SUT Journal of Mathematics
Vol. 50, No. 1 (2014), 25–45

# Inequalities for Power Series in Banach Algebras

**Silvestru Sever Dragomir**[1,2]

(Received December 11, 2013; Revised September 11, 2014)

**Abstract.** For any $x, y \in \mathcal{B}$, a unital Banach algebra and $n \geq 1$ we show that

$$\|y^n - x^n\| \leq n \|y - x\| \int_0^1 \|(1-t)x + ty\|^{n-1} dt.$$

Upper bounds for quantities such as

$$\|f(x) - f(y)\|, \|f(xy) - f(yx)\|,$$

and

$$\left\| \frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2}\right) \right\|$$

that can naturally be associated with the analytic function $f(\lambda) := \sum_{j=0}^{\infty} \alpha_j \lambda^j$ defined on the open disk $D(0, R)$ and the elements $x$ and $y$ of the unital Banach algebra $\mathcal{B}$ are given. Some applications for functions of interest such as the exponential map on $\mathcal{B}$ and the resolvent function are provided as well.

*AMS* 2010 *Mathematics Subject Classification.* 47A63; 47A99.

*Key words and phrases.* Banach algebras, power series, Lipschitz type inequalities, Jensen type inequalities.

## §1. Introduction

Let $\mathcal{B}$ be an algebra over $\mathbb{C}$. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\| : \mathcal{B} \to [0, \infty)$ such that $(\mathcal{B}, \|\cdot\|)$ is a normed space, and, further:

$$\|ab\| \leq \|a\| \|b\|$$

for any $a, b \in \mathcal{B}$. The normed algebra $(\mathcal{B}, \|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm.*

We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\| = 1$.

Let $\mathcal{B}$ be a unital algebra. An element $a \in \mathcal{B}$ is *invertible* if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element $b$ is unique; it is called the *inverse* of $a$ and written $a^{-1}$ or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by Inv $\mathcal{B}$. If $a, b \in$Inv $\mathcal{B}$ then $ab \in$Inv $\mathcal{B}$ and $(ab)^{-1} = b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a \in \mathcal{B}$ and $\lim_{n\to\infty} \|a^n\|^{1/n} < 1$, then $1 - a \in$Inv $\mathcal{B}$;

(ii) $\{b \in \mathcal{B}\colon\ \|1 - b\| < 1\} \subset$Inv $\mathcal{B}$;

(iii) Inv$\mathcal{B}$ is an *open subset* of $\mathcal{B}$;

(iv) The map Inv$\mathcal{B} \ni a \longmapsto a^{-1} \in$Inv $\mathcal{B}$ is continuous.

For simplicity, we denote $\lambda 1$, where $\lambda \in \mathbb{C}$ and $1$ is the identity of $\mathcal{B}$, by $\lambda$. The *resolvent set* of $a \in \mathcal{B}$ is defined by

$$\rho(a) := \{\lambda \in \mathbb{C} :\ \lambda - a \in \text{Inv } \mathcal{B}\};$$

the *spectrum* of $a$ is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of $a$ is $R_a : \rho(a) \to$Inv $\mathcal{B}$,

$$R_a(\lambda) := (\lambda - a)^{-1}.$$

For each $\lambda, \gamma \in \rho(a)$ we have the identity

$$R_a(\gamma) - R_a(\lambda) = (\lambda - \gamma) R_a(\lambda) R_a(\gamma).$$

We also have that

$$\sigma(a) \subset \{\lambda \in \mathbb{C} :\ |\lambda| \leq \|a\|\}.$$

The *spectral radius* of $a$ is defined as

$$\nu(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}.$$

If $a, b$ are *commuting* elements in $\mathcal{B}$, i.e. $ab = ba$, then

$$\nu(ab) \leq \nu(a)\nu(b) \text{ and } \nu(a + b) \leq \nu(a) + \nu(b).$$

Let $\mathcal{B}$ a unital Banach algebra and $a \in \mathcal{B}$. Then

(i) The resolvent set $\rho(a)$ is open in $\mathbb{C}$;

