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Instability of solitary waves for nonlinear Schrödinger equations of derivative type

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Dedicated to Professor Nakao Hayashi on his sixtieth birthday

Abstract. We study the orbital stablity and instability of solitary wave solutions for nonlinear Schrödinger equations of derivative type.

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§1. Introduction

In this paper, we study the instability of solitary wave solutions for nonlinear Schrödinger equations of the form

$$i\partial_t u = -\partial_x^2 u - i|u|^2\partial_x u - b|u|^4 u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}, \tag{1.1}$$

where $b \ge 0$ is a constant. Eq. (1.1) appears in various areas of physics such as plasma physics, nonlinear optics, and so on (see, e.g., [12, 13] and also Introduction of [16]). It is known that (1.1) has a two parameter family of solitary wave solutions

$$u_\omega(t,x) = e^{i\omega_0 t}\phi_\omega(x-\omega_1 t), \tag{1.2}$$

where $\omega = (\omega_0, \omega_1) \in \Omega := \{(\omega_0,\omega_1)\in\mathbb{R}^2 : \omega_1^2 < 4\omega_0\}$, $\gamma = 1 + \frac{16}{3}b$,

$$\phi_\omega(x) = \tilde\phi_\omega(x)\exp\left(i\frac{\omega_1}{2}x - \frac{i}{4}\int_{-\infty}^x |\tilde\phi_\omega(\eta)|^2\,d\eta\right), \tag{1.3}$$

$$\tilde\phi_\omega(x) = \left\{\frac{2(4\omega_0-\omega_1^2)}{-\omega_1 + \sqrt{\omega_1^2+\gamma(4\omega_0-\omega_1^2)}\cosh(\sqrt{4\omega_0-\omega_1^2}\,x)}\right\}^{1/2}. \tag{1.4}$$

Here, we note that $\phi_\omega(x)$ is a solution of

$$-\partial_x^2\phi + \omega_0\phi + \omega_1 i\partial_x\phi - i|\phi|^2\partial_x\phi - b|\phi|^4\phi = 0, \quad x \in \mathbb{R}, \tag{1.5}$$

and $\tilde{\phi}_\omega(x)$ is a solution of

$$-\partial_x^2\phi + \frac{4\omega_0 - \omega_1^2}{4}\phi + \frac{\omega_1}{2}|\phi|^2\phi - \frac{3}{16}\gamma|\phi|^4\phi = 0, \quad x \in \mathbb{R}. \tag{1.6}$$

For v, $w \in L^2(\mathbb{R}) = L^2(\mathbb{R}, \mathbb{C})$, we define

$$(v, w)_{L^2} = \Re \int_{\mathbb{R}} v(x)\overline{w(x)}\, dx,$$

and regard $L^2(\mathbb{R})$ as a real Hilbert space. Similarly, $H^1(\mathbb{R}) = H^1(\mathbb{R}, \mathbb{C})$ is regarded as a real Hilbert space with inner product

$$(v, w)_{H^1} = (v, w)_{L^2} + (\partial_x v, \partial_x w)_{L^2}.$$

We define the energy $E : H^1(\mathbb{R}) \to \mathbb{R}$ by

$$E(v) = \frac{1}{2}\|\partial_x v\|_{L^2}^2 - \frac{1}{4}(i|v|^2\partial_x v, v)_{L^2} - \frac{b}{6}\|v\|_{L^6}^6. \tag{1.7}$$

Then, we have

$$E'(v) = -\partial_x^2 v - i|v|^2\partial_x v - b|v|^4 v,$$

and (1.1) can be written in a Hamiltonian form $i\partial_t u = E'(u)$ in $H^{-1}(\mathbb{R})$.

For $\theta = (\theta_0, \theta_1) \in \mathbb{R}^2$ and $v \in H^1(\mathbb{R})$, we define

$$T(\theta)v(x) = e^{i\theta_0}v(x - \theta_1) \quad (x \in \mathbb{R}). \tag{1.8}$$

Note that the energy E is invariant under T, i.e.,

$$E(T(\theta)v) = E(v), \quad \theta \in \mathbb{R}^2,\ v \in H^1(\mathbb{R}), \tag{1.9}$$

and that the solitary wave solution (1.2) is written as $u_\omega(t) = T(\omega t)\phi_\omega$.

The Cauchy problem for (1.1) is locally well-posed in the energy space $H^1(\mathbb{R})$ (see [16] and also [7, 8, 9]). For any $u_0 \in H^1(\mathbb{R})$, there exist $T_{\max} \in (0, \infty]$ and a unique solution $u \in C([0, T_{\max}), H^1(\mathbb{R}))$ of (1.1) with $u(0) = u_0$ such that either $T_{\max} = \infty$ or $T_{\max} < \infty$ and $\lim_{t \to T_{\max}} \|u(t)\|_{H^1} = \infty$. Moreover, the solution $u(t)$ satisfies

$$E(u(t)) = E(u_0), \quad Q_0(u(t)) = Q_0(u_0), \quad Q_1(u(t)) = Q_1(u_0)$$

for all $t \in [0, T_{\max})$, where Q_0 and Q_1 are defined by

$$Q_0(v) = \frac{1}{2}\|v\|_{L^2}^2, \quad Q_1(v) = \frac{1}{2}(i\partial_x v, v)_{L^2}. \tag{1.10}$$

For $\varepsilon > 0$, we define

$$U_\varepsilon(\phi_\omega) = \{u \in H^1(\mathbb{R}) : \inf_{\theta \in \mathbb{R}^2} \|u - T(\theta)\phi_\omega\|_{H^1} < \varepsilon\}.$$

Then, the stability and instability of solitary waves are defined as follows.

Definition 1. We say that the solitary wave solution $T(\omega t)\phi_\omega$ of (1.1) is *stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $u_0 \in U_\delta(\phi_\omega)$, then the solution $u(t)$ of (1.1) with $u(0) = u_0$ exists for all $t \geq 0$, and $u(t) \in U_\varepsilon(\phi_\omega)$ for all $t \geq 0$. Otherwise, $T(\omega t)\phi_\omega$ is said to be *unstable*.

For the case $b = 0$, Colin and Ohta [2] proved that the solitary wave solution $T(\omega t)\phi_\omega$ of (1.1) is stable for all $\omega \in \Omega$ (see also [6, 20]). We remark that the instability of solitary waves for (1.1) is not studied in previous papers [2, 6, 20]. For a recent result on a generalized derivative nonlinear Schrödinger equation, see [10].

