

## A shorter proof of Thomassen’s theorem on Tutte paths in plane graphs

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**Abstract.** A graph is said to be *Hamiltonian-connected* if there exists a Hamiltonian path between any given pair of distinct vertices. In 1983, Thomassen proved that every 4-connected plane graph is Hamiltonian-connected, using the concept of *Tutte subgraph*. In this paper, we give a new proof to Thomassen’s theorem.

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### §1. Introduction

In 1956, Tutte [9] proved that every 4-connected plane graph contains a Hamiltonian cycle. Extending the technique of Tutte, Thomassen [8] proved that every 4-connected plane graph is *Hamiltonian-connected*, i.e. there is a Hamiltonian path between any given pair of distinct vertices. For the proof of these two results, they considered a stronger concept, called a *Tutte subgraph*.

Let  $T$  be a subgraph of a graph  $G$ . A  $T$ -bridge of  $G$  is either (i) an edge of  $G - E(T)$  with both end vertices on  $T$  or (ii) a subgraph of  $G$  induced by the edges in a component of  $G - V(T)$  and all edges from that component to  $T$ . A  $T$ -bridge with the former type is said to be *trivial*, while the latter is *non-trivial*. For a  $T$ -bridge  $B$  of  $G$ , the vertices in  $B \cap T$  are the *attachments* of  $B$  (on  $T$ ). We say that  $T$  is a *Tutte subgraph* in  $G$  if every  $T$ -bridge of  $G$  has at most three attachments on  $T$ . For another subgraph  $C$  of  $G$ , the subgraph  $T$  is a  $C$ -*Tutte subgraph* in  $G$  if  $T$  is a Tutte subgraph in  $G$  and every  $T$ -bridge of  $G$  containing an edge of  $C$  has at most two attachments on  $T$ . A *Tutte path* (respectively, a *Tutte cycle*) in a graph is a path (respectively, a cycle) which is a Tutte subgraph. For a connected plane graph  $G$ , the boundary walk of

the outer face is called the *outer walk* of  $G$ . Furthermore, if it is a cycle, then it is the *outer cycle* of  $G$ .

Thomassen [8] proved the following result. (Although Thomassen's proof in [8] contained a small omission, it was corrected by Chiba and Nishizeki [1].)

**Theorem 1.1** (Thomassen [8]). *Let  $G$  be a 2-connected plane graph, let  $C$  be the outer cycle of  $G$ , let  $x \in V(C)$ , let  $y \in V(G) - \{x\}$ , and let  $e \in E(C)$ . Then  $G$  has a  $C$ -Tutte path from  $x$  to  $y$  through  $e$ .*

It is easy to see that Theorem 1.1 implies the result stating that every 4-connected plane graph is Hamiltonian-connected. In fact, for a given pair of distinct vertices  $x$  and  $y$  in a given 4-connected plane graph  $G$ , we first specify a facial cycle  $C$  incident with  $x$  and an edge  $e$  in  $C$ . Actually, we can specify the edge  $e$  (and the cycle  $C$ ) so that neither  $x$  nor  $y$  is an end vertex of  $e$ . Then we consider the graph  $G$  so that  $C$  is the outer cycle, and it follows from Theorem 1.1 that  $G$  has a  $C$ -Tutte path  $T$  from  $x$  to  $y$  through  $e$ . By the choice of the edge  $e$ , we have  $|T| \geq 4$ . If there exists a non-trivial  $T$ -bridge  $B$  of  $G$ , then the attachments of  $B$  form a cut set of order at most 3 that separates  $T - (B \cap T)$  from  $B - (B \cap T)$ , which contradicts that  $G$  is 4-connected. Therefore, there exists no non-trivial  $T$ -bridge of  $G$ , which means that  $T$  is a Hamiltonian path. This shows that  $G$  is Hamiltonian-connected, and we are done.

Note that in several papers, finding Tutte subgraphs is a crucial method to show Hamiltonicity or other related properties of graphs on surfaces. See for example, [2, 3, 4, 5, 6, 7, 10]. The purpose of this note is to give a simpler proof to Theorem 1.1. Indeed, we show the following statement.

