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Two-factor experiment with split units constructed by a BIBRC

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Abstract. We construct a nested row-column design with split units for a two-factor experiment. We use a balanced incomplete block design with nested rows and columns (BIBRC for short) repeatedly for the whole plot treatments and we use a proper block design for the subplot treatments. We give the stratum efficiency factors for such a nested row-column design with split units, which has the general balance property.

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§1. Introduction

We consider a two-factor experiment of split-plot type with b blocks, in which the first factor A occurs at v_1 levels $A_1, A_2, \cdots, A_{v_1}$ and the second factor B occurs at v_2 levels $B_1, B_2, \cdots, B_{v_2}$. Each block is divided into k_1 rows and k_2 columns and these k_1k_2 units are treated as whole plots. Additionally, each whole plot is divided into k_3 subplots. The levels of A are arranged on the whole plots (called whole plot treatments), while the levels of B are arranged on the subplots (called subplot treatments). Such a design is called a nested row-column design with split units.

The nested row-column designs with split units are often used in biological, agricultural and environmental sciences. For the nested row-column design with split units, a mixed linear model with fixed treatment effects and random block, row, column, whole plot and subplot effects was considered by [4]. The hth factorial treatment combination effect τ_h is defined by

$$\tau_h = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij}$$

for $h = (i-1)v_2+j$, $i = 1, 2, \cdots, v_1$ and $j = 1, 2, \cdots, v_2$, where μ is the general mean, α_i denotes the main effect of the ith whole plot treatment A_i, β_j denotes the main effect of the jth subplot treatment B_j and $(\alpha\beta)_{ij}$ denotes the interaction effect of A_i and B_j. Here $\sum_{i=1}^{v_1} \alpha_i = 0$, $\sum_{j=1}^{v_2} \beta_j = 0$, $\sum_{i=1}^{v_1} (\alpha\beta)_{ij} = 0$ for $j = 1, 2, \cdots, v_2$ and $\sum_{j=1}^{v_2} (\alpha\beta)_{ij} = 0$ for $i = 1, 2, \cdots, v_1$. The mixed linear model results from a four-step randomization, i.e., the randomization of blocks, the randomization of rows within each block, the randomization of columns within each block and the randomization of subplots within each whole plot. This kind of randomization leads us to an experiment with orthogonal block structure as defined by [9] and [10] and the multistratum analysis proposed by [2], [9] and [10] can be applied to the analysis of data in the experiment. In this case, we have five strata, except zero stratum connected with the general mean only, (I) inter-block stratum, (II) inter-row stratum, (III) inter-column stratum, (IV) inter-whole plot stratum and (V) inter-subplot stratum. The statistical properties of the nested row-column design with split units are strictly connected with the eigenvalues and the eigenvectors of the stratum information matrices for the treatment combinations. The stratum information matrices $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{A}_4$ and $\mathbf{A}_5$ are given by

$$\mathbf{A}_1 = \frac{1}{k_1k_2k_3}\mathbf{N}_0\mathbf{N}_0' - \frac{1}{n}\boldsymbol{r}\boldsymbol{r}', \quad \mathbf{A}_2 = \frac{1}{k_2k_3}\mathbf{N}_1\mathbf{N}_1' - \frac{1}{k_1k_2k_3}\mathbf{N}_0\mathbf{N}_0',$$

$$\mathbf{A}_3 = \frac{1}{k_1k_3}\mathbf{N}_2\mathbf{N}_2' - \frac{1}{k_1k_2k_3}\mathbf{N}_0\mathbf{N}_0',$$

$$\mathbf{A}_4 = \frac{1}{k_3}\mathbf{N}_3\mathbf{N}_3' - \frac{1}{k_1k_3}\mathbf{N}_2\mathbf{N}_2' - \frac{1}{k_2k_3}\mathbf{N}_1\mathbf{N}_1' + \frac{1}{k_1k_2k_3}\mathbf{N}_0\mathbf{N}_0', \tag{1.1}$$