(ii) For any *bounded linear functionals* $\lambda : \mathcal{B} \to \mathbb{C}$, the function $\lambda \circ R_a$ is analytic on $\rho(a)$;

(iii) The spectrum $\sigma(a)$ is compact and nonempty in $\mathbb{C}$;

(iv) We have
$$\nu(a)=\lim_{n\to\infty}\|a^n\|^{1/n}.$$

Let $f$ be an analytic functions on the open disk $D(0,R)$ given by the *power series*
$$f(\lambda):=\sum_{j=0}^{\infty}\alpha_j\lambda^j \ \ (|\lambda|<R).$$
If $\nu(a)<R$, then the series $\sum_{j=0}^{\infty}\alpha_j a^j$ converges in the Banach algebra $\mathcal{B}$ because $\sum_{j=0}^{\infty}|\alpha_j|\left\|a^j\right\|<\infty$, and we can define $f(a)$ to be its sum. Clearly $f(a)$ is well defined and there are many examples of important functions on a Banach algebra $\mathcal{B}$ that can be constructed in this way. For instance, the *exponential map* on $\mathcal{B}$ denoted exp and defined as
$$\exp a:=\sum_{j=0}^{\infty}\frac{1}{j!}a^j \text{ for each } a\in\mathcal{B}.$$
If $\mathcal{B}$ is not commutative, then many of the familiar properties of the exponential function from the scalar case do not hold. The following key formula is valid, however with the additional hypothesis of commutativity for $a$ and $b$ from $\mathcal{B}$
$$\exp(a+b)=\exp(a)\exp(b).$$
In a general Banach algebra $\mathcal{B}$ it is difficult to determine the elements in the range of the exponential map $\exp(\mathcal{B})$, i.e. the element which have a "*logarithm*". However, it is easy to see that if $a$ is an element in $\mathcal{B}$ such that $\|1-a\|<1$, then $a$ is in $\exp(\mathcal{B})$. That follows from the fact that if we set
$$b=-\sum_{n=1}^{\infty}\frac{1}{n}(1-a)^n,$$
then the series converges absolutely and, as in the scalar case, substituting this series into the series expansion for $\exp(b)$ yields $\exp(b)=a$.

Concerning other basic definitions and facts in the theory of Banach algebras, the reader can consult the classical books [9] and [21].

Let $\mathcal{B}(H)$ be the Banach algebra of bounded linear operators on a separable complex Hilbert space $H$. The absolute value of an operator $A$ is the positive operator $|A|$ defined as $|A|:=(A^*A)^{1/2}$.

It is known that [3] in the infinite-dimensional case the map $f(A):=|A|$ is not *Lipschitz continuous* on $\mathcal{B}(H)$ with the usual operator norm, i.e. there is no constant $L>0$ such that
$$\||A|-|B|\|\le L\|A-B\|$$

for any $A, B \in \mathcal{B}(H)$.

However, as shown by Farforovskaya in [10], [11] and Kato in [16], the following inequality holds

$$\||A| - |B|\| \leq \frac{2}{\pi} \|A - B\| \left(2 + \log\left(\frac{\|A\| + \|B\|}{\|A - B\|}\right)\right) \tag{1.1}$$

for any $A, B \in \mathcal{B}(H)$ with $A \neq B$.

If the operator norm is replaced with *Hilbert-Schmidt norm* $\|C\|_{HS} := (trC^*C)^{1/2}$ of an operator $C$, then the following inequality is true [1]

$$\||A| - |B|\|_{HS} \leq \sqrt{2} \|A - B\|_{HS} \tag{1.2}$$

for any $A, B \in \mathcal{B}(H)$.

The coefficient $\sqrt{2}$ is best possible for a general $A$ and $B$. If $A$ and $B$ are restricted to be selfadjoint, then the best coefficient is 1.

