performed by reducing to the investigation of the mean curvature flow starting from the lift of the submanifold to the pseudo-Hilbert space $H^0([0,1],\mathfrak{g})$.

Appendix

In this appendix, we shall give models of isoparametric submanifolds in the Hilbert space. For its purpose, we suffice to give examples of polar actions on the Hilbert space because the principal orbits of of the polar action are isoparametric. First we recall the notion of the affine Kac-Moody Lie algebra and the affine Kac-Moody Lie group. Let $\mathfrak{g}$ be a simple Lie algebra over the field $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$ and σ an automorphism of $\mathfrak{g}$. Then the twisted loop algebra $L(\mathfrak{g},\sigma)$ is defined as

$$L(\mathfrak{g},\sigma) := \{u \in C^\infty(\mathbb{R},\mathfrak{g}) \,|\, u(t+2\pi)=\sigma(u(t)) \text{ for all } t \in \mathbb{R}\}$$

equipped with the bracket product $[\ ,\ ]_0$ defined by $[u,v]_0(t) := [u(t),v(t)]$ $(t \in \mathbb{R})$ for $u,v \in L(\mathbb{R},\mathfrak{g})$. Let $\langle\ ,\ \rangle$ be the Killing form of $\mathfrak{g}$. Define an inner product $\langle\ ,\ \rangle_0$ of $L(\mathbb{R},\mathfrak{g})$ by $\langle u,v\rangle_0 := \displaystyle\int_0^{2\pi} \langle u(t),v(t)\rangle dt$ $(u,v \in L(\mathfrak{g},\sigma))$. For $\lambda \in \mathbb{F}\setminus\{0\}$, define a skew-symmetric bilnear form ω_λ on $L(\mathfrak{g},\sigma)$ by $\omega_\lambda(u,v) := \lambda\langle u',v\rangle_0$ $(u,v \in L(\mathfrak{g},\sigma))$. *The affine Kac-Moody Lie algebra* $\widehat{L}(\mathfrak{g},\sigma)$ is defined as $\widehat{L}(\mathfrak{g},\sigma) := L(\mathfrak{g},\sigma)+\mathbb{F}c+\mathbb{F}d$ equipped with the bracket product $[\ ,\ ]$ defined by

$$\begin{aligned}
&[u,v] := [u,v]_0 + \omega_\lambda(u,v)c\\
&[d,u] := u'\\
&[c,x] := 0
\end{aligned}$$

for all $u,v \in L(\mathfrak{g},\sigma)$ and all $x \in \widehat{L}(\mathfrak{g},\sigma)$. Note that $\widehat{L}(\mathfrak{g},\sigma)$ is not simple. The isomorphism class of $\widehat{L}(\mathfrak{g},\sigma)$ is independent of the choice of λ. Also, for automorphisms σ_i $(i=1,2)$ of $\mathfrak{g}$, $\widehat{L}(\mathfrak{g},\sigma_1) \cong \widehat{L}(\mathfrak{g},\sigma_2)$ if and only if σ_1 and σ_2 are conjugate in $\mathrm{Aut}(\mathfrak{g})/\mathrm{Int}(\mathfrak{g})$. Hence we may replace σ by an element of finite order. In the sequel, we assume that $\mathbb{F}=\mathbb{R}$, $\mathfrak{g}$ is compact and that σ is finite order. Let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$ and denote by the same symbol the involution of G inducing from σ. Then the twisted loop group $L(G,\sigma)$ is defined as

$$L(G,\sigma) := \{g \in C^\infty(\mathbb{R},G) \,|\, g(t+2\pi)=\sigma(g(t)) \text{ for any } t \in \mathbb{R}\},$$

which is a Frechet Lie group with Lie algebra $L(\mathfrak{g},\sigma)$. The affine Kac-Moody Lie group $\widehat{L}(G,\sigma)$ is defined as a torus bundle over $L(G,\sigma)$ as follows. Let $\widetilde{\omega}_\lambda$