In this paper, we consider the case $b > 0$, and prove the following.

Theorem 1. *Let $b > 0$. Then there exists $\kappa = \kappa(b) \in (0, 1)$ such that the solitary wave solution $T(\omega t)\phi_\omega$ of (1.1) is stable if $-2\sqrt{\omega_0} < \omega_1 < 2\kappa\sqrt{\omega_0}$, and unstable if $2\kappa\sqrt{\omega_0} < \omega_1 < 2\sqrt{\omega_0}$.*

Remark 1. Let $b > 0$, $\gamma = 1 + \dfrac{16}{3}b$, and

$$g(\xi) = \frac{2(\gamma - 1)}{\xi} \tan^{-1} \frac{1 + \sqrt{1 + \xi^2}}{\xi}, \quad \xi \in (0, \infty). \tag{1.11}$$

Then, $g : (0, \infty) \to (0, \infty)$ is strictly decreasing and bijective. Thus, for any $b > 0$, there exists a unique $\hat{\xi} = \hat{\xi}(b) \in (0, \infty)$ such that $g(\hat{\xi}) = 1$. The constant κ in Theorem 1 is given by $\kappa = (1 + \hat{\xi}^2/\gamma)^{-1/2}$ (see Lemma 1 below).

Remark 2. The sufficient condition $-2\sqrt{\omega_0} < \omega_1 < 2\kappa\sqrt{\omega_0}$ for stability of $T(\omega t)\phi_\omega$ is equivalent to $Q_1(\phi_\omega) > 0$, and the sufficient condition $2\kappa\sqrt{\omega_0} < \omega_1 < 2\sqrt{\omega_0}$ for instability is equivalent to $Q_1(\phi_\omega) < 0$ (see Lemma 1 and Proof of Theorem 1 below). We also remark that $E(\phi_\omega) = -\dfrac{\omega_1}{2} Q_1(\phi_\omega)$ for all $\omega \in \Omega$.

Remark 3. We do not study the borderline case $\omega_1 = 2\kappa\sqrt{\omega_0}$ in this paper, and leave it as an open problem. Note that $E(\phi_\omega) = Q_1(\phi_\omega) = 0$ in the case $\omega_1 = 2\kappa\sqrt{\omega_0}$. For related results for one-parameter family of solitary waves in borderline cases, see [1, 15, 14, 11].

Remark 4. It is not known whether (1.1) has finite time blowup solutions or not. It will be interesting to study relations between unstable solitary wave solutions obtained in Theorem 1 and the existence of blowup solutions for (1.1). For a recent progress in this direction, see Wu [18, 19].

For $\omega \in \Omega$, we define the action $S_\omega : H^1(\mathbb{R}) \to \mathbb{R}$ by

$$S_\omega(v) = E(v) + \sum_{j=0}^{1} \omega_j Q_j(v),$$

where E, Q_0 and Q_1 are defined by (1.7) and (1.10). Note that $Q_0'(v) = v$, $Q_1'(v) = i\partial_x v$, and that (1.5) is equivalent to $S_\omega'(\phi) = 0$.

We also define a function $d : \Omega \to \mathbb{R}$ by

$$d(\omega) = S_\omega(\phi_\omega) = E(\phi_\omega) + \sum_{j=0}^{1} \omega_j Q_j(\phi_\omega).$$

Then, we have

$$d'(\omega) = (\partial_{\omega_0} d(\omega), \partial_{\omega_1} d(\omega)) = (Q_0(\phi_\omega), Q_1(\phi_\omega)),$$

and the Hessian matrix $d''(\omega)$ of $d(\omega)$ is given by

$$d''(\omega) = \begin{bmatrix} \partial_{\omega_0}^2 d(\omega) & \partial_{\omega_1}\partial_{\omega_0} d(\omega) \\ \partial_{\omega_0}\partial_{\omega_1} d(\omega) & \partial_{\omega_1}^2 d(\omega) \end{bmatrix} = \begin{bmatrix} \partial_{\omega_0} Q_0(\phi_\omega) & \partial_{\omega_1} Q_0(\phi_\omega) \\ \partial_{\omega_0} Q_1(\phi_\omega) & \partial_{\omega_1} Q_1(\phi_\omega) \end{bmatrix}.$$

To prove Theorem 1, we use the following sufficient conditions for stability and instability in terms of the Hessian matrix $d''(\omega)$ (see [5]).

Theorem 2. *Let $\omega \in \Omega$. If the matrix $d''(\omega)$ has a positive eigenvalue, then the solitary wave solution $T(\omega t)\phi_\omega$ of (1.1) is stable.*

Theorem 3. *Let $\omega \in \Omega$. If $d''(\omega)$ is negative definite (all eigenvalues of $d''(\omega)$ are negative), then the solitary wave solution $T(\omega t)\phi_\omega$ of (1.1) is unstable.*

Theorem 2 can be proved in the same way as in Colin and Ohta [2], and we omit the proof. We give the proof of Theorem 3 in Section 3 below. As we stated above, the instability of solitary waves for (1.1) has not been studied in previous papers [2, 6, 20].

Moreover, by the explicit form (1.3) with (1.4) of ϕ_ω, and by elementary computations, we have the following.

Lemma 1. *Let $b > 0$ and $\gamma = 1 + \frac{16}{3}b$. For $\omega \in \Omega$, we have*

$$Q_0(\phi_\omega) = \frac{4}{\sqrt{\gamma}} \tan^{-1} \frac{\omega_1 + \sqrt{\omega_1^2 + \gamma(4\omega_0 - \omega_1^2)}}{\sqrt{\gamma(4\omega_0 - \omega_1^2)}},$$

$$Q_1(\phi_\omega) = \frac{1}{\gamma^{3/2}} \left\{ \sqrt{\gamma(4\omega_0 - \omega_1^2)} - 2(\gamma - 1)\omega_1 \tan^{-1} \frac{\omega_1 + \sqrt{\omega_1^2 + \gamma(4\omega_0 - \omega_1^2)}}{\sqrt{\gamma(4\omega_0 - \omega_1^2)}} \right\},$$

$$\det[d''(\omega)] = \frac{-4Q_1(\phi_\omega)}{\sqrt{4\omega_0 - \omega_1^2}\, \{\omega_1^2 + \gamma(4\omega_0 - \omega_1^2)\}}.$$

Theorem 1 follows from Theorems 2 and 3, Lemma 1 and Remark 1.