It has been shown in [3] that, if $A$ is an invertible operator, then for all operators $B$ in a neighborhood of $A$ we have

$$\||A| - |B|\| \leq a_1 \|A - B\| + a_2 \|A - B\|^2 + O\left(\|A - B\|^3\right) \tag{1.3}$$

where

$$a_1 = \left\|A^{-1}\right\| \|A\| \text{ and } a_2 = \left\|A^{-1}\right\| + \left\|A^{-1}\right\|^3 \|A\|^2.$$

In [2] the author also obtained the following *Lipschitz type inequality*

$$\|f(A) - f(B)\| \leq f'(a) \|A - B\| \tag{1.4}$$

where $f$ is an *operator monotone function* on $(0, \infty)$ and $A, B \geq aI_H > 0$.

One of the central problems in perturbation theory is to find bounds for

$$\|f(A) - f(B)\|$$

in terms of $\|A - B\|$ for different classes of measurable functions $f$ for which the function of operator can be defined. For some results on this topic, see [4], [12] and the references therein.

In this paper, motivated by the above considerations, we establish some upper bounds for the following quantities

$$\|f(x) - f(y)\|, \|f(xy) - f(yx)\|,$$
$$\left\|\frac{f(x) + f(y)}{2} - f\left(\frac{x + y}{2}\right)\right\|$$

and

$$\left\|\frac{f(x^2) + f(y^2)}{2} - f(xy)\right\|$$

that can naturally be associated with the analytic function $f(\lambda) := \sum_{j=0}^{\infty} \alpha_j \lambda^j$ defined on the open disk $D(0, R)$ and the elements $x$ and $y$ of the unital Banach algebra $\mathcal{B}$.

Some applications for functions of interest such as the exponential map on $\mathcal{B}$ and the resolvent function are provided as well.

## §2. Some Lipschitz Type Inequalities

The following result for powers holds.

**Theorem 1.** *For any $x, y \in \mathcal{B}$ and $n \geq 1$ we have*

$$\|y^n - x^n\| \leq n \|y - x\| \int_0^1 \|(1-t)x + ty\|^{n-1} dt. \tag{2.1}$$

*Proof.* We use the identity (see for instance [5, p. 254])

$$a^n - b^n = \sum_{j=0}^{n-1} a^{n-1-j}(a-b)b^j \tag{2.2}$$

that holds for any $a, b \in \mathcal{B}$ and $n \geq 1$.

For $x, y \in \mathcal{B}$ we consider the function $\varphi : [0,1] \to \mathcal{B}$ defined by $\varphi(t) = [(1-t)x + ty]^n$. For $t \in (0,1)$ and $\varepsilon \neq 0$ with $t + \varepsilon \in (0,1)$ we have from (2.2) that

$$\begin{aligned}\varphi(t+\varepsilon) - \varphi(t) &= [(1-t-\varepsilon)x + (t+\varepsilon)y]^n - [(1-t)x + ty]^n \\ &= \varepsilon \sum_{j=0}^{n-1} [(1-t-\varepsilon)x + (t+\varepsilon)y]^{n-1-j}(y-x)[(1-t)x + ty]^j.\end{aligned}$$

Dividing with $\varepsilon \neq 0$ and taking the limit over $\varepsilon \to 0$ we have in the norm topology of $\mathcal{B}$ that

$$\begin{aligned}\varphi'(t) &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[\varphi(t+\varepsilon) - \varphi(t)] \\ &= \sum_{j=0}^{n-1} [(1-t)x + ty]^{n-1-j}(y-x)[(1-t)x + ty]^j.\end{aligned} \tag{2.3}$$

Integrating on $[0,1]$ we get from (2.3) that

$$\int_0^1 \varphi'(t)\,dt = \sum_{j=0}^{n-1} \int_0^1 [(1-t)x + ty]^{n-1-j}(y-x)[(1-t)x + ty]^j\,dt$$

and since

$$\int_0^1 \varphi'(t)\,dt = \varphi(1) - \varphi(0) = y^n - x^n$$

then we get the following *equality of interest*

$$y^n - x^n = \sum_{j=0}^{n-1} \int_0^1 [(1-t)x + ty]^{n-1-j}(y-x)[(1-t)x+ty]^j\,dt$$

for any $x, y \in \mathcal{B}$ and $n \geq 1$.