be the left invariant 2-form on $L(G,\sigma)$ obtained from ω_λ, which is closed. By retaking λ if necessary, we may assume that $\frac{1}{2\pi}\widetilde{\omega}_\lambda$ is an integral cohomology class. Is determined uniquely a S^1-bundle over $L(G,\sigma)$ with a connection whose curvature form is equal to $\widetilde{\omega}_\lambda$ (up to the bundle isomorphicness). Denote by $\widetilde{L}(G,\sigma)$ the total space of lthis S^1-bundle, which is a Frechet Lie group with Lie algebra $\widetilde{L}(\mathfrak{g},\sigma) := L(\mathfrak{g},\sigma) + \mathbb{R}c$. We define an action of $\mathbb{R}$ on $L(G,\sigma)$ by $(s\cdot u)(t) := u(t+s)$ $(t\in\mathbb{R})$. This action is lifted to $\widetilde{L}(G,\sigma)$ because it leaves $\widetilde{\omega}_\lambda$ invariantly. Furthermore, since σ is of finite order, this lifted action of $\mathbb{R}$ descends to an action of S^1. *The affine Kac-Moody Lie group* $\widehat{L}(G,\sigma)$ is the semi-direct product $S^1 \ltimes \widetilde{L}(G,\sigma)$ defined in terms of this S^1-action, which is a Frechet Lie group. Define an inner product $\langle\ ,\ \rangle_K$ of $\widehat{L}(\mathfrak{g},\sigma)$ by

$$\langle u_1\alpha_1 c + \beta_1 d,\, u_2 + \alpha_2 c + \beta_2 d\rangle_K := -\langle u_1, u_2\rangle_0 + \alpha_1\beta_2 + \beta_1\alpha_2 \\ (u_i \in L(G,\sigma),\ \alpha_i,\beta_i \in \mathbb{R}\ (i=1,2)).$$

It is shown that $(\widehat{L}(G,\sigma), \langle\ ,\ \rangle_K)$ is a Lorentzian symmetric space. Next we recall the notion of the affine Kac-Moody symmetric space. Let $\widehat{\rho}$ be an involution of $\widehat{L}(G,\sigma)$. This involution $\widehat{\rho}$ is said to be of *the first kind* (resp. *the second kind*) if $\widehat{\rho}c = c$ (resp. $\widehat{\rho}c = -c$). For simplicity, set $\widehat{G} := \widehat{L}(G,\sigma)$ and $\widehat{K} := \operatorname{Fix}\widehat{\rho}$. Let $\widehat{\mathfrak{g}}$ (resp. $\widehat{\mathfrak{k}}$) be the Lie algebra of $\widehat{G}$ (resp. $\widehat{K}$) and $\widehat{\mathfrak{p}}$ the (-1)-eigenspace decomposition of $\widehat{\rho}_{*e}$, which is identified with $T_{e\widehat{K}}(\widehat{G}/\widehat{K})$. Give $\widehat{G}/\widehat{K}$ the $\widehat{G}$-invariant (Lorentzian) metric obtained from the restriction $\langle\ ,\ \rangle_K$ to $\widehat{\mathfrak{p}}$. Then the space $\widehat{G}/\widehat{K}$ is a Lorentzian symmetric space. This space $\widehat{G}/\widehat{K}$ is called the *affine Kac-Moody symmetric space of the first (resp. the second) kind* if $\widehat{\rho}$ is of the first (resp. the second) kind. In the sequel, we treat only affine Kac-Moody symmetric spaces of the second kind. For (general) Frechet manifolds, the inverse function theorem is not valid. R.S. Hamilton introduced the class of *tame Frechet manifolds*. He showed that the inverse function theorem is valid for tame Frechet manifolds. On the other hand, B. Popescu [58] showed that affine Kac-Moody symmetric spaces are tame Frechet manifolds. Define a submanifold $\mathbb{R}^\infty_\sigma$ in the affine Kac-Moody Lie algebra $\widehat{L}(\mathfrak{g},\sigma)$ by

$$\mathbb{R}^\infty_\sigma := \{u - (1 - \frac{1}{2}\langle u,u\rangle_0)c + d \,|\, u \in L(\mathfrak{g},\sigma)\},$$

which is a horosphere in the infinite dimensional hyperbolic space $H^\infty_\sigma := \{u + \alpha c + d \,|\, u \in L(\mathfrak{g},\sigma),\ \alpha\in\mathbb{R}\}\,(\subset \widehat{L}(\mathfrak{g},\sigma))$ and is isometric to $L(\mathfrak{g},\sigma)(\cong H^0([0,1],\mathfrak{g}))$.