$$\mathbf{A}_5 = \boldsymbol{r}^\delta - \frac{1}{k_3}\mathbf{N}_3\mathbf{N}_3',$$

where $n = bk_1k_2k_3$, $v = v_1v_2$, $\boldsymbol{r}$ is the $v \times 1$ vector of replications of the treatment combinations, $\boldsymbol{r}^\delta$ is the diagonal matrix with the diagonal elements equal to these replications and $\mathbf{N}_0$, $\mathbf{N}_1$, $\mathbf{N}_2$ and $\mathbf{N}_3$ are the incidence matrices for the treatment combinations vs. blocks, rows, columns and whole plots, respectively. Here we assume that the treatment combinations A_iB_j ($i = 1, 2, \cdots, v_1$, $j = 1, 2, \cdots, v_2$) are ordered lexicographically.

A generally balanced design was firstly introduced by [9] and [10], for which the stratum information matrices are spanned by a common set of eigenvectors. Let ξ_{fh} be an eigenvalue of the stratum information matrix $\mathbf{A}_f$ corresponding to an eigenvector $\boldsymbol{s}_h$ with respect to $\boldsymbol{r}^\delta$, i.e.,

$$\mathbf{A}_f\boldsymbol{s}_h = \xi_{fh}\boldsymbol{r}^\delta\boldsymbol{s}_h$$

for $f = 1, 2, 3, 4, 5$ and $h = 0, 1, \cdots, v-1$, where

$$\text{(1.2)} \qquad \boldsymbol{s}_h' \boldsymbol{r}^\delta \boldsymbol{s}_{h'} = \begin{cases} 1, & h = h', \\ 0, & h \neq h'. \end{cases}$$

If (1.2) is satisfied, we say that $\boldsymbol{s}_0, \boldsymbol{s}_1, \cdots, \boldsymbol{s}_{v-1}$ are mutually $\boldsymbol{r}^\delta$-orthonormal. Since $\mathbf{A}_f \mathbf{1}_v = \mathbf{0}$ for $f = 1, 2, 3, 4, 5$, $\frac{1}{\sqrt{n}} \mathbf{1}_v$ may be chosen as the first eigenvector $\boldsymbol{s}_0$, where $\mathbf{1}_v$ is the $v \times 1$ vector of unit elements. Then, a basic contrast in the treatment effects (see [11]) is defined by $\boldsymbol{c}_h' \boldsymbol{\tau}$ for $h = 1, 2, \cdots, v-1$, where $\boldsymbol{c}_h = \boldsymbol{r}^\delta \boldsymbol{s}_h$ and $\boldsymbol{\tau}$ is the $v \times 1$ vector of the treatment effects. The eigenvalue ξ_{fh} can be identified as a stratum efficiency factor of the design concerning estimation of the hth basic contrast in the fth stratum for $f = 1, 2, 3, 4, 5$ and $h = 1, 2, \cdots, v-1$ (see [2]).

[5] and [6] constructed nested row-column designs with split units, using a Youden square repeatedly for the whole plot treatments, and they gave the stratum efficiency factors for such nested row-column designs. In this paper, we use a BIBRC in stead of a Youden square to construct a nested row-column design with split units and we give the stratum efficiency factors for such a nested row-column design, which has the general balance property. These designs constructed here are useful in practice, since they have smaller block sizes than the designs constructed by a Youden square.

§2. A BIBRC

Let V be a set of v treatments and let $\mathcal{B}$ be a collection of b subsets (called blocks) of V. A design $(V, \mathcal{B})$ is called a balanced incomplete block design and it is denoted by BIBD(v, b, r, k, λ), if each block contains k treatments, every treatment occurs in precisely r blocks and every pair of distinct treatments occurs in precisely λ blocks.