Taking the norm and utilizing the properties of Bochner integral for vector valued functions (see for instance [18, p. 21]) we have

$$\begin{aligned} \|y^n - x^n\| &\leq \sum_{j=0}^{n-1} \left\| \int_0^1 [(1-t)x+ty]^{n-1-j}(y-x)[(1-t)x+ty]^j\,dt \right\| \\ &\leq \sum_{j=0}^{n-1} \int_0^1 \left\| [(1-t)x+ty]^{n-1-j}(y-x)[(1-t)x+ty]^j \right\| dt \\ &\leq \sum_{j=0}^{n-1} \int_0^1 \left\| [(1-t)x+ty]^{n-1-j} \right\| \|y-x\| \left\| [(1-t)x+ty]^j \right\| dt \\ &\leq \sum_{j=0}^{n-1} \int_0^1 \|(1-t)x+ty\|^{n-1-j} \|y-x\| \|(1-t)x+ty\|^j\,dt \\ &= n\|y-x\| \int_0^1 \|(1-t)x+ty\|^{n-1}\,dt \end{aligned} \tag{2.4}$$

for any $x, y \in \mathcal{B}$ and $n \geq 1$. □

**Remark 1.** Utilising the Hermite-Hadamard inequality for convex functions

$$\frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right],$$

(see for instance [8, p. 2]) we have the sequence of inequalities

$$\begin{aligned} \|y^n - x^n\| &\leq n\|y-x\| \int_0^1 \|(1-t)x+ty\|^{n-1}\,dt \\ &\leq \frac{1}{2}n\|y-x\| \left[\left\|\frac{x+y}{2}\right\|^{n-1} + \frac{\|x\|^{n-1} + \|y\|^{n-1}}{2}\right] \\ &\leq \frac{1}{2}n\|y-x\| \left[\|x\|^{n-1} + \|y\|^{n-1}\right] \\ &\leq n\|y-x\| \max\left\{\|x\|^{n-1}, \|y\|^{n-1}\right\}. \end{aligned} \tag{2.5}$$

For other Hermite-Hadamard type inequalities that may be utilized to obtain such upper bounds, see for instance [6], [7], [14], [15], [19], [20], [22], [23], [24], [25] and [26]. The details are not presented here.

We also have

$$
\begin{aligned}
(2.6)\qquad \|y^n - x^n\| &\le n\|y-x\| \int_0^1 \|(1-t)x + ty\|^{n-1} dt \\
&\le n\|y-x\| \int_0^1 ((1-t)\|x\| + t\|y\|)^{n-1} dt \\
&\le \frac{1}{2} n\|y-x\| \left[\left(\frac{\|x\|+\|y\|}{2}\right)^{n-1} + \frac{\|x\|^{n-1}+\|y\|^{n-1}}{2}\right] \\
&\le \frac{1}{2} n\|y-x\| \left[\|x\|^{n-1}+\|y\|^{n-1}\right] \\
&\le n\|y-x\| \max\left\{\|x\|^{n-1}, \|y\|^{n-1}\right\}.
\end{aligned}
$$

We observe that if $\|y\| \neq \|x\|$, then by the change of variable $s = (1-t)\|x\| + t\|y\|$ we have

$$
\begin{aligned}
\int_0^1 ((1-t)\|x\| + t\|y\|)^{n-1} dt &= \frac{1}{\|y\| - \|x\|} \int_{\|x\|}^{\|y\|} s^{n-1} ds \\
&= \frac{1}{n} \cdot \frac{\|y\|^n - \|x\|^n}{\|y\| - \|x\|}.
\end{aligned}
$$