Proof of Theorem 1. Let $\omega \in \Omega$. If $\omega_1 \leq 0$, then by Lemma 1, we have $Q_1(\phi_\omega) > 0$ and $\det[d''(\omega)] < 0$. Thus, the matrix $d''(\omega)$ has one positive eigenvalue and one negative eigenvalue. Therefore, by Theorem 2, $T(\omega t)\phi_\omega$ is stable.

Next, we consider the case $\omega_1 > 0$. We put $\xi = \sqrt{\gamma\left(\frac{4\omega_0}{\omega_1^2} - 1\right)}$. Then, by Lemma 1, we have

$$Q_1(\phi_\omega) = \frac{1}{\gamma}\sqrt{4\omega_0 - \omega_1^2}\,\{1 - g(\xi)\},$$

where $g(\xi)$ is defined by (1.11) in Remark 1.

If $g(\xi) < 1$, then $Q_1(\phi_\omega) > 0$ and $\det[d''(\omega)] < 0$. Thus, $d''(\omega)$ has a positive eigenvalue, and by Theorem 2, $T(\omega t)\phi_\omega$ is stable.

On the other hand, if $g(\xi) > 1$, then $Q_1(\phi_\omega) < 0$ and $\det[d''(\omega)] > 0$. Moreover, since

$$\partial_{\omega_0}^2 d(\omega) = \partial_{\omega_0} Q_0(\phi_\omega) = \frac{-4\omega_1}{\sqrt{4\omega_0 - \omega_1^2}\{\gamma(4\omega_0 - \omega_1^2) + \omega_1^2\}} < 0,$$

we see that $d''(\omega)$ is negative definite. Thus, it follows from Theorem 3 that $T(\omega t)\phi_\omega$ is unstable.

Finally, by Remark 1, we see that $g(\xi) < 1$ is equivalent to $\omega_1 < 2\kappa\sqrt{\omega_0}$, and that $g(\xi) > 1$ is equivalent to $\omega_1 > 2\kappa\sqrt{\omega_0}$. □

The rest of the paper is organized as follows. In Section 2, we give a variational characterization of ϕ_ω. This part is essentially the same as Section 3 of [2], so we omit the details. In Section 3, we give the proof of Theorem 3. We divide the proof into two parts. In Subsection 3.1, we prove that if $d''(\omega)$ is negative definite, then there exists an unstable direction ψ. In Subsection 3.2, we prove the instability of $T(\omega t)\phi_\omega$ using the variational characterization of ϕ_ω and the unstable direction ψ.

§2. Variational characterization

In this section, we give a variational characterization of ϕ_ω. Although ϕ_ω is given by (1.3) and (1.4) explicitly, we need such a variational characterization to prove stability and instability of solitary wave solutions $T(\omega t)\phi_\omega$.

Throughout this section, we assume that $b > 0$. The case $b = 0$ is studied in Section 3 of [2], and the proof for the case $b > 0$ is almost the same as that for $b = 0$, so we will omit the details.

For $\omega \in \Omega$, we define

$$L_\omega(v) = \|\partial_x v\|_{L^2}^2 + \omega_0 \|v\|_{L^2}^2 + \omega_1 (i\partial_x v, v)_{L^2},$$
$$S_\omega(v) = \frac{1}{2} L_\omega(v) - \frac{1}{4}(i|v|^2 \partial_x v, v)_{L^2} - \frac{b}{6}\|v\|_{L^6}^6,$$
$$K_\omega(v) = L_\omega(v) - (i|v|^2 \partial_x v, v)_{L^2} - b\|v\|_{L^6}^6,$$

and consider the following minimization problem:

$$\mu(\omega) = \inf\{S_\omega(v) : v \in H^1(\mathbb{R}) \setminus \{0\},\ K_\omega(v) = 0\}. \tag{2.1}$$

Note that (1.5) is equivalent to $S_\omega'(\phi) = 0$ and that $K_\omega(v) = \partial_\lambda S_\omega(\lambda v)|_{\lambda=1}$.

We also define

$$\tilde{S}_\omega(v) = S_\omega(v) - \frac{1}{4} K_\omega(v) = \frac{1}{4} L_\omega(v) + \frac{b}{12}\|v\|_{L^6}^6.$$

Lemma 2. *Let $\omega \in \Omega$.*

(1) *There exists a constant $C_1 = C_1(\omega) > 0$ such that $L_\omega(v) \ge C_1 \|v\|_{H^1}^2$ for all $v \in H^1(\mathbb{R})$.*

(2) *$\mu(\omega) > 0$.*

(3) *If $v \in H^1(\mathbb{R})$ satisfies $K_\omega(v) < 0$, then $\mu(\omega) < \tilde{S}_\omega(v)$.*

Proof. (1) See Lemma 7 (1) of [2].

(2) Let $v \in H^1(\mathbb{R}) \setminus \{0\}$ satisfy $K_\omega(v) = 0$. Then, by (1) and the Sobolev inequality, there exists $C_2 > 0$ such that

$$C_1 \|v\|_{H^1}^2 \le L_\omega(v) = (i|v|^2 \partial_x v, v)_{L^2} + b\|v\|_{L^6}^6$$
$$\le \|\partial_x v\|_{L^2} \|v\|_{L^6}^3 + b\|v\|_{L^6}^6 \le \frac{C_1}{2}\|v\|_{H^1}^2 + C_2 \|v\|_{H^1}^6.$$

Since $v \ne 0$, we have $\|v\|_{H^1}^4 \ge \frac{C_1}{2C_2}$. Thus, we have

$$\mu(\omega) = \inf\{\tilde{S}_\omega(v) : v \in H^1(\mathbb{R}) \setminus \{0\},\ K_\omega(v) = 0\}$$
$$\ge \frac{1}{4} \inf\{L_\omega(v) : v \in H^1(\mathbb{R}) \setminus \{0\},\ K_\omega(v) = 0\} \ge \frac{C_1}{4}\sqrt{\frac{C_1}{2C_2}} > 0.$$

(3) Let $v \in H^1(\mathbb{R}) \setminus \{0\}$ satisfy $K_\omega(v) < 0$. Then, there exists $\lambda_1 \in (0,1)$ such that

$$K_\omega(\lambda_1 v) = \lambda_1^2 L_\omega(v) - \lambda_1^4 (i|v|^2 \partial_x v, v)_{L^2} - \lambda_1^6 b \|v\|_{L^6}^6 = 0.$$

Since $v \ne 0$, we have

$$\mu(\omega) \le \tilde{S}_\omega(\lambda_1 v) = \frac{\lambda_1^2}{4} L_\omega(v) + \frac{\lambda_1^6 b}{12}\|v\|_{L^6}^6 < \tilde{S}_\omega(v).$$

This completes the proof. □

Let $\mathcal{M}_\omega$ be the set of all minimizers for (2.1), i.e.,

$$\mathcal{M}_\omega = \{\varphi \in H^1(\mathbb{R}) \setminus \{0\} : S_\omega(\varphi) = \mu(\omega),\ K_\omega(\varphi) = 0\}.$$

Then, we obtain the following.