Here we give examples of the affine Kac-Moody symmetric spaces.

Example 1. Let σ be an automorphism of G and $\widehat{\rho}$ the involution of the second kind of the affine Kac-Moody Lie group $\widehat{L}(G\times G, \sigma\times\sigma^{-1})$ satisfying $\widehat{\rho}_{*e}(c) =$

$-c$, $\widehat{\rho}_{*e}(d) = -d$ and $\widehat{\rho}_{*e}(u,v)(t) := (v(-t), u(-t))$ $((u,v) \in L(\mathfrak{g}\times\mathfrak{g}, \sigma_{*e}\times\sigma_{*e}^{-1}))$. Then the affine Kac-Moody symmetric space $\widehat{L}(G\times G, \sigma\times\sigma^{-1})/\mathrm{Fix}\,\widehat{\rho}$ is isometric to the affine Kac-Moody Lie group $\widehat{L}(G\times G, \sigma\times\sigma^{-1})$. For simplicity, set $\widehat{G\times G} := \widehat{L}(G\times G, \sigma\times\sigma^{-1})$, $\widehat{K} := \mathrm{Fix}\,\widehat{\rho}$, $\widehat{\mathfrak{g}\times\mathfrak{g}} := \widehat{L}(\mathfrak{g}\times\mathfrak{g}, \sigma_{*e}\times\sigma_{*e}^{-1})$ and $\widehat{\mathfrak{k}} := \mathrm{Fix}\,\widehat{\rho}_{*e}$. Denote by $\widehat{\mathfrak{p}}$ the (-1)-eigenspace of $\widehat{\rho}_{*e}$ and set $V := \widehat{\mathfrak{p}}\cap\mathbb{R}^{\infty}_{\sigma\times\sigma^{-1}}$, where $\mathbb{R}^{\infty}_{\sigma\times\sigma^{-1}}$ is the horosphere in the infinite dimensional hyperbolic space in $\widehat{\mathfrak{g}\times\mathfrak{g}}$ corresponding to the above $\mathbb{R}^{\infty}_{\sigma}$. Also, set $G(\sigma) := \{(g,\sigma(g)) \mid g\in G\}$. The isotropy action of $\widehat{G\times G}/\widehat{K}$ leaves V invariantly and the restriction of this action to V is equivalent to the gauge action $P(G, G(\sigma)) \curvearrowright H^0([0,1],\mathfrak{g})$.

Example 2. Let $\rho_{\pm}$ be involutions of G and set $\sigma := \rho_-\circ\rho_+$. Let $\widehat{\rho}$ be the involution of the second kind of the affine Kac-Moody Lie group $\widehat{L}(G,\sigma)$ satisfying $\widehat{\rho}(g)(t) = \rho_+(g(-t))$ $(g\in L(G,\sigma))$. For simplicity, set $\widehat{G} := \widehat{L}(G,\sigma)$, $\widehat{K} := \mathrm{Fix}\,\widehat{\rho}$, $\widehat{\mathfrak{g}} := \widehat{L}(\mathfrak{g}, \sigma_{*e})$ and $\widehat{\mathfrak{k}} := \mathrm{Fix}\,\widehat{\rho}_{*e}$. Denote by $\widehat{\mathfrak{p}}$ the (-1)-eigenspace of $\widehat{\rho}_{*e}$ and set $V := \widehat{\mathfrak{p}}\cap\mathbb{R}^{\infty}_{\sigma}$. Also, set $K_{\pm} := \mathrm{Fix}\,\rho_{\pm}$. Denote by $\widehat{\mathrm{Ad}}$ the adjoint representation of $\widehat{G}$. The isotropy action $\widehat{\mathrm{Ad}}(\widehat{K}) : \widehat{\mathfrak{p}} \to \widehat{\mathfrak{p}}$ of $\widehat{G}/\widehat{K}$ leaves V invariantly and the restriction of this action to V is equivalent to the gauge action $P(G, K_+\times K_-) \curvearrowright H^0([0,1],\mathfrak{g})$.