If $\|y\| = \|x\|$, then

$$
\int_0^1 ((1-t)\|x\| + t\|y\|)^{n-1} dt = \|x\|^{n-1}.
$$

Utilising these observations we then get the following divided difference inequality for $x \neq y$

$$
\begin{aligned}
(2.7)\qquad \frac{\|y^n - x^n\|}{\|y-x\|} &\le n \int_0^1 \|(1-t)x + ty\|^{n-1} dt \\
&\le \begin{cases} \frac{\|y\|^n - \|x\|^n}{\|y\| - \|x\|} & \text{if } \|y\| \neq \|x\|, \\ \\ n\|x\|^{n-1} & \text{if } \|y\| = \|x\|. \end{cases}
\end{aligned}
$$

**Remark 2.** We observe that the quantity $n\int_0^1 \|(1-t)x + ty\|^{n-1} dt$, which might be difficult to compute in various examples of Banach algebras, has got the simpler bounds

$$
B_1(x,y) := \frac{1}{2} n \left[\left\|\frac{x+y}{2}\right\|^{n-1} + \frac{\|x\|^{n-1}+\|y\|^{n-1}}{2}\right]
$$

and

$$B_2(x,y) := \begin{cases} \frac{\|y\|^n - \|x\|^n}{\|y\|-\|x\|} & \text{if } \|y\| \neq \|x\|, \\ \\ n\|x\|^{n-1} & \text{if } \|y\| = \|x\|. \end{cases}$$

It is natural then to ask which of these bounds is better?

Let $m \geq 1$. Then

$$B_1(x,y) = \frac{1}{2}m\left[\left\|\frac{x+y}{2}\right\|^{m-1} + \frac{\|x\|^{m-1} + \|y\|^{m-1}}{2}\right]$$

and

$$B_2(x,y) = \begin{cases} \|y\|^{m-1} + \|y\|^{m-2}\|x\| + ... + \|x\|^{m-1} & \text{if } \|y\| \neq \|x\|, \\ \\ m\|x\|^{m-1} & \text{if } \|y\| = \|x\|. \end{cases}$$

If we take $y = tx$ with $\|x\| = 1$ and $|t| \neq 1$ then we get

$$B_1(t) = \frac{1}{2}m\left[\left|\frac{1+t}{2}\right|^{m-1} + \frac{1+|t|^{m-1}}{2}\right]$$

and

$$B_2(t) = |t|^{m-1} + ... + |t| + 1.$$

If we take $m = 4$ and plot the difference

$$d(t) := 2\left(\left|\frac{t+1}{2}\right|^3 + \frac{1+|t|^3}{2}\right) - \left(|t|^3 + |t|^2 + |t| + 1\right)$$

on the interval $[-8, 8]$, then we can conclude that some time the first bound is better than the second, while other time the conclusion is the other way around. The details for the plot are nor presented here.

Now, by the help of power series $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ we can naturally construct another power series which will have as coefficients the absolute values of the coefficients of the original series, namely, $f_a(\lambda) := \sum_{n=0}^{\infty} |\alpha_n| \lambda^n$. It is obvious that this new power series will have the same radius of convergence as the original series. We also notice that if all coefficients $\alpha_n \geq 0$, then $f_a = f$.

The following result is valid.