Lemma 3. *For any $\omega \in \Omega$, we have $\mathcal{M}_\omega = \{T(\theta)\phi_\omega : \theta \in \mathbb{R}^2\}$. In particular, if $v \in H^1(\mathbb{R})$ satisfies $K_\omega(v) = 0$ and $v \neq 0$, then $S_\omega(\phi_\omega) \leq S_\omega(v)$.*

The proof of Lemma 3 is almost the same as that of Lemma 10 of [2], so we omit it.

The following lemma plays an important role in the proof of Lemma 12.

Lemma 4. *If $v \in H^1(\mathbb{R})$ satisfies $\langle K'_\omega(\phi_\omega), v\rangle = 0$, then $\langle S''_\omega(\phi_\omega)v, v\rangle \geq 0$.*

Proof. Let $v \in H^1(\mathbb{R})$ satisfy $\langle K'_\omega(\phi_\omega), v\rangle = 0$. Since $K_\omega(\phi_\omega) = 0$ and $\langle K'_\omega(\phi_\omega), \phi_\omega\rangle \neq 0$, by the implicit function theorem, there exist a constant $\delta > 0$ and a C^2-function $\gamma : (-\delta, \delta) \to \mathbb{R}$ such that $\gamma(0) = 0$ and

$$K_\omega(\phi_\omega + sv + \gamma(s)\phi_\omega) = 0, \quad s \in (-\delta, \delta). \tag{2.2}$$

Taking δ smaller if necessary, we also have $\phi_\omega + sv + \gamma(s)\phi_\omega \neq 0$ for $s \in (-\delta, \delta)$.

Differentiating (2.2) at $s = 0$, we have

$$0 = \langle K'_\omega(\phi_\omega), v\rangle + \gamma'(0)\langle K'_\omega(\phi_\omega), \phi_\omega\rangle.$$

Since $\langle K'_\omega(\phi_\omega), v\rangle = 0$ and $\langle K'_\omega(\phi_\omega), \phi_\omega\rangle \neq 0$, we have $\gamma'(0) = 0$.

Moreover, since $\phi_\omega \in \mathcal{M}_\omega$ by Lemma 3, it follows from (2.2) that the function $s \mapsto S_\omega(\phi_\omega + sv + \gamma(s)\phi_\omega)$ has a local minimum at $s = 0$. Thus, we have

$$\begin{aligned}
0 &\leq \frac{d^2}{ds^2} S_\omega(\phi_\omega + sv + \gamma(s)\phi_\omega)\big|_{s=0} \\
&= \langle S''_\omega(\phi_\omega)(v + \gamma'(0)\phi_\omega), v + \gamma'(0)\phi_\omega\rangle + \langle S'_\omega(\phi_\omega), \gamma''(0)\phi_\omega\rangle \\
&= \langle S''_\omega(\phi_\omega)v, v\rangle.
\end{aligned}$$

This completes the proof. □

§3. Proof of Theorem 3

In this section, we give the proof of Theorem 3. We divide the proof into two parts. In Subsection 3.1, we prove that if $d''(\omega)$ is negative definite, then there exists an unstable direction ψ (see Lemma 6). In Subsection 3.2, we prove the instability of $T(\omega t)\phi_\omega$ using the variational characterization of ϕ_ω and the unstable direction ψ (see Proposition 1). Theorem 3 follows from Lemma 6 and Proposition 1.

3.1. Existence of unstable direction

Lemma 5. $\langle S''_\omega(\phi_\omega)\phi_\omega, \phi_\omega\rangle < 0$.

Proof. Since the function

$$(0,\infty) \ni \lambda \mapsto S_\omega(\lambda\phi_\omega) = \frac{\lambda^2}{2}L_\omega(\phi_\omega) - \frac{\lambda^4}{4}(i|\phi_\omega|^2\partial_x\phi_\omega, \phi_\omega)_{L^2} - \frac{\lambda^6\, b}{6}\|\phi_\omega\|_{L^6}^6$$

has a strictly local maximum at $\lambda = 1$, we have

$$0 > \frac{d^2}{d\lambda^2}S_\omega(\lambda\phi_\omega)\big|_{\lambda=1} = \langle S''_\omega(\phi_\omega)\phi_\omega, \phi_\omega\rangle.$$

This completes the proof. □

Lemma 6. *Assume that $d''(\hat\omega)$ is negative definite. Then there exists $\psi \in H^1(\mathbb{R})$ such that*

$$\langle Q'_0(\phi_{\hat\omega}), \psi\rangle = \langle Q'_1(\phi_{\hat\omega}), \psi\rangle = 0, \quad \langle S''_{\hat\omega}(\phi_{\hat\omega})\psi, \psi\rangle < 0.$$

Proof. For (s,ω) near $(0,\hat\omega)$ in $\mathbb{R}\times\Omega$, we define

$$F(s,\omega) := \begin{bmatrix} Q_0(s\phi_{\hat\omega} + \phi_\omega) - Q_0(\phi_{\hat\omega}) \\ Q_1(s\phi_{\hat\omega} + \phi_\omega) - Q_1(\phi_{\hat\omega}) \end{bmatrix}.$$

Then, we have $F(0,\hat\omega) = 0$. Moreover, since $D_\omega F(0,\hat\omega) = d''(\hat\omega)$ is negative definite and invertible, by the implicit function theorem, there exist a constant $\delta > 0$ and a C^1-function $\gamma : (-\delta,\delta) \to \Omega$ such that $\gamma(0) = \hat\omega$ and

$$Q_0(s\phi_{\hat\omega} + \phi_{\gamma(s)}) = Q_0(\phi_{\hat\omega}), \quad Q_1(s\phi_{\hat\omega} + \phi_{\gamma(s)}) = Q_1(\phi_{\hat\omega})$$

for $s \in (-\delta,\delta)$. We define $\varphi_s := s\phi_{\hat\omega} + \phi_{\gamma(s)}$ for $s \in (-\delta,\delta)$, and

$$w_j := \partial_{\omega_j}\phi_\omega|_{\omega=\hat\omega} \quad (j = 0,1), \quad \psi := \partial_s\varphi_s|_{s=0} = \phi_{\hat\omega} + \sum_{j=0}^{1}\gamma'_j(0)w_j.$$