A BIBRC was firstly introduced by [12] and many authors (for example, [1], [3] and [8]) gave the construction methods of BIBRCs. Let $(V, \mathcal{B})$ be a design with v treatments and b blocks, where each block is divided into p rows and q columns. Further define $\lambda_R(i, j)$ to be the number of rows in blocks in which a pair of distinct treatments i and j occurs together. Also $\lambda_C(i, j)$ and $\lambda_B(i, j)$ are defined similarly for columns and blocks. The design $(V, \mathcal{B})$ is called a BIBRC if the following conditions are satisfied:

(i) every treatment occurs at most once in each block (said to be binary),
(ii) every treatment occurs in precisely r blocks,
(iii) for any pair of distinct treatments i and j,

$$\lambda = p\lambda_R(i, j) + q\lambda_C(i, j) - \lambda_B(i, j)$$

is a constant independent of the treatments i and j.

Furthermore if the following condition (iv) is satisfied, the BIBRC is said to be completely balanced (see [7]).

(iv) for any pair of distinct treatments i and j, $\lambda_R(i,j)$, $\lambda_C(i,j)$ and $\lambda_B(i,j)$ are constants independent of the treatments i and j, simply denoted by λ_R, λ_C and λ_B, respectively.

The condition (iv) implies that $(V, \mathcal{B})$ is a BIBD(v, b, r, pq, λ_B), $(V, \mathcal{B}_R)$ is a BIBD$(v,\ bp, r, q, \lambda_R)$ and $(V, \mathcal{B}_C)$ is a BIBD(v, bq, r, p, λ_C), where $\mathcal{B}_R$ and $\mathcal{B}_C$ denote the collections of bp rows and bq columns of $\mathcal{B}$. Hereafter we consider a completely balanced BIBRC and we denote it by BIBRC$(v, b, r, p, q, \lambda_B, \lambda_R, \lambda_C)$.

Example 2.1. Let $V = \{1, 2, 3, 4, 5, 6, 7\}$ and let $\mathcal{B}$ be a collection of the following blocks:

$$\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 6&5&3\\\hline\end{array}\quad\begin{array}{|c|c|c|}\hline 2&3&5\\\hline 7&6&4\\\hline\end{array}\quad\begin{array}{|c|c|c|}\hline 3&4&6\\\hline 1&7&5\\\hline\end{array}\quad\begin{array}{|c|c|c|}\hline 4&5&7\\\hline 2&1&6\\\hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline 5&6&1\\\hline 3&2&7\\\hline\end{array}\quad\begin{array}{|c|c|c|}\hline 6&7&2\\\hline 4&3&1\\\hline\end{array}\quad\begin{array}{|c|c|c|}\hline 7&1&3\\\hline 5&4&2\\\hline\end{array}$$

Then $(V, \mathcal{B})$ is a BIBRC(7,7,6,2,3,5,2,1). □

§3. A nested row-column design with split units constructed by a BIBRC

Now we construct a nested row-column design with split units. The whole plot treatments occur in a BIBRC$(v_A, b_A, r_A, p, q, \lambda_B, \lambda_R, \lambda_C)$ $(V, \mathcal{B})$ and the subplot treatments occur in a block design D_B with v_B treatments, b_B blocks, k_B treatments in each block and the vector of replications of the treatments $\boldsymbol{r}_B$, where D_B may not be binary. We construct a nested row-column design $\mathcal{D}$ with split units using the BIBRC repeatedly embedding each block of D_B in all whole plots of the BIBRC. The parameters of $\mathcal{D}$ are $v_1 = v_A$, $v_2 = v_B$, $b = b_A b_B$, $k_1 = p$, $k_2 = q$ and $k_3 = k_B$. Then, the concurrence matrices $\mathbf{N}_0\mathbf{N}_0'$, $\mathbf{N}_1\mathbf{N}_1'$, $\mathbf{N}_2\mathbf{N}_2'$, $\mathbf{N}_3\mathbf{N}_3'$ and the vector of replications of the treatment combinations $\boldsymbol{r}$ of $\mathcal{D}$ are expressed as