**Theorem 1.2.** *Let  $G$  be a connected plane graph and let  $C$  be the outer walk of  $G$ . Then both of the following hold:*

- (I) *Let  $x \in V(C)$ ,  $y \in V(G) - \{x\}$  and  $e \in E(C)$ . If  $G$  has a path from  $x$  to  $y$  through  $e$ , then  $G$  has a  $C$ -Tutte path from  $x$  to  $y$  through  $e$ .*
- (II) *Let  $x, y \in V(C)$  with  $x \neq y$ , and  $S \subset V(C) - \{x, y\}$  with  $|S| \leq 2$ .*
  - (a) *Suppose that  $|S| = 1$  and  $C$  has a subpath  $Q_1$  from  $z$  to  $y$  with  $x \in V(Q_1)$ , where  $\{z\} = S$ . Then  $G$  has a path  $T$  from  $x$  to  $y$  such that  $V(T) \cap S = \emptyset$  and  $T \cup S$  is a  $Q_1$ -Tutte subgraph in  $G$ .*
  - (b) *Suppose that  $|S| = 2$  and  $C$  has a subpath  $Q_2$  from  $x$  to  $y$  with  $V(Q_2) \cap S = \emptyset$ . Then  $G$  has a path  $T$  from  $x$  to  $y$  such that  $V(T) \cap S = \emptyset$  and  $T \cup S$  is a  $Q_2$ -Tutte subgraph in  $G$ .*

Note that statement (I) implies Theorem 1.1 as an immediate corollary and statement (II-b) is exactly Theorem (2.4) in [5]. Our new ideas to prove Theorem 1.2 (I) are the following;

- Not assuming 2-connectedness.  
Since Theorem 1.1 deals with 2-connected graphs only, in order to use the induction hypothesis, we have to use some tricks to make reduced graphs 2-connected. In fact, the proof in [8] first consider 2-cuts with certain conditions, and reduce a given graph, if such a 2-cut exists, with several cases depending on the properties of the 2-cut. On the other hand, since we only assume the connectedness in Theorem 1.2, we do not need to consider the step. Instead, we have to consider whether there exists a cut vertex (see Claim 1), but the proof of it is much easier than to deal with 2-cuts with certain conditions.
- A shorter proof to statement (II).  
In order to show statement (I), we need a statement like (II). Thomassen [8] (and other researchers, for example in [1, 5]) has considered a block decomposition to show statement (II), together with the induction hypothesis for statement (I). In this paper, we actually give a shorter proof to statement (II), directly using the induction hypothesis for statement (I). Because of that, we could reduce the length of the proof.

Notice also that statement (I) guarantees the existence of a  $C$ -Tutte path connecting two given vertices through a given edge, which easily implies the existence of a  $C$ -Tutte path connecting two given vertices through a given vertex. In Section 2, we will often use the latter statement when we consider the induction hypothesis.

Let  $P$  be a path or a cycle with fixed direction. For two vertices  $x$  and  $y$  in  $P$ ,  $P[x, y]$  denotes the subpath of  $P$  from  $x$  to  $y$  (along the direction).

## §2. Proof of Theorem 1.2

We prove Theorem 1.2 by using simultaneous induction on the order of  $G$ . Actually, we will show the following two statements.

- Theorem 1.2 (I) holds for a graph  $G$  if both Theorems 1.2 (I) and (II) hold for all graphs  $G'$  with  $|G'| < |G|$ .
- Theorem 1.2 (II) holds for a graph  $G$  if Theorem 1.2 (I) holds for all graphs  $G'$  with  $|G'| \leq |G|$ .

Note that both Theorems 1.2 (I) and (II) clearly hold for all graphs of order at most 3. Hence proving the above two statements completes the proof of Theorem 1.2.

*Proof.* We show statement (i). Let  $G$  be a plane graph, let  $C$  be the outer walk of  $G$ , let  $x \in V(C)$ , let  $y \in V(G) - \{x\}$ , and let  $e \in E(C)$ . Suppose that  $G$  has a path from  $x$  to  $y$  through  $e$ . We first show the following claim.