Then, for $j = 0, 1$, we have

$$(3.1) \qquad 0 = \frac{d}{ds}Q_j(\varphi_s)|_{s=0} = \langle Q'_j(\phi_{\hat\omega}), \psi\rangle = \langle Q'_j(\phi_{\hat\omega}), \phi_{\hat\omega}\rangle + \sum_{k=0}^{1}\gamma'_k(0)\langle Q'_j(\phi_{\hat\omega}), w_k\rangle.$$

Moreover, differentiating

$$0 = S'_\omega(\phi_\omega) = E'(\phi_\omega) + \sum_{k=0}^{1}\omega_k Q'_k(\phi_\omega),$$

with respect to ω_j for $j=0,1$, we have

$$0 = E''(\phi_\omega)(\partial_{\omega_j}\phi_\omega) + \sum_{k=0}^{1} \omega_k Q_k''(\phi_\omega)(\partial_{\omega_j}\phi_\omega) + Q_j'(\phi_\omega) \tag{3.2}$$
$$= S_\omega''(\phi_\omega)(\partial_{\omega_j}\phi_\omega) + Q_j'(\phi_\omega).$$

By (3.1) and (3.2), we have

$$\begin{aligned}
\langle S_{\hat\omega}''(\phi_{\hat\omega})\psi,\psi\rangle &= \langle S_{\hat\omega}''(\phi_{\hat\omega})\phi_{\hat\omega},\phi_{\hat\omega}\rangle + 2\sum_{j=0}^{1}\gamma_j'(0)\langle S_{\hat\omega}''(\phi_{\hat\omega})w_j,\phi_{\hat\omega}\rangle \\
&\qquad + \sum_{j,k=0}^{1}\gamma_j'(0)\gamma_k'(0)\langle S_{\hat\omega}''(\phi_{\hat\omega})w_j,w_k\rangle \\
&= \langle S_{\hat\omega}''(\phi_{\hat\omega})\phi_{\hat\omega},\phi_{\hat\omega}\rangle - 2\sum_{j=0}^{1}\gamma_j'(0)\langle Q_j'(\phi_{\hat\omega}),\phi_{\hat\omega}\rangle - \sum_{j,k=0}^{1}\gamma_j'(0)\gamma_k'(0)\langle Q_j'(\phi_{\hat\omega}),w_k\rangle \\
&= \langle S_{\hat\omega}''(\phi_{\hat\omega})\phi_{\hat\omega},\phi_{\hat\omega}\rangle + \sum_{j,k=0}^{1}\gamma_j'(0)\gamma_k'(0)\langle Q_j'(\phi_{\hat\omega}),w_k\rangle \\
&= \langle S_{\hat\omega}''(\phi_{\hat\omega})\phi_{\hat\omega},\phi_{\hat\omega}\rangle + \sum_{j,k=0}^{1}\gamma_j'(0)\gamma_k'(0)\,\partial_{\omega_j}\partial_{\omega_k}d(\hat\omega).
\end{aligned}$$

Since $d''(\hat\omega)$ is negative definite, it follows from Lemma 5 that

$$\langle S_{\hat\omega}''(\phi_{\hat\omega})\psi,\psi\rangle \le \langle S_{\hat\omega}''(\phi_{\hat\omega})\phi_{\hat\omega},\phi_{\hat\omega}\rangle < 0.$$

This completes the proof. □

3.2. Proof of instability

In this subsection, we prove the following.

Proposition 1. *Let $\omega\in\Omega$, and assume that there exists $\psi\in H^1(\mathbb{R})$ such that*

$$\langle Q_0'(\phi_\omega),\psi\rangle = \langle Q_1'(\phi_\omega),\psi\rangle = 0, \quad \langle S_\omega''(\phi_\omega)\psi,\psi\rangle < 0. \tag{3.3}$$

Then, the solitary wave solution $T(\omega t)\phi_\omega$ of (1.1) *is unstable.*

To prove Proposition 1, we use the argument of Gonçalves Ribeiro [3] (see also [17, 4]) with some modifications. Throughout this subsection, we fix $\omega\in\Omega$, and assume that $\psi\in H^1(\mathbb{R})$ satisfies (3.3).

Lemma 7. *There exists a constant* $\lambda_0 > 0$ *such that*

$$S_\omega(\phi_\omega + \lambda\psi) < S_\omega(\phi_\omega)$$

for all $\lambda \in (-\lambda_0, 0) \cup (0, \lambda_0)$.

Proof. By Taylor's expansion, for $\lambda \in \mathbb{R}$, we have

$$\begin{aligned} &S_\omega(\phi_\omega + \lambda\psi) \\ &= S_\omega(\phi_\omega) + \lambda\langle S'_\omega(\phi_\omega), \psi\rangle + \lambda^2 \int_0^1 (1-s)\langle S''_\omega(\phi_\omega + s\lambda\psi)\psi, \psi\rangle\, ds \\ &= S_\omega(\phi_\omega) + \lambda^2 \int_0^1 (1-s)\langle S''_\omega(\phi_\omega + s\lambda\psi)\psi, \psi\rangle\, ds. \end{aligned}$$

Since $\langle S''_\omega(\phi_\omega)\psi, \psi\rangle < 0$, by the continuity of $\lambda \mapsto \langle S''_\omega(\phi_\omega + \lambda\psi)\psi, \psi\rangle$, there exists $\lambda_0 > 0$ such that

$$\langle S''_\omega(\phi_\omega + \lambda\psi)\psi, \psi\rangle \le \frac{1}{2}\langle S''_\omega(\phi_\omega)\psi, \psi\rangle$$

for all $\lambda \in (-\lambda_0, \lambda_0)$. Thus, for $\lambda \in (-\lambda_0, 0) \cup (0, \lambda_0)$, we have

$$S_\omega(\phi_\omega + \lambda\psi) \le S_\omega(\phi_\omega) + \frac{\lambda^2}{4}\langle S''_\omega(\phi_\omega)\psi, \psi\rangle < S_\omega(\phi_\omega).$$

This completes the proof. □

For $u \in H^1(\mathbb{R})$, we define

$$T'_0 u = iu, \quad T'_1 u = -\partial_x u.$$

Then, by (1.8) and (1.10), we have

$$\partial_{\theta_j} T(\theta)u = T(\theta)T'_j u = T'_j T(\theta)u, \quad \langle Q'_j(u), v\rangle = (T'_j u, iv)_{L^2} \tag{3.4}$$

for $\theta = (\theta_0, \theta_1) \in \mathbb{R}^2$, u, $v \in H^1(\mathbb{R})$ and $j = 0, 1$. We denote $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.