$$\mathbf{N}_0\mathbf{N}_0' = \mathbf{N}_{A_B}\mathbf{N}_{A_B}' \otimes \mathbf{N}_B\mathbf{N}_B', \quad \mathbf{N}_1\mathbf{N}_1' = \mathbf{N}_{A_R}\mathbf{N}_{A_R}' \otimes \mathbf{N}_B\mathbf{N}_B', \tag{3.1}$$

$$\mathbf{N}_2\mathbf{N}_2' = \mathbf{N}_{A_C}\mathbf{N}_{A_C}' \otimes \mathbf{N}_B\mathbf{N}_B', \quad \mathbf{N}_3\mathbf{N}_3' = r_A\mathbf{I}_{v_A} \otimes \mathbf{N}_B\mathbf{N}_B' \tag{3.2}$$

and

$$\boldsymbol{r} = r_A\mathbf{1}_{v_A} \otimes \boldsymbol{r}_B, \tag{3.3}$$

where $\mathbf{N}_{A_B}$, $\mathbf{N}_{A_R}$ and $\mathbf{N}_{A_C}$ are the incidence matrices of $(V,\mathcal{B})$, $(V,\mathcal{B}_R)$ and $(V,\mathcal{B}_C)$ of the BIBRC, $\mathbf{N}_B$ is the incidence matrix of D_B, $\mathbf{I}_{v_A}$ is the identity matrix of order v_A and $\otimes$ denotes the Kronecker product of two matrices.

Example 3.1. Let $D_B = (V_B, \mathcal{B}_B)$, where $V_B = \{1,2,3\}$ and $\mathcal{B}_B = \{\{1,2,3\}, \{1,2,3\}, \{1,3,3\}, \{2,3,3\}\}$. Using the BIBRC given in Example 2.1 and D_B, we construct a nested row-column design $\mathcal{D}$ with split units, which has 28 blocks. The following four blocks of $\mathcal{D}$ are obtained by embedding four blocks of D_B in all whole plots of the first block of the BIBRC.

A_1			A_2			A_4		
B_1	B_2	B_3	B_1	B_2	B_3	B_1	B_2	B_3
A_6			A_5			A_3		
B_1	B_2	B_3	B_1	B_2	B_3	B_1	B_2	B_3

A_1			A_2			A_4		
B_1	B_2	B_3	B_1	B_2	B_3	B_1	B_2	B_3
A_6			A_5			A_3		
B_1	B_2	B_3	B_1	B_2	B_3	B_1	B_2	B_3

A_1			A_2			A_4		
B_1	B_3	B_3	B_1	B_3	B_3	B_1	B_3	B_3
A_6			A_5			A_3		
B_1	B_3	B_3	B_1	B_3	B_3	B_1	B_3	B_3

A_1			A_2			A_4		
B_2	B_3	B_3	B_2	B_3	B_3	B_2	B_3	B_3
A_6			A_5			A_3		
B_2	B_3	B_3	B_2	B_3	B_3	B_2	B_3	B_3

Similarly, the remaining blocks of $\mathcal{D}$ are obtained from the remaining blocks of the BIBRC. □