**Claim 1.** *If  $G$  has a cut vertex, then  $G$  has a  $C$ -Tutte path from  $x$  to  $y$  through  $e$ .*

*Proof.* Suppose contrary that  $G$  has a cut vertex  $v$ . Let  $G'_1$  be a component of  $G - v$  with  $V(G'_1) \cap V(C) \neq \emptyset$ , let  $G_1$  be the subgraph of  $G$  induced by  $V(G'_1) \cup \{v\}$ , and let  $G_2 = G - V(G'_1)$ . Note that  $|G_1|, |G_2| < |G|$ . If  $V(G_2 - v) \cap V(C) \neq \emptyset$ , then we can use the symmetry between  $G_1$  and  $G_2$ , and hence by symmetry, we may assume that  $x \in V(G_1)$ ; Otherwise, that is, if  $V(G_2 - v) \cap V(C) = \emptyset$ , then  $x \in V(G_1)$  since  $x \in V(C)$ . In either case, we have  $x \in V(G_1)$ . Let  $C_1$  be the restriction of  $C$  into  $G_1$ .

Assume that  $y$  is also contained in  $G_1$ . Since  $G$  has a path from  $x$  to  $y$  through  $e$ , the edge  $e$  is also contained in  $G_1$  and  $G_1$  has a path from  $x$  to  $y$  through  $e$ . Since we assumed that Theorem 1.2 (I) holds for all graphs  $G'$  with  $|G'| < |G|$ ,  $G_1$  has a  $C_1$ -Tutte path  $T$  from  $x$  to  $y$  through  $e$ . Then we can easily see that  $T$  is a  $C$ -Tutte path in  $G$  from  $x$  to  $y$  through  $e$ , and we are done.

Hence we may assume that  $y$  is contained in  $G_2 - v$ . Let  $C_2$  be the restriction of  $C$  into  $G_2$ . Here we assume that  $e$  is contained in  $G_1$ , but we can show the other case (when  $e$  is contained in  $G_2$ ) by the almost same arguments. Since  $G$  has a path from  $x$  to  $y$  through  $e$ ,  $G_1$  has a path from  $x$  to  $v$  through  $e$  and  $G_2$  has a path from  $v$  to  $y$ . Since we assumed that Theorem 1.2 (I) holds for all graphs  $G'$  with  $|G'| < |G|$ ,  $G_1$  has a  $C_1$ -Tutte path  $T_1$  from  $x$  to  $v$  through  $e$  and  $G_2$  has a  $C_2$ -Tutte path  $T_2$  from  $v$  to  $y$ . Then letting  $T = T_1 \cup T_2$ , we see that  $T$  is a  $C$ -Tutte path in  $G$  from  $x$  to  $y$  through  $e$ , and we are done. This completes the proof of Claim 1.  $\square$

Throughout the rest of the proof of statement (i), it follows from Claim 1 that we may assume that  $G$  is 2-connected. Thus,  $C$  is a cycle. We fix the direction of  $C$  in the clockwise order.

Let  $e = u_1u_2$ . If  $\{x, y\} = \{u_1, u_2\}$ , then the edge  $xy$  itself is a  $C$ -Tutte path in  $G$ , and we are done. Thus, we may assume that  $\{x, y\} \neq \{u_1, u_2\}$ . This and symmetry imply that  $y \neq u_1, u_2$ , since otherwise, we can change the roles of  $x$  and  $y$ . Furthermore, it follows from the symmetry between  $u_1$  and  $u_2$  that we may assume that  $G$  has a path from  $x$  to  $y$  through  $u_1$  and then  $u_2$ . Then the subpath  $C[x, u_1]$  of  $C$ , which is directed in the clockwise order, satisfies that  $G - V(C[x, u_1])$  has a path from  $u_2$  to  $y$ .

Let  $H$  be the component of  $G - V(C[x, u_1])$  containing  $u_2$  and  $y$ , and let  $C_H$  be the outer walk of  $H$ . Take a vertex  $w$  in  $V(C_H) \cap V(C)$  so that there exists a path in  $H$  from  $u_2$  to  $y$  through  $w$ , and  $w$  is as close to  $x$  on  $C[u_2, x]$