**Corollary 2.** *Let $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. For any $x, y \in \mathcal{B}$ with $\|x\|, \|y\| < R$ we have*

$$\|f(y) - f(x)\| \leq \|y - x\| \int_0^1 f_a'(\|(1-t)x + ty\|)\, dt. \tag{2.8}$$

*Proof.* Now, for any $m \geq 1$, by making use of the inequality (2.1) we have

$$\begin{aligned} &\left\| \sum_{n=0}^{m} \alpha_n y^n - \sum_{n=0}^{m} \alpha_n x^n \right\| = \left\| \sum_{n=1}^{m} \alpha_n \left( y^n - x^n \right) \right\| \\ &\leq \sum_{n=1}^{m} |\alpha_n| \, \| y^n - x^n \| \\ &\leq \| y - x \| \sum_{n=1}^{m} n \, |\alpha_n| \int_0^1 \| (1-t)\, x + ty \|^{n-1} \, dt \\ &= \| y - x \| \int_0^1 \left( \sum_{n=1}^{m} n \, |\alpha_n| \, \| (1-t)\, x + ty \|^{n-1} \right) dt. \end{aligned} \tag{2.9}$$

Moreover, since $\|x\|, \|y\| < R$, then the series $\sum_{n=0}^{\infty} \alpha_n y^n$, $\sum_{n=0}^{\infty} \alpha_n x^n$ and

$$\sum_{n=1}^{\infty} n \, |\alpha_n| \, \| (1-t)\, x + ty \|^{n-1}$$

are convergent and

$$\sum_{n=0}^{\infty} \alpha_n y^n = f(y), \sum_{n=0}^{\infty} \alpha_n x^n = f(x)$$

while

$$\sum_{n=1}^{\infty} n \, |\alpha_n| \, \| (1-t)\, x + ty \|^{n-1} = f_a' \left( \| (1-t)\, x + ty \| \right).$$

Therefore, by taking the limit over $m \to \infty$ in the inequality (2.9) we deduce the desired result (2.8). □

**Remark 3.** We observe that $f_a'$ is monotonic nondecreasing and convex on the interval $[0, R)$ and since the function $\psi(t) := \| (1-t)\, x + ty \|$ is convex on $[0,1]$ we have that $f_a' \circ \psi$ is also convex on $[0,1]$. Utilising the Hermite-Hadamard inequality for convex functions (see for instance [8, p. 2]) we have the sequence of inequalities

$$\begin{aligned} \| f(y) - f(x) \| &\leq \| y - x \| \int_0^1 f_a' \left( \| (1-t)\, x + ty \| \right) dt \\ &\leq \frac{1}{2} \| y - x \| \left[ f_a' \left( \left\| \frac{x+y}{2} \right\| \right) + \frac{f_a'(\|x\|) + f_a'(\|y\|)}{2} \right] \\ &\leq \frac{1}{2} \| y - x \| \left[ f_a'(\|x\|) + f_a'(\|y\|) \right] \\ &\leq \| y - x \| \max \left\{ f_a'(\|x\|), f_a'(\|y\|) \right\}. \end{aligned} \tag{2.10}$$

We also have

$$\begin{aligned}(2.11)\quad \|f(y)-f(x)\| &\le \|y-x\|\int_0^1 f_a'(\|(1-t)x+ty\|)\,dt \\ &\le \|y-x\|\int_0^1 f_a'((1-t)\|x\|+t\|y\|)\,dt \\ &\le \frac{1}{2}\|y-x\|\left[f_a'\left(\frac{\|x\|+\|y\|}{2}\right)+\frac{f_a'(\|x\|)+f_a'(\|y\|)}{2}\right] \\ &\le \frac{1}{2}\|y-x\|\left[f_a'(\|x\|)+f_a'(\|y\|)\right] \\ &\le \|y-x\|\max\left\{f_a'(\|x\|),f_a'(\|y\|)\right\}.\end{aligned}$$

We observe that if $\|y\|\ne\|x\|$, then by the change of variable $s=(1-t)\|x\|+t\|y\|$ we have

$$\int_0^1 f_a'((1-t)\|x\|+t\|y\|)\,dt=\frac{f_a(\|y\|)-f_a(\|x\|)}{\|y\|-\|x\|}.$$