In order to find the stratum efficiency factors for $\mathcal{D}$, it is necessary to find the common eigenvectors of the stratum information matrices $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{A}_4$ and $\mathbf{A}_5$ given in (1.1) with respect to $\boldsymbol{r}^{\delta}$, i.e., the common eigenvectors of the concurrence matrices $\mathbf{N}_0\mathbf{N}_0'$, $\mathbf{N}_1\mathbf{N}_1'$, $\mathbf{N}_2\mathbf{N}_2'$ and $\mathbf{N}_3\mathbf{N}_3'$ given in (3.1) and (3.2) with respect to $\boldsymbol{r}^{\delta}$, where $\boldsymbol{r}$ is given in (3.3). To find the common eigenvectors of $\mathbf{N}_0\mathbf{N}_0'$, $\mathbf{N}_1\mathbf{N}_1'$, $\mathbf{N}_2\mathbf{N}_2'$ and $\mathbf{N}_3\mathbf{N}_3'$, we consider the eigenvectors of $\mathbf{N}_{A_B}\mathbf{N}_{A_B}'$, $\mathbf{N}_{A_R}\mathbf{N}_{A_R}'$ and $\mathbf{N}_{A_C}\mathbf{N}_{A_C}'$ of the BIBRC and the eigenvectors of $\mathbf{N}_B\mathbf{N}_B'$ of D_B.

By the definition of the BIBRC,

$$(3.4)\quad \mathbf{N}_{A_B}\mathbf{N}_{A_B}' = (r_A - \lambda_B)\mathbf{I}_{v_A} + \lambda_B \mathbf{J}_{v_A}, \quad \mathbf{N}_{A_R}\mathbf{N}_{A_R}' = (r_A - \lambda_R)\mathbf{I}_{v_A} + \lambda_R \mathbf{J}_{v_A}$$

and

$$\mathbf{N}_{A_C}\mathbf{N}'_{A_C} = (r_A - \lambda_C)\mathbf{I}_{v_A} + \lambda_C \mathbf{J}_{v_A} \tag{3.5}$$

hold, where $\mathbf{J}_{v_A}$ is the $v_A \times v_A$ matrix with every element unity. From (3.4) and (3.5), it is well known that $\mathbf{N}_{A_B}\mathbf{N}'_{A_B}$, $\mathbf{N}_{A_R}\mathbf{N}'_{A_R}$ and $\mathbf{N}_{A_C}\mathbf{N}'_{A_C}$ have common mutually orthonormal eigenvectors, which are denoted by $\boldsymbol{x}_0, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_{v_A-1}$ with $\boldsymbol{x}_0 = \frac{1}{\sqrt{v_A}}\mathbf{1}_{v_A}$. The eigenvalues of $\mathbf{N}_{A_B}\mathbf{N}'_{A_B}$, $\mathbf{N}_{A_R}\mathbf{N}'_{A_R}$ and $\mathbf{N}_{A_C}\mathbf{N}'_{A_C}$ corresponding to $\boldsymbol{x}_0$ are $r_A pq$, $r_A q$ and $r_A p$, and corresponding to $\boldsymbol{x}_i$ ($i = 1, 2, \cdots, v_A - 1$) are $r_A - \lambda_B$, $r_A - \lambda_R$ and $r_A - \lambda_C$, respectively. These eigenvalues and eigenvectors are summarized in the following table:

Table 3.1. Eigenvalues and common eigenvectors of $\mathbf{N}_{A_B}\mathbf{N}'_{A_B}$, $\mathbf{N}_{A_R}\mathbf{N}'_{A_R}$ and $\mathbf{N}_{A_C}\mathbf{N}'_{A_C}$

Eigenvalues			Common eigenvectors
$\mathbf{N}_{A_B}\mathbf{N}'_{A_B}$	$\mathbf{N}_{A_R}\mathbf{N}'_{A_R}$	$\mathbf{N}_{A_C}\mathbf{N}'_{A_C}$	
$r_A pq$	$r_A q$	$r_A p$	$\boldsymbol{x}_0 = \frac{1}{\sqrt{v_A}}\mathbf{1}_{v_A}$
$r_A - \lambda_B$	$r_A - \lambda_R$	$r_A - \lambda_C$	$\boldsymbol{x}_i$ ($i = 1, 2, \cdots, v_A - 1$)

Furthermore, the eigenvalue and the corresponding eigenvector of $\mathbf{N}_B\mathbf{N}'_B$ with respect to $\boldsymbol{r}_B^\delta$ are denoted by ω_j and $\boldsymbol{y}_j$ for $j = 0, 1, \cdots, v_B - 1$ with $\omega_0 = k_B$ and $\boldsymbol{y}_0 = \frac{1}{\sqrt{n_B}}\mathbf{1}_{v_B}$, where $\boldsymbol{y}_0, \boldsymbol{y}_1, \cdots, \boldsymbol{y}_{v_B-1}$ are mutually $\boldsymbol{r}_B^\delta$-orthonormal and $n_B = \mathbf{1}'_{v_B}\boldsymbol{r}_B$.