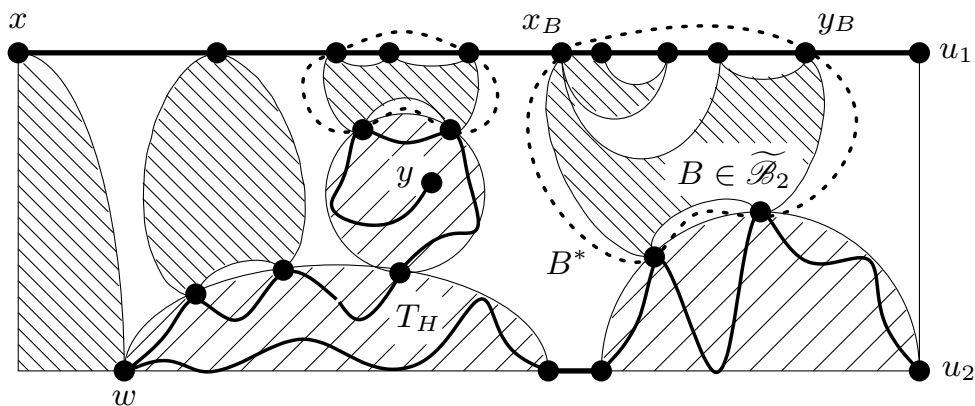


Figure 1: The  $C_H$ -Tutte path  $T_H$  and non-trivial  $(C[x, u_1] \cup T_H)$ -bridges of  $G$ . (The outer rectangle represents the outer cycle  $C$  of  $G$ . The regions with falling diagonals stroke from top left to bottom right represent non-trivial  $(C[x, u_1] \cup T_H)$ -bridges of  $G$  having an attachment on  $C[x, u_1]$ . The regions bounded by dotted curves are graphs  $B^*$  for  $B \in \tilde{\mathcal{B}}_2$ .)

as possible under the condition. Since  $u_2$  can play the role of  $w$  except for the last condition, such a vertex  $w$  exists. Since we assumed that Theorem 1.2 (I) holds for all graphs  $G'$  with  $|G'| < |G|$ ,  $H$  has a  $C_H$ -Tutte path  $T_H$  from  $u_2$  to  $y$  through  $w$ . See Figure 1.

We claim that we can take such a path  $T_H$  so that  $|T_H| \geq 3$ , unless  $u_2$  and  $y$  are connected by an edge that is a cut-edge of  $H$ . Suppose that  $|T_H| \leq 2$ . Since  $y \neq u_2$ , we have  $|T_H| = 2$ ,  $T_H$  consists of only  $u_2$  and  $y$ , and  $w = u_2$  or  $w = y$ . If there exists a vertex  $w'$  such that  $w' \neq u_2, y$  and  $H$  has a path from  $u_2$  to  $y$  through  $w'$ , then instead of the path  $T_H$ , we can take a  $C_H$ -Tutte path  $T'_H$  from  $u_2$  to  $y$  through  $w'$ . Note that  $T'_H$  has the properties same as  $T_H$  together with the condition  $|T'_H| \geq 3$ , and the claim holds. Therefore, there does not exist such a vertex  $w'$ . We can easily see that these properties imply that  $u_2$  and  $y$  are connected by an edge that is a cut-edge of  $H$ . Then the claim holds.

Let  $\mathcal{B}_1$  (and resp.  $\mathcal{B}_2$ ) be the set of non-trivial  $(C[x, u_1] \cup T_H)$ -bridges  $B$  of  $G$  such that  $B$  has exactly one (resp. at least two) attachments on  $C[x, u_1]$ . For any  $B \in \mathcal{B}_2$ , let  $x_B$  and  $y_B$  be the attachments of  $B$  on  $C[x, u_1]$  such that  $x_B$  is as close to  $x$  on  $C[x, u_1]$  as possible and  $y_B$  is as close to  $u_1$  on  $C[x, u_1]$  as possible. Note that  $x_B \neq y_B$  and  $C[x_B, y_B]$  is contained in  $C[x, u_1]$  for any  $B \in \mathcal{B}_2$ . For  $B, B' \in \mathcal{B}_2$ , we write  $B' \preceq B$  if either (i)  $B = B'$ , or (ii)  $B'$  is contained in the disk bounded by  $Q \cup C[x_B, y_B]$ , where  $Q$  is a path in  $B$  connecting  $x_B$  and  $y_B$ . Since  $G$  is a plane graph and  $C[x, u_1]$  is a subpath of the outer cycle  $C$  of  $G$ , the binary relation  $\preceq$  is a partial order on  $\mathcal{B}_2$ . Let  $\tilde{\mathcal{B}}_2$

be the set of maximal elements of  $\mathcal{B}_2$  with respect to the partial order  $\preceq$ . By the planarity, we have the following claim.