Lemma 8. *There exist a constant* $\varepsilon_0 > 0$ *and a* C^1*-function*

$$\alpha = (\alpha_0, \alpha_1) : U_{\varepsilon_0}(\phi_\omega) \to \mathbb{T} \times \mathbb{R}$$

such that $\alpha(\phi_\omega) = 0$, *and*

(1) $\alpha(T(\xi)u) = \alpha(u) + \xi$ *for all* $u \in U_{\varepsilon_0}(\phi_\omega)$ *and* $\xi \in \mathbb{T} \times \mathbb{R}$.

(2) $(T'_j u, T(\alpha(u))\phi_\omega)_{L^2} = 0$ *for all* $u \in U_{\varepsilon_0}(\phi_\omega)$ *and* $j = 0$, 1.

(3) *There exists* $\rho > 0$ *such that*

$$\sum_{j,k=0}^{1} (T_j' u, T(\alpha(u)) T_k' \phi_\omega)_{L^2} \zeta_j \zeta_k \ge \rho |\zeta|^2$$

for all $u \in U_{\varepsilon_0}(\phi_\omega)$ *and* $\zeta = (\zeta_0, \zeta_1) \in \mathbb{R}^2$.

Proof. See Section 3 of [3]. □

For $u \in U_{\varepsilon_0}(\phi_\omega)$, we define

$$H(u) = [h_{jk}(u)]_{j,k=0,1}, \quad h_{jk}(u) = (T_j' u, T(\alpha(u)) T_k' \phi_\omega)_{L^2}.$$

Then, by Lemma 8 (1), we have

$$h_{jk}(T(\xi)u) = (T(\xi) T_j' u, T(\alpha(u) + \xi) T_k' \phi_\omega)_{L^2} = h_{jk}(u) \tag{3.5}$$

for $u \in U_{\varepsilon_0}(\phi_\omega)$ and $\xi \in \mathbb{T} \times \mathbb{R}$.

Moreover, differentiating Lemma 8 (2) with respect to u, we have

$$\sum_{k=0}^{1} h_{jk}(u) \langle \alpha_k'(u), w \rangle = (T(\alpha(u)) T_j' \phi_\omega, w)_{L^2} \tag{3.6}$$

for $u \in U_{\varepsilon_0}(\phi_\omega)$, $w \in H^1(\mathbb{R})$ and $j = 0, 1$. By Lemma 8 (3), the matrix $H(u)$ is invertible, and we denote the inverse $H(u)^{-1}$ by $G(u) = [g_{jk}(u)]$. Then, there exists a constant $C > 0$ such that

$$|g_{jk}(u)| \le C \text{ for all } u \in U_{\varepsilon_0}(\phi_\omega),\ j,k = 0,1. \tag{3.7}$$

For $j = 0, 1$ and $u \in U_{\varepsilon_0}(\phi_\omega)$, we define

$$a_j(u) := \sum_{k=0}^{1} g_{jk}(u) T(\alpha(u)) T_k' \phi_\omega.$$

Since $\phi_\omega \in H^2(\mathbb{R})$, we see that $a_j(u) \in H^1(\mathbb{R})$, it follows from (3.6) that

$$\langle \alpha_j'(u), w \rangle = (a_j(u), w)_{L^2}, \quad w \in H^1(\mathbb{R}).$$

By (3.5) and Lemma 8 (1), for $j = 0, 1$, we have

$$a_j(T(\xi)u) = T(\xi) a_j(u) \text{ for all } u \in U_{\varepsilon_0}(\phi_\omega),\ \xi \in \mathbb{T} \times \mathbb{R}. \tag{3.8}$$

Moreover, by (3.7), there exists a constant $C > 0$ such that

$$\|a_j(u)\|_{H^1} \le C \text{ for all } u \in U_{\varepsilon_0}(\phi_\omega),\ j = 0,1. \tag{3.9}$$

Next, for $u \in U_{\varepsilon_0}(\phi_\omega)$, we define

$$A(u) = (iu, T(\alpha(u))\psi)_{L^2}, \tag{3.10}$$

$$q(u) = T(\alpha(u))\psi + \sum_{j=0}^{1} \left(iu, T(\alpha(u))T_j'\psi\right)_{L^2} ia_j(u). \tag{3.11}$$

Then, since ψ, $a_0(u)$, $a_1(u) \in H^1(\mathbb{R})$, we see that $q(u) \in H^1(\mathbb{R})$.

Lemma 9. *For $u \in U_{\varepsilon_0}(\phi_\omega)$,*

(1) $A(T(\xi)u) = A(u)$, $q(T(\xi)u) = T(\xi)q(u)$ *for all* $\xi \in \mathbb{T} \times \mathbb{R}$.

(2) $\langle A'(u), w\rangle = (q(u), iw)_{L^2}$ *for* $w \in H^1(\mathbb{R})$.

(3) $q(\phi_\omega) = \psi$.

(4) $\langle Q_j'(u), q(u)\rangle = 0$ *for* $j = 0, 1$.