Combining the above eigenvectors, we consider four sets of vectors as follows:

$$(1)\quad \frac{1}{\sqrt{v_A}}\mathbf{1}_{v_A} \otimes \frac{1}{\sqrt{n_B}}\mathbf{1}_{v_B}, \qquad (2)\quad \boldsymbol{x}_i \otimes \frac{1}{\sqrt{n_B}}\mathbf{1}_{v_B},$$

$$(3)\quad \frac{1}{\sqrt{v_A}}\mathbf{1}_{v_A} \otimes \boldsymbol{y}_j, \qquad (4)\quad \boldsymbol{x}_i \otimes \boldsymbol{y}_j$$

for $i = 1, 2, \cdots, v_A - 1$ and $j = 1, 2, \cdots, v_B - 1$. It is easily checked that the vectors of (1)-(4) are mutually $\boldsymbol{r}^\delta$-orthonormal and the total number of the vectors is $v_A v_B$. We show that the vectors of (1)-(4) are the common eigenvectors of $\mathbf{N}_0\mathbf{N}'_0$, $\mathbf{N}_1\mathbf{N}'_1$, $\mathbf{N}_2\mathbf{N}'_2$ and $\mathbf{N}_3\mathbf{N}'_3$ with respect to $\boldsymbol{r}^\delta$ and we find the corresponding eigenvalues of $\mathbf{N}_0\mathbf{N}'_0$, $\mathbf{N}_1\mathbf{N}'_1$, $\mathbf{N}_2\mathbf{N}'_2$ and $\mathbf{N}_3\mathbf{N}'_3$.
Firstly, we take into account the matrix $\mathbf{N}_0\mathbf{N}'_0$. For (1), we have, from (3.1),

$$\begin{aligned}
\mathbf{N}_0\mathbf{N}'_0\left(\frac{1}{\sqrt{v_A}}\mathbf{1}_{v_A} \otimes \frac{1}{\sqrt{n_B}}\mathbf{1}_{v_B}\right) &= \left(\mathbf{N}_{A_B}\mathbf{N}'_{A_B}\frac{1}{\sqrt{v_A}}\mathbf{1}_{v_A}\right) \otimes \left(\mathbf{N}_B\mathbf{N}'_B\frac{1}{\sqrt{n_B}}\mathbf{1}_{v_B}\right) \\
&= r_A pq\frac{1}{\sqrt{v_A}}\mathbf{1}_{v_A} \otimes k_B\frac{1}{\sqrt{n_B}}\boldsymbol{r}_B \\
&= pqk_B\left(r_A\mathbf{I}_{v_A} \otimes \boldsymbol{r}_B^\delta\right)\left(\frac{1}{\sqrt{v_A}}\mathbf{1}_{v_A} \otimes \frac{1}{\sqrt{n_B}}\mathbf{1}_{v_B}\right).
\end{aligned}$$