The affine Kac-Moody symmetric spaces are classified as follows.

Theorem A.1([17,18]). *The affine Kac-Moody symmetric spaces as in Examples 1 and 2 are all of the affine Kac-Moody symmetric space.*

C. L. Terng and G. Thorbergsson proved the following fact.

Theorem A.2([62,63,64]). *Let G/K be a symmetric space of compact type and H be a symmetric subgroup. Then the gauge action $P(G, H\times K) \curvearrowright H^0([0,1],\mathfrak{g})$ be a polar action (hence the principal orbits of this action are isoparametric). In particular, the restriction of the isotropy representation of an Affine Kac-Moody symmetric space $\widehat{L}(G,\sigma)/\widehat{K}$ to $V := \widehat{\mathfrak{p}}\cap(\mathbb{R}^{\infty}_{\sigma})$ is a polar action.*

Conjecture 1. *Is any polar action on a Hilbert space equivalent to the restriction of the isotropy representation of an Affine Kac-Moody symmetric space $\widehat{L}(G,\sigma)/\widehat{K}$ to $V := \widehat{\mathfrak{p}}\cap(\mathbb{R}^{\infty}_{\sigma})$?*

According to Theorem A.1, this conjecture can be restated as follows.

Conjecture 1′. *Is any polar action on a Hilbert space equivalent to one of polar actions of the following two types:*

(I) $P(G, K_+ \times K_-) \curvearrowright H^0([0,1], \mathfrak{g})$, *where* G *is a compact simple Lie group and* $K_\pm$*'s are the fixed point groups of some involutions* $\rho_\pm$ *of* G.

(II) $P(G, G(\sigma)) \curvearrowright H^0([0,1], \mathfrak{g})$, *where* G *is a compact simple Lie group,* σ *is an involution of* G *and* $G(\sigma) := \{(g, \sigma(g)) \,|\, g \in G\}$?

References

[1] J. Berndt, S. Console and C. Olmos, *Submanifolds and holonomy*, Chapman & Hall/CRC Research Notes in Mathematics, 434, Chapman & Hall/CRC Boca Raton, FL, 2003.

[2] J. Berndt and H. Tamaru, *Homogeneous codimension one foliations on noncompact symmetric space*, J. Differential Geom. **63** (2003), 1–40.

[3] J. Berndt and H. Tamaru, *Cohomogeneity one actions on noncompact symmetric spaces with a totally geodesic singular orbit*, Tohoku Math. J. (2) **56** (2004), 163–177.

[4] J. Berndt and H. Tamaru, Cohomogeneity one actions on noncompact symmetric spaces of rank one, Trans. Amer. Math. Soc. **359** (2007), 3425–3438.

[5] J. Berndt and L. Vanhecke, Curvature-adapted submanifolds, Nihonkai Math. J. **3** (1992), 177–185.

[6] M. Brück, Equifocal famlies in symmetric spaces of compact type, J.reine angew. Math. **515** (1999), 73–95.

[7] E. Cartan, Familles de surfaces isoparamétriques dans les espaces á courbure constante, Ann. Mat. Pura Appl. **17** (1938), 177–191.

[8] U. Christ, Homogeneity of equifocal submanifolds, J. Differential Geom. **62** (2002), 1–15.

[9] A. J. Di Scala and C. Olmos, The geometry of homogeneous submanifolds of hyperbolic space, Math. Z. **237** (2001), 199-209.

[10] H. Ewert, Equifocal submanifolds in Riemannian symmetric spaces, Doctoral thesis.

[11] L. Geatti, Invariant domains in the complexfication of a noncompact Riemannian symmetric space, J. Algebra **251** (2002), 619–685.

[12] L. Geatti, Complex extensions of semisimple symmetric spaces, manuscripta math. **120** (2006) 1-25.

[13] O. Goertsches and G. Thorbergsson, On the Geometry of the orbits of Hermann actions, Geom. Dedicata **129** (2007) 101–118.

[14] C. Gorodski and E. Heintze, Homogeneous structures and rigidity of isoparametric submanifolds in Hilbert space, J. Fixed Point Theory Appl. **11** (2012) 93-136.