Proof. (1) By Lemma 8 (1), we have

$$\begin{aligned} A(T(\xi)u) &= (iT(\xi)u, T(\alpha(u)+\xi)\psi)_{L^2} \\ &= (iT(\xi)u, T(\xi)T(\alpha(u))\psi)_{L^2} = A(u). \end{aligned}$$

Moreover, by (3.8), we have

$$\begin{aligned} q(T(\xi)u) &= T(\xi)T(\alpha(u))\psi + \sum_{j=0}^{1} \left(iT(\xi)u, T(\xi)T(\alpha(u))T_j'\psi\right)_{L^2} ia_j(T(\xi)u) \\ &= T(\xi)q(u). \end{aligned}$$

(2) For $u \in U_{\varepsilon_0}(\phi_\omega)$ and $w \in H^1(\mathbb{R})$, we have

$$\begin{aligned} \langle A'(u), w\rangle &= (iw, T(\alpha(u))\psi)_{L^2} + \sum_{j=0}^{1} \langle \alpha_j'(u), w\rangle \left(iu, T(\alpha(u))T_j'\psi\right)_{L^2} \\ &= (iw, T(\alpha(u))\psi)_{L^2} + \sum_{j=0}^{1} \left(iu, T(\alpha(u))T_j'\psi\right)_{L^2} (a_j(u), w)_{L^2} \\ &= (q(u), iw)_{L^2}. \end{aligned}$$

(3) By (3.4) and the assumption (3.3), we have

$$(i\phi_\omega, T_j'\psi)_{L^2} = (T_j'\phi_\omega, i\psi)_{L^2} = \langle Q_j'(\phi_\omega), \psi\rangle = 0.$$

Moreover, since $\alpha(\phi_\omega) = 0$, by (3.11), we have $q(\phi_\omega) = \psi$.

(4) For $u \in H^2(\mathbb{R}) \cap U_{\varepsilon_0}(\phi_\omega)$, by (1) and (2), we have

$$0 = \partial_{\xi_j} A(T(\xi)u)\big|_{\xi=0} = \langle A'(u), T_j' u\rangle = (q(u), iT_j' u)_{L^2}.$$

By density argument, we have $(q(u), iT_j' u)_{L^2} = 0$ for all $u \in U_{\varepsilon_0}(\phi_\omega)$.

Thus, we have $\langle Q_j'(u), q(u)\rangle = (T_j' u, iq(u))_{L^2} = 0$ for $u \in U_{\varepsilon_0}(\phi_\omega)$. □

For $u \in U_{\varepsilon_0}(\phi_\omega)$, we define

$$P(u) := \langle E'(u), q(u)\rangle.$$

We remark that by Lemma 9 (4), we have

$$P(u) = \langle S_\omega'(u), q(u)\rangle, \quad u \in U_{\varepsilon_0}(\phi_\omega). \tag{3.12}$$

Lemma 10. *Let I be an interval of $\mathbb{R}$. Let $u \in C(I, H^1(\mathbb{R})) \cap C^1(I, H^{-1}(\mathbb{R}))$ be a solution of* (1.1)*, and assume that $u(t) \in U_{\varepsilon_0}(\phi_\omega)$ for all $t \in I$. Then,*

$$\frac{d}{dt} A(u(t)) = P(u(t))$$

for all $t \in I$.

Proof. By Lemma 4.6 of [4] and Lemma 9 (2), we see that $t \mapsto A(u(t))$ is a C^1-function on I, and

$$\frac{d}{dt} A(u(t)) = \langle i\partial_t u(t), q(u(t))\rangle$$

for all $t \in I$. Since $u(t)$ is a solution of (1.1), we have

$$\langle i\partial_t u(t), q(u(t))\rangle = \langle E'(u(t)), q(u(t))\rangle = P(u(t))$$

for all $t \in I$. This completes the proof. □

Lemma 11. *There exist constants $\lambda_1 > 0$ and $\varepsilon_1 \in (0, \varepsilon_0)$ such that*

$$S_\omega(u + \lambda q(u)) \le S_\omega(u) + \lambda P(u)$$

for all $\lambda \in (-\lambda_1, \lambda_1)$ and $u \in U_{\varepsilon_1}(\phi_\omega)$.

Proof. For $u \in U_{\varepsilon_0}(\phi_\omega)$ and $\lambda \in \mathbb{R}$, by Taylor's expansion, we have

$$S_\omega(u + \lambda q(u)) = S_\omega(u) + \lambda P(u) + \lambda^2 \int_0^1 (1-s) R(\lambda s, u)\, ds, \tag{3.13}$$

where we used (3.12) and put

$$R(\lambda, u) := \langle S_\omega''(u + \lambda q(u)) q(u), q(u)\rangle.$$

Here, we remark that

$$P(T(\xi)u) = \langle S_\omega'(T(\xi)u), T(\xi)q(u)\rangle = P(u),$$
$$R(\lambda, T(\xi)u) = \langle S_\omega''(T(\xi)(u+\lambda q(u)))T(\xi)q(u), T(\xi)q(u)\rangle = R(\lambda, u)$$

for $\xi \in \mathbb{T}\times\mathbb{R}$, $\lambda \in \mathbb{R}$ and $u \in H^1(\mathbb{R})$. Moreover, since

$$R(0,\phi_\omega) = \langle S_\omega''(\phi_\omega)q(\phi_\omega), q(\phi_\omega)\rangle = \langle S_\omega''(\phi_\omega)\psi, \psi\rangle < 0,$$

by the continuity of $R(\lambda, u)$ with respect to λ and u, there exist constants $\lambda_1 > 0$ and $\varepsilon_1 \in (0,\varepsilon_0)$ such that $R(\lambda, u) < 0$ for all $\lambda \in (-\lambda_1, \lambda_1)$ and $u \in U_{\varepsilon_1}(\phi_\omega)$. Thus, by (3.13), we have

$$S_\omega(u+\lambda q(u)) \le S_\omega(u) + \lambda P(u)$$

for all $\lambda \in (-\lambda_1,\lambda_1)$ and $u \in U_{\varepsilon_1}(\phi_\omega)$. □

Lemma 12. *There exist constants $\varepsilon_2 \in (0,\varepsilon_1)$ and $\lambda_2 \in (0,\lambda_1)$ that satisfy the following. For any $u \in U_{\varepsilon_2}(\phi_\omega)$, there exists $\Lambda(u) \in (-\lambda_2,\lambda_2)$ such that*

$$K_\omega\left(u+\Lambda(u)q(u)\right) = 0, \quad u+\Lambda(u)q(u) \neq 0.$$

Proof. First, since $\langle S_\omega''(\phi_\omega)\psi,\psi\rangle < 0$, by Lemma 4, we have $\langle K_\omega'(\phi_\omega), \psi\rangle \neq 0$. Thus, without loss of generality, we may assume that $\langle K_\omega'(\phi_\omega),\psi\rangle > 0$.