The corresponding eigenvalue is pqk_B.
For (2), we have

$$\mathbf{N}_0\mathbf{N}_0'\left(\boldsymbol{x}_i\otimes\frac{1}{\sqrt{n_B}}\mathbf{1}_{v_B}\right)=\left(\mathbf{N}_{A_B}\mathbf{N}_{A_B}'\boldsymbol{x}_i\right)\otimes\left(\mathbf{N}_B\mathbf{N}_B'\frac{1}{\sqrt{n_B}}\mathbf{1}_{v_B}\right)$$
$$=((r_A-\lambda_B)\boldsymbol{x}_i)\otimes k_B\frac{1}{\sqrt{n_B}}\boldsymbol{r}_B$$
$$=\frac{(r_A-\lambda_B)k_B}{r_A}\left(r_A\mathbf{I}_{v_A}\otimes\boldsymbol{r}_B^{\delta}\right)\left(\boldsymbol{x}_i\otimes\frac{1}{\sqrt{n_B}}\mathbf{1}_{v_B}\right).$$

The corresponding eigenvalue is $(r_A-\lambda_B)k_B/r_A$.
For (3), we have

$$\mathbf{N}_0\mathbf{N}_0'\left(\frac{1}{\sqrt{v_A}}\mathbf{1}_{v_A}\otimes\boldsymbol{y}_j\right)=\left(\mathbf{N}_{A_B}\mathbf{N}_{A_B}'\frac{1}{\sqrt{v_A}}\mathbf{1}_{v_A}\right)\otimes\left(\mathbf{N}_B\mathbf{N}_B'\boldsymbol{y}_j\right)$$
$$=r_Apq\frac{1}{\sqrt{v_A}}\mathbf{1}_{v_A}\otimes\omega_j\boldsymbol{r}_B^{\delta}\boldsymbol{y}_j=pq\omega_j\left(r_A\mathbf{I}_{v_A}\otimes\boldsymbol{r}_B^{\delta}\right)\left(\frac{1}{\sqrt{v_A}}\mathbf{1}_{v_A}\otimes\boldsymbol{y}_j\right).$$

The corresponding eigenvalue is $pq\omega_j$.
For (4), we have

$$\mathbf{N}_0\mathbf{N}_0'\left(\boldsymbol{x}_i\otimes\boldsymbol{y}_j\right)=\left(\mathbf{N}_{A_B}\mathbf{N}_{A_B}'\boldsymbol{x}_i\right)\otimes\left(\mathbf{N}_B\mathbf{N}_B'\boldsymbol{y}_j\right)$$
$$=((r_A-\lambda_B)\boldsymbol{x}_i)\otimes\omega_j\boldsymbol{r}_B^{\delta}\boldsymbol{y}_j=\frac{(r_A-\lambda_B)\omega_j}{r_A}\left(r_A\mathbf{I}_{v_A}\otimes\boldsymbol{r}_B^{\delta}\right)\left(\boldsymbol{x}_i\otimes\boldsymbol{y}_j\right).$$

The corresponding eigenvalue is $(r_A-\lambda_B)\omega_j/r_A$.
Similarly, from (3.1) and (3.2), we can show that the vectors of (1)-(4) are also the eigenvectors of $\mathbf{N}_1\mathbf{N}_1'$, $\mathbf{N}_2\mathbf{N}_2'$ and $\mathbf{N}_3\mathbf{N}_3'$ with respect to $\boldsymbol{r}^{\delta}$ and we summarize the corresponding eigenvalues of $\mathbf{N}_0\mathbf{N}_0'$, $\mathbf{N}_1\mathbf{N}_1'$, $\mathbf{N}_2\mathbf{N}_2'$ and $\mathbf{N}_3\mathbf{N}_3'$ in the following:

Table 3.2. Eigenvalues and common eigenvectors of $\mathbf{N}_0\mathbf{N}_0'$, $\mathbf{N}_1\mathbf{N}_1'$, $\mathbf{N}_2\mathbf{N}_2'$ and $\mathbf{N}_3\mathbf{N}_3'$