[15] J. Hahn, Isoparametric hypersurfaces in the pseudo-Riemannian space form, Math. Z. **187** (1984) 195–208.

[16] J. Hahn, Isotropy representations of semi-simple symmetric spaces and homogeneous hypersurfaces, J. Math. Soc. Japan **40** (1988) 271–288.

[17] E. Heintze, Toward symmetric spaces of affine Kac-Moody type, International J. of Geom. Methods in Modern Physics **3** (2006) 881-898.

[18] E. Heintze and C. Gro, Finite order automorphisms and real frorms of affine Kac-Moody algebras in the smooth and algebraic category, Memoires of AMS **219** 2012, no. 1030.

[19] E. Heintze and X. Liu, A splitting theorem for isoparametric submanifolds in Hilbert space, J. Differential Geom. **45** (1997), 319–335.

[20] E. Heintze and X. Liu, Homogeneity of infinite dimensional isoparametric submanifolds, Ann. of Math. **149** (1999), 149–181.

[21] E. Heintze, X. Liu and C. Olmos, Isoparametric submanifolds and a Chevalley type restriction theorem, Integrable systems, geometry, and topology, 151–190, AMS/IP Stud. Adv. Math. 36, Amer. Math. Soc., Providence, RI, 2006.

[22] E. Heintze and C. Olmos, Normal holonomy groups and s-representations, Indiana Univ. Math. J. **41** (1992), 869–874.

[23] E. Heintze, C. Olmos and G. Thorbergsson, Submanifolds with constant prinicipal curvatures and normal holonomy groups, Intern. J. Math. **2** (1991), 167–175.

[24] E. Heintze, R. S. Palais, C. L. Terng and G. Thorbergsson, Hyperpolar actions on symmetric spaces, Geometry, topology and physics, 214–245 Conf. Proc. Lecture Notes Geom. Topology **4**, Internat. Press, Cambridge, Mass., 1995.

[25] S. Helgason, Differential geometry, Lie groups and symmetric spaces, Pure Appl. Math. 80, Academic Press, New York, 1978.

[26] M. C. Hughes, Complex reflection groups, Comm. Algebra **18** (1990), 3999–4029.

[27] S. M. B. Kashani, Isoparametric functions and submanifolds, Glasgow Math. J. **35** (1993), 145–152.

[28] S. Kobayashi and K. Nomizu, Foundations of differential geometry, Interscience Tracts in Pure and Applied Mathematics 15, Vol. II, New York, 1969.

[29] N. Koike, Proper isoparametric semi-Riemannian submanifolds in a semi-Riemannian space form, Tsukuba J. Math. **13** (1989) 131–146.

[30] N. Koike, On proper Fredholm submanifolds in a Hilbert space arising from submanifolds in a symmetric space, Japan. J. Math. **28** (2002) 61–80.

[31] N. Koike, Submanifold geometries in a symmetric space of non-compact type and a pseudo-Hilbert space, Kyushu J. Math. **58** (2004), 167–202.

[32] N. Koike, Complex equifocal submanifolds and infinite dimensional anti-Kaehlerian isoparametric submanifolds, Tokyo J. Math. **28** (2005), 201–247.

[33] N. Koike, Actions of Hermann type and proper complex equifocal submanifolds, Osaka J. Math. **42** (2005) 599–611.

[34] N. Koike, A splitting theorem for proper complex equifocal submanifolds, Tohoku Math. J. **58** (2006), 393–417.

[35] N. Koike, Complex hyperpolar actions with a totally geodesic orbit, Osaka J. Math. **44** (2007), 491–503.

[36] N. Koike, The homogeneous slice theorem for the complete complexification of a proper complex equifocal submanifold, Tokyo J. Math. **33** (2010), 1–30.

[37] N. Koike, Examples of a complex hyperpolar action without singular orbit, Cubo A Math. J. **12** (2010), 127–144.

[38] N. Koike, Hermann type actions on a pseudo-Riemannian symmetric space, Tsukuba J. Math. **34** (2010), 137–172.

[39] N. Koike, On curvature-adapted and proper complex equifocal submanifolds, Kyungpook Math. J. **50** (2010) 509–536.