**Claim 2.** *For any  $B, B' \in \widetilde{\mathcal{B}}_2$  with  $B \neq B'$ ,  $C[x_B, y_B]$  and  $C[x_{B'}, y_{B'}]$  are edge-disjoint.*

For  $B \in \mathcal{B}_1 \cup \widetilde{\mathcal{B}}_2$ , let  $S_B = V(B) \cap V(T_H)$ . If  $B \in \mathcal{B}_1$ , then let  $B^* = B$ , and if  $B \in \widetilde{\mathcal{B}}_2$ , then let  $B^*$  be the subgraph of  $G$  induced by the union of all elements  $B' \in \mathcal{B}_2$  such that  $B' \preceq B$ , together with  $C[x_B, y_B]$ . Let  $C_B$  be the outer walk of  $B^*$ . Note that if  $B \in \widetilde{\mathcal{B}}_2$ , then  $C[x_B, y_B]$  is a subpath of  $C_B$  with  $V(C[x_B, y_B]) \cap S_B = \emptyset$ . We need the following claim.

**Claim 3.** *For any  $B \in \mathcal{B}_1 \cup \widetilde{\mathcal{B}}_2$ , we have  $|S_B| \leq 2$ . Furthermore, if  $B^*$  contains an edge of  $C[u_2, x] - \{x\}$ , then  $|S_B| \leq 1$*

*Proof.* Let  $B \in \mathcal{B}_1 \cup \widetilde{\mathcal{B}}_2$ . If  $S_B = \emptyset$ , then the claim holds. Hence suppose that  $S_B \neq \emptyset$ . Since  $B$  has an attachment on  $C[x, u_1]$ ,  $B - V(C[x, u_1])$  is a  $T_H$ -bridge of  $H$  containing an edge of  $C_H$ . Since  $T_H$  is a  $C_H$ -Tutte path in  $H$ , we have  $|S_B| \leq 2$ . So, the first statement holds.

Suppose that  $B^*$  contains an edge of  $C[u_2, x] - \{x\}$  and  $|S_B| = 2$  for some  $B \in \mathcal{B}_1 \cup \widetilde{\mathcal{B}}_2$ . Since  $B$  has an attachment on  $C[x, u_1]$  and  $B^*$  contains an edge of  $C[u_2, x] - \{x\}$ , it follows from the definition of  $B^*$  that  $B$  contains an edge of  $C[w, x] - \{x\}$ . Let  $\{z_1, z_2\} = S_B$ , and by symmetry, we may assume that  $T_H$  passes through  $z_1$  first and then  $z_2$ . Since  $B$  is a  $(C[x, u_1] \cup T_H)$ -bridge,  $B - V(C[x, u_1])$  is connected, and hence there exists a path a contradiction. Thus, there exists a path  $Q$  in  $B - V(C[x, u_1])$  from  $z_1$  to  $z_2$  through a vertex  $w'$  for some  $w' \in (V(B) - \{x, z_1, z_2\}) \cap V(C[u_2, x])$ .

Since  $w \in V(T_H)$ , it follows from the choice of  $w'$  that  $w' \in V(C[w, x]) - \{w, x\}$ . Then connecting  $T_H[u_2, z_1]$ ,  $Q$  and  $T_H[z_2, y]$ , we can find a path in  $H$  from  $u_2$  to  $y$  through  $w'$ , which contradicts the choice of  $w$ .  $\square$

Let  $B \in \widetilde{\mathcal{B}}_2$ . If  $|T_H| \geq 3$ , then  $V(T_H) - S_B \neq \emptyset$  since  $|S_B| \leq 2$  by Claim 3. Suppose that  $|T_H| = 2$ . In this case,  $u_2$  and  $y$  are connected by an edge that is a cut-edge of  $H$ . This implies that at least one of  $u_2$  and  $y$  is not an attachment of  $B^*$ , and hence  $|S_B| \leq 1$ . In either case,  $V(T_H) - S_B \neq \emptyset$ . Since  $V(T_H) - S_B \subset V(G) - V(B^*)$ , we obtain  $|B^*| < |G|$ . Since we assumed that Theorems 1.2 (I) and (II) hold for all graphs  $G'$  with  $|G'| < |G|$ ,  $B^*$  has a path  $T_B$  from  $x_B$  to  $y_B$  with  $V(T_B) \cap S_B = \emptyset$  such that  $T_B \cup S_B$  is a  $C_B$ -Tutte subgraph in  $B^*$  if  $S_B = \emptyset$  (by Theorem (I)), or such that  $T_B \cup S_B$  is a  $Q_B$ -Tutte subgraph in  $B^*$  if  $|S_B| = 1$ , where  $Q_B$  is a subpath of  $C_B$  from  $z_B$  to  $y_B$  with  $x_B \in V(Q_B)$  and  $\{z_B\} = S_B$  (by Theorem (II-a)), or such that  $T_B \cup S_B$  is a  $C[x_B, y_B]$ -Tutte subgraph in  $B^*$  if  $|S_B| = 2$  (by Theorem (II-b)). Note that when  $|S_B| = 1$ , then  $E(C[x_B, y_B]) \subset E(C_B[z_B, y_B])$  and  $x_B \in V(C_B[z_B, y_B])$ .