Eigenvalues				Common eigenvectors
$\mathbf{N}_0\mathbf{N}_0'$	$\mathbf{N}_1\mathbf{N}_1'$	$\mathbf{N}_2\mathbf{N}_2'$	$\mathbf{N}_3\mathbf{N}_3'$	
pqk_B	qk_B	pk_B	k_B	(1)
$(r_A-\lambda_B)k_B/r_A$	$(r_A-\lambda_R)k_B/r_A$	$(r_A-\lambda_C)k_B/r_A$	k_B	(2)
$pq\omega_j$	$q\omega_j$	$p\omega_j$	ω_j	(3)
$(r_A-\lambda_B)\omega_j/r_A$	$(r_A-\lambda_R)\omega_j/r_A$	$(r_A-\lambda_C)\omega_j/r_A$	ω_j	(4)

The vectors of (1)-(4) are also the common eigenvectors of the stratum information matrices $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{A}_4$ and $\mathbf{A}_5$ with respect to $\boldsymbol{r}^{\delta}$, which means $\mathcal{D}$ has the general balance property, and the eigenvectors of (2), (3) and (4) define the basic contrasts among the main effects of the whole plot treatments, the main effects of the subplot treatments and the interaction effects of the whole plot treatments and the subplot treatments, respectively. By use of (1.1) and Table 3.2, we give the stratum efficiency factors for $\mathcal{D}$ in the following table:

Table 3.3. Stratum efficiency factors for $\mathcal{D}$

Type of contrasts	Number of contrasts	Strata				
		I	II	III	IV	V
A	$v_A - 1$	Q_{B_0}	$pQ_{R_0} - Q_{B_0}$	$qQ_{C_0} - Q_{B_0}$	Q	0
B	1	ω_j/k_B	0	0	0	$1 - \omega_j/k_B$
$A \times B$	$(v_A - 1) \times 1$	Q_{B_j}	$pQ_{R_j} - Q_{B_j}$	$qQ_{C_j} - Q_{B_j}$	Q_j	$1 - \omega_j/k_B$

for $j = 1, 2, \cdots, v_B - 1$, where

$$Q_{B_0} = \frac{r_A - \lambda_B}{r_A pq}, \quad Q_{R_0} = \frac{r_A - \lambda_R}{r_A pq}, \quad Q_{C_0} = \frac{r_A - \lambda_C}{r_A pq},$$

$$Q_{B_j} = \frac{\omega_j}{k_B} Q_{B_0}, \quad Q_{R_j} = \frac{\omega_j}{k_B} Q_{R_0}, \quad Q_{C_j} = \frac{\omega_j}{k_B} Q_{C_0},$$

$$Q = 1 - pQ_{R_0} - qQ_{C_0} + Q_{B_0}, \quad Q_j = \frac{\omega_j}{k_B} Q.$$

In Table 3.3, we see that, for example, the basic contrasts among the interaction effects of the whole plot treatments and the subplot treatments are estimated from the strata I, II, III, IV and V with respective efficiency factors Q_{B_j}, $pQ_{R_j} - Q_{B_j}$, $qQ_{C_j} - Q_{B_j}$, Q_j and $1 - \omega_j/k_B$ for $i = 1, 2, \cdots, v_A - 1$ and $j = 1, 2, \cdots, v_B - 1$. Table 3.3 is used in order to improve the estimators for the basic contrasts in the treatment effects combining the estimators obtained from the strata I, II, III, IV and V. This procedure was proposed by [2], [9] and [10].

Example 3.2. For the nested row-column design $\mathcal{D}$ with split units constructed in Example 3.1 with $v_A = 7$, $r_A = 6$, $p = 2$, $q = 3$, $\lambda_B = 5$, $\lambda_R = 2$, $\lambda_C = 1$, $v_B = 3$, $k_B = 3$ and $\omega_1 = \omega_2 = 1/3$, using Table 3.3, we have the stratum efficiency factors as follows:

Type of contrasts	Number of contrasts	Strata				
		I	II	III	IV	V
A	6	1/36	7/36	14/36	14/36	0
B	2	1/9	0	0	0	8/9
$A \times B$	12	1/324	7/324	14/324	14/324	8/9

□

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