[40] N. Koike, Collapse of the mean curvature flow for equifocal submanifolds, Asian J. Math. **15** (2011), 101-128.

[41] N. Koike, Homogeneity of anti-Kaehler isoparametric submanifolds, Tokyo J. Math. **37** (2014), 1–20.

[42] N. Koike, A Cartan type identity for isoparametric hypersurfaces in symmetric spaces Tohoku Math. J. **66** (2014), 435–454.

[43] N. Koike, Collapse of the mean curvature flow for isoparametric submanifolds in non-compact symmetric spaces, Kodai Math. J. **37** (2014), 355–382.

[44] N. Koike, The constancy of principal curvatures of curvature-adapted submanifolds in symmetric spaces, Differential Geom. Appl. **35** (2014), 103–113.

[45] N. Koike, Homogeneity of anti-Kaehler isoparametric submanifolds II, arXiv:math.DG/1404.0827v1.

[46] N. Koike, The classifications of certain kind of isoparametric submanifolds in non-compact symmetric spaces, arXiv:math.DG/1209.1933v2.

[47] A. Kollross, A classification of hyperpolar and cohomogeneity one actions, Trans. Amer. Math. Soc. **354** (2002), 571–612.

[48] X. Liu and C. L. Terng, The mean curvature flow for isoparametric submanifolds, Duke Math. J. **147** (2009), 157-179.

[49] M. Magid, Lorentzian isoparametric hypersurfaces, Pacific J. Math. **118** (1985) 165–197.

[50] C. Olmos, The normal holonomy group, Proc. Amer. Math. Soc. **110** (1990), 813–818.

[51] C. Olmos and A. Will, Normal holonomy in Lorentzian space and submanifold geometry, Indiana Univ. Math. J. **50** (2001), 1777-1788.

[52] B. O'Neill, Semi-Riemannian Geometry, with applications to relativity, Pure Appl. Math. 103, Academic Press, New York, 1983.

[53] T. Ohshima and J. Sekiguchi, The restricted root system of a semisimple symmetric pair, Group representations and systems of differential equations (Tokyo, 1982), 433–497, Adv. Stud. Pure Math. **4**, North-Holland, Amsterdam, 1984.

[54] R. S. Palais, Morse theory on Hilbert manifolds, Topology **2** (1963), 299–340.

[55] R. S. Palais and C. L. Terng, A general theory of canonical forms, Trans. Amer. Math. Soc. **300** (1987), 771–789.

[56] R. S. Palais and C. L. Terng, Critical point theory and submanifold geometry, Lecture Notes in Math. **1353**, Springer-Verlag, Berlin, 1988.

[57] A. Z. Petrov, Einstein spaces, Pergamon Press, 1969.

[58] W. Rossmann, The structures of semisimple symmetric spaces, Canad. J. Math. **31** (1979), 157–180.

[59] B. Popescu, Infinite dimensional symmetric spaces, Thesis, Augsburg 2005.

[60] R. Szöke, Canonical complex structures associated to connections and complexifications of Lie groups, Math. Ann. **329** (2004), 553–591.

[61] C. L. Terng, Isoparametric submanifolds and their Coxeter groups, J. Differential Geom. **21** (1985), 79–107.

[62] C. L. Terng, Proper Fredholm submanifolds of Hilbert space, J. Differential Geom. **29** (1989), 9–47.

[63] C. L. Terng, Polar actions on Hilbert space, J. Geom. Anal. **5** (1995), 129–150.

[64] C. L. Terng and G. Thorbergsson, Submanifold geometry in symmetric spaces, J. Differential Geom. **42** (1995), 665–718.

[65] G. Thorbergsson, Isoparametric foliations and their buildings, Ann of Math. **133** (1991), 429–446.

[66] H. Wu, Holonomy groups of indefinite metrics, Pacific J. Math. **20** (1967), 351–392.

[67] B. Wu, Isoparametric submanifolds of hyperbolic spaces, Trans. Amer. Math. Soc. **331** (1992), 609–626.

Naoyuki Koike
Department of Mathematics, Tokyo University of Science
1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan
E-mail: koike@ma.kagu.tus.ac.jp