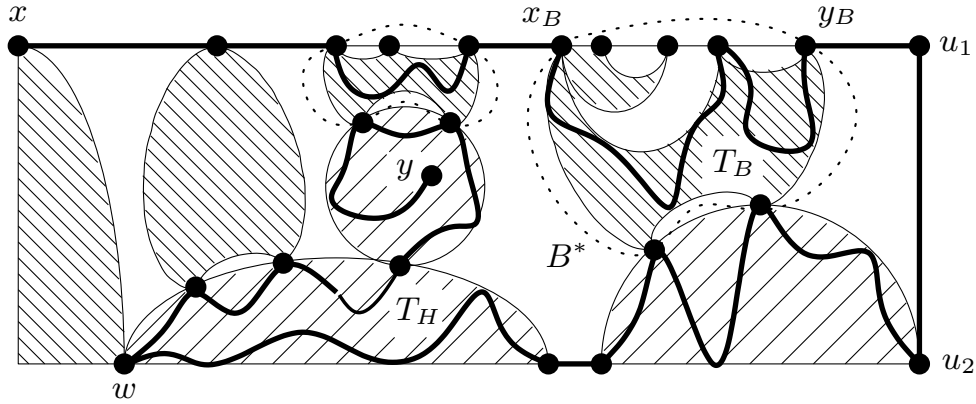


Figure 2: The  $C$ -Tutte path  $T$  in  $G$ .

In particular, in either case,  $T_B \cup S_B$  is a  $C[x_B, y_B]$ -Tutte subgraph in  $B^*$ .

Let

$$T = \left( C[x, u_1] - \bigcup_{B \in \widetilde{\mathcal{B}}_2} C[x_B, y_B] \right) \cup \bigcup_{B \in \widetilde{\mathcal{B}}_2} T_B \cup \{u_1 u_2\} \cup T_H.$$

See Figure 2. By Claim 2,  $T$  is a path in  $G$  from  $x$  to  $y$  through  $e$ . We will show that  $T$  is a  $C$ -Tutte path in  $G$ .

Let  $D$  be a non-trivial  $T$ -bridge of  $G$ . Note that either (i)  $D$  is a non-trivial  $(T_B \cup S_B)$ -bridge of  $B^*$  for some  $B \in \widetilde{\mathcal{B}}_2$ , or (ii)  $D$  is a non-trivial  $(C[x, u_1] \cup T_H)$ -bridge of  $G$  having at most one attachment on  $C[x, u_1]$ .

Suppose first that  $D$  satisfies (i). Since  $T_B \cup S_B$  is a  $C[x_B, y_B]$ -Tutte subgraph in  $B^*$ ,  $D$  has at most three attachments, and at most two attachments if  $D$  contains an edge in  $C[x_B, y_B]$ . Hence if  $E(C) \cap E(B^*) \subseteq E(C[x_B, y_B])$ , then we are done. Thus, suppose that  $(E(C) \cap E(B^*)) - E(C[x_B, y_B]) \neq \emptyset$ . Then  $B^*$  contains an edge of  $C[u_2, x] - \{x\}$ , and hence by Claim 3,  $|S_B| \leq 1$ . This implies that  $D$  has at most two attachments on  $T_B \cup S_B$  if  $S_B = \emptyset$ ; otherwise, that is, if  $|S_B| = 1$ , then  $D$  contains an edge in  $Q_B$  (see the definition of  $Q_B$ ), and hence  $D$  also has at most two attachments on  $T_B \cup S_B$ . Note that  $E(C) \cap E(B^*) \subset E(C_B)$ , and furthermore  $E(C) \cap E(B^*) = E(Q_B)$  if  $|S_B| = 1$ . Then in either case, we obtain that  $D$  has at most three attachments on  $T$  and at most two attachments on  $T$  if  $D$  contains an edge of  $C$ .