For $u \in U_{\varepsilon_0}(\phi_\omega)$ and $\lambda \in \mathbb{R}$, we have

$$K_\omega(u+\lambda q(u)) = K_\omega(u) + \lambda \int_0^1 \langle K_\omega'(u+s\lambda q(u)), q(u)\rangle\, ds. \tag{3.14}$$

Since $\langle K_\omega'(\phi_\omega), q(\phi_\omega)\rangle = \langle K_\omega'(\phi_\omega),\psi\rangle > 0$, by the continuity of the function $\langle K_\omega'(u+\lambda q(u)), q(u)\rangle$ with respect to λ and u, there exist constants $\lambda_2 \in (0,\lambda_1)$ and $\varepsilon_2 \in (0,\varepsilon_1)$ such that

$$\langle K_\omega'(u+\lambda q(u)), q(u)\rangle \ge \frac{1}{2}\langle K_\omega'(\phi_\omega),\psi\rangle \tag{3.15}$$

for all $\lambda \in [-\lambda_2,\lambda_2]$ and $u \in U_{\varepsilon_2}(\phi_\omega)$. Moreover, since $K_\omega(\phi_\omega) = 0$, taking ε_2 smaller if necessary, we have

$$|K_\omega(u)| < \frac{\lambda_2}{2}\langle K_\omega'(\phi_\omega),\psi\rangle, \quad u \in U_{\varepsilon_2}(\phi_\omega). \tag{3.16}$$

Let $u \in U_{\varepsilon_2}(\phi_\omega)$. If $K_\omega(u) < 0$, then it follows from (3.14)–(3.16) that

$$\begin{aligned} K_\omega(u+\lambda_2 q(u)) &= K_\omega(u) + \lambda_2 \int_0^1 \langle K_\omega'(u+s\lambda_2 q(u)), q(u)\rangle\, ds \\ &> -\frac{\lambda_2}{2}\langle K_\omega'(\phi_\omega),\psi\rangle + \frac{\lambda_2}{2}\langle K_\omega'(\phi_\omega),\psi\rangle = 0. \end{aligned}$$

Since the function $\lambda \mapsto K_\omega(u+\lambda q(u))$ is continuous, there exists $\Lambda(u) \in (0, \lambda_2)$ such that

$$K_\omega(u + \Lambda(u)q(u)) = 0. \tag{3.17}$$

Similarly, if $K_\omega(u) > 0$, then we have

$$\begin{aligned} K_\omega(u - \lambda_2 q(u)) &= K_\omega(u) - \lambda_2 \int_0^1 \langle K_\omega'(u - s\lambda_2 q(u)), q(u)\rangle \, ds \\ &< \frac{\lambda_2}{2}\langle K_\omega'(\phi_\omega), \psi\rangle - \frac{\lambda_2}{2}\langle K_\omega'(\phi_\omega), \psi\rangle = 0. \end{aligned}$$

Thus, there exists $\Lambda(u) \in (-\lambda_2, 0)$ such that (3.17). If $K_\omega(u) = 0$, taking $\Lambda(u) = 0$, (3.17) is satisfied.

Finally, by (3.9) and (3.11), taking λ_2 and ε_2 smaller if necessary, we have $u + \Lambda(u)q(u) \neq 0$ for all $u \in U_{\varepsilon_2}(\phi_\omega)$. This completes the proof. □

Lemma 13. *Let λ_2 and ε_2 be the positive constants given in Lemma 12. Then,*

$$S_\omega(\phi_\omega) \leq S_\omega(u) + \lambda_2 |P(u)|$$

for all $u \in U_{\varepsilon_2}(\phi_\omega)$.

Proof. By Lemma 12, for any $u \in U_{\varepsilon_2}(\phi_\omega)$, there exists $\Lambda(u) \in (-\lambda_2, \lambda_2)$ such that $K_\omega(u + \Lambda(u)q(u)) = 0$ and $u + \Lambda(u)q(u) \neq 0$. Then, it follows from Lemma 3 that

$$S_\omega(\phi_\omega) \leq S_\omega(u + \Lambda(u)q(u)), \quad u \in U_{\varepsilon_2}(\phi_\omega). \tag{3.18}$$

Thus, by Lemma 11 and (3.18), for $u \in U_{\varepsilon_2}(\phi_\omega)$, we have

$$\begin{aligned} S_\omega(\phi_\omega) &\leq S_\omega\left(u + \Lambda(u)q(u)\right) \leq S_\omega(u) + \Lambda(u)P(u) \\ &\leq S_\omega(u) + |\Lambda(u)||P(u)| \leq S_\omega(u) + \lambda_2 |P(u)|. \end{aligned}$$

This completes the proof. □

We are now in a position to give the Proof of Proposition 1.

Proof of Proposition 1. Suppose that $T(\omega t)\phi_\omega$ is stable. For λ close to 0, let $u_\lambda(t)$ be the solution of (1.1) with $u_\lambda(0) = \phi_\omega + \lambda\psi$. Since $T(\omega t)\phi_\omega$ is stable, there exists $\lambda_3 \in (0, \lambda_0)$ such that if $|\lambda| < \lambda_3$, then $u_\lambda(t) \in U_{\varepsilon_2}(\phi_\omega)$ for all $t \geq 0$. Moreover, by the definition (3.10) of A, there exists $C_1 > 0$ such that $|A(v)| \leq C_1$ for all $v \in U_{\varepsilon_2}(\phi_\omega)$.

Let $\lambda \in (-\lambda_3, 0) \cup (0, \lambda_3)$. Then, by Lemma 7, we have

$$\delta_\lambda := S_\omega(\phi_\omega) - S_\omega(u_\lambda(0)) > 0.$$

Moreover, by Lemma 13 and the conservation of S_ω, we have

$$0 < \delta_\lambda = S_\omega(\phi_\omega) - S_\omega(u_\lambda(t)) \le \lambda_2 |P(u_\lambda(t))|, \quad t \ge 0.$$

Since $t \mapsto P(u_\lambda(t))$ is continuous, we see that either (i) $P(u_\lambda(t)) \ge \delta_\lambda/\lambda_2$ for all $t \ge 0$, or (ii) $P(u_\lambda(t)) \le -\delta_\lambda/\lambda_2$ for all $t \ge 0$. Moreover, by Lemma 10, we have

$$\frac{d}{dt} A(u_\lambda(t)) = P(u_\lambda(t)), \quad t \ge 0.$$

Therefore, we see that $A(u_\lambda(t)) \to \infty$ as $t \to \infty$ for the case (i), and $A(u_\lambda(t)) \to -\infty$ as $t \to \infty$ for the case (ii). This contradicts the fact that $|A(u_\lambda(t))| \le C_1$ for all $t \ge 0$. Hence, $T(\omega t)\phi_\omega$ is unstable. □

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