Suppose next that  $D$  satisfies (ii). Since  $|S_D| \leq 2$ ,  $D$  has at most three attachments on  $C[x, u_1] \cup T_H$  such that at most one of them is on  $C[x, u_1]$ . Furthermore, if  $D$  contains an edge of  $C$ , then by the planarity of  $G$ ,  $D$  contains an edge of  $C[u_2, x] - \{x\}$ . Then  $D$  has at most two attachments by the same arguments as in the previous paragraph.

These imply that  $T$  is a  $C$ -Tutte path in  $G$  from  $x$  to  $y$  through  $e$ , and we complete the proof of statement (i).  $\square$

*Proof.* We here show statement (ii). Let  $G$  be a plane graph and let  $C$  be the outer walk of  $G$ . Let  $x, y \in V(C)$  with  $x \neq y$ , and  $S \subset V(C) - \{x, y\}$  with  $|S| \leq 2$ .

(a) Suppose first that  $|S| = 1$  and  $C$  has a subpath  $Q_1$  from  $z$  to  $y$  with  $x \in V(Q_1)$ , where  $S = \{z\}$ . Let  $G^*$  be the graph obtained from  $G$  by adding an edge connecting  $z$  and  $y$  so that  $Q_1$  and the edge  $zy$  form the outer walk of  $G^*$ , say  $C^*$ . Since we assumed that Theorem 1.2 (I) holds for all graphs  $G'$  with  $|G'| \leq |G|$ ,  $G^*$  has a  $C^*$ -Tutte path  $T^*$  from  $z$  to  $x$  through the edge  $zy$ . Let  $T$  be the path obtained from  $T^*$  by deleting  $z$ . Note that  $T$  is a path in  $G$  from  $x$  to  $y$  with  $V(T) \cap S = \emptyset$ . Since  $E(Q_1) = E(C^*) - \{zy\}$ ,  $T \cup S$  is a  $Q_1$ -Tutte subgraph in  $G$ . This completes the proof of (a).

(b) Suppose that  $|S| = 2$  and  $C$  has a subpath  $Q_2$  from  $x$  to  $y$  with  $V(Q_2) \cap S = \emptyset$ . Let  $\{z_1, z_2\} = S$ , and by symmetry, we may assume that  $x, z_1, z_2, y$  appear in  $C$  in this clockwise order. Let  $G^*$  be the graph obtained from  $G$  by deleting  $z_2$  and adding an edge connecting  $z_1$  and  $x$  so that  $Q_2$ , the edge  $z_1x$  and all neighbors of  $z_2$  appear in the outer walk of  $G^*$ , say  $C^*$ . Since we assumed that Theorem 1.2 (I) holds for all graphs  $G'$  with  $|G'| \leq |G|$ ,  $G^*$  has a  $C^*$ -Tutte path  $T^*$  from  $z_1$  to  $y$  through the edge  $z_1x$ . Let  $T$  be the path obtained from  $T^*$  by deleting  $z_1$ . Note that  $V(T) \cap S = \emptyset$ . Let  $B$  be a  $(T \cup S)$ -bridge of  $G$ . If  $z_2$  is not an attachment of  $B$ , then  $B$  has at most three attachments and at most two attachments if  $B$  contains an edge of  $C^*$ . Notice that  $E(Q_2) \subset E(C^*)$ . On the other hand, if  $z_2$  is an attachment of  $B$ , then  $B - z_2$  is a  $T^*$ -bridge of  $G^*$  containing an edge of  $C^*$  because all neighbors of  $z_2$  appear in  $C^*$ . Thus,  $B - z_2$  has at most two attachments on  $T \cup \{z_1\}$ , and at most three attachments on  $T \cup S$  one of which is  $z_2$ . In particular, by the planarity of  $G$  and the fact that  $Q_2$  is a subpath of the outer walk of  $G$ ,  $B$  contains no edge of  $Q_2$ . These imply that  $T \cup S$  is a  $Q_2$ -Tutte subgraph in  $G$ , and this completes the proof of (b).  $\square$

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