If $\|y\|=\|x\|$, then

$$\int_0^1 f_a'((1-t)\|x\|+t\|y\|)\,dt=f_a'(\|x\|).$$

Utilising these observations we then get the following divided difference inequality for $x\ne y$

$$(2.12)\qquad \frac{\|f(y)-f(x)\|}{\|y-x\|} \le \int_0^1 f_a'(\|(1-t)x+ty\|)\,dt \le \begin{cases}\frac{f_a(\|y\|)-f_a(\|x\|)}{\|y\|-\|x\|} & \text{if } \|y\|\ne\|x\|, \\ f_a'(\|x\|) & \text{if } \|y\|=\|x\|.\end{cases}$$

If $\|x\|,\|y\|\le M<R$, then from either of the inequalities (2.5) or (2.6) we have the Lipschitz type inequality

$$(2.13)\qquad \|f(y)-f(x)\|\le f_a'(M)\|y-x\|.$$

The following result for generalized commutator holds:

**Corollary 3.** *Let $f(\lambda)=\sum_{n=0}^{\infty}\alpha_n\lambda^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. For any $x,y\in\mathcal{B}$ with $\|xy\|,\|yx\|<R$ we have*

$$(2.14)\qquad \|f(xy)-f(yx)\|\le\|xy-yx\|\int_0^1 f_a'(\|(1-t)xy+tyx\|)\,dt.$$

Since $f_a'$ is monotonic nondecreasing and convex on the interval $[0, R)$, then

$$\begin{aligned}
&\|f(xy) - f(yx)\| \\
&\le \|xy - yx\| \int_0^1 f_a'(\|(1-t)xy + tyx\|)\,dt \\
&\le \frac{1}{2}\|xy - yx\| \left[ f_a'\left(\left\|\frac{xy+yx}{2}\right\|\right) + \frac{f_a'(\|xy\|) + f_a'(\|yx\|)}{2} \right] \\
&\le \frac{1}{2}\|xy - yx\| \left[ f_a'(\|xy\|) + f_a'(\|yx\|) \right] \\
&\le \|xy - yx\| \max\left\{ f_a'(\|xy\|), f_a'(\|yx\|) \right\} \\
&\le \|xy - yx\| f_a'(\|x\|\,\|y\|)
\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}
&\|f(xy) - f(yx)\| \\
&\le \|xy - yx\| \int_0^1 f_a'(\|(1-t)xy + tyx\|)\,dt \\
&\le \|xy - yx\| \int_0^1 f_a'((1-t)\|xy\| + t\|yx\|)\,dt \\
&\le \frac{1}{2}\|xy - yx\| \left[ f_a'\left(\frac{\|xy\| + \|yx\|}{2}\right) + \frac{f_a'(\|xy\|) + f_a'(\|yx\|)}{2} \right] \\
&\le \frac{1}{2}\|xy - yx\| \left[ f_a'(\|xy\|) + f_a'(\|yx\|) \right] \\
&\le \|xy - yx\| \max\left\{ f_a'(\|xy\|), f_a'(\|yx\|) \right\} \\
&\le \|xy - yx\| f_a'(\|x\|\,\|y\|).
\end{aligned} \tag{2.16}$$

If $\|x\|, \|y\| \le M < R^{1/2}$, then from the inequalities (2.15) we get the simpler inequality

$$\|f(yx) - f(xy)\| \le f_a'\left(M^2\right) \|yx - xy\|. \tag{2.17}$$

## §3. Bounds for the Jensen Difference

In this section we establish some bounds for the norm of the *Jensen difference*, namely, the quantity

$$\left\| \frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2}\right) \right\|,$$

where $x, y \in \mathcal{B}$ and $f$ is a function defined on the Banach algebra $\mathcal{B}$.

**Theorem 4.** *Let $f(\lambda)=\sum_{n=0}^{\infty}\alpha_n\lambda^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. For any $x,y\in\mathcal{B}$ with $\|x\|,\|y\|<R$ we have*

$$\left\|\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right\| \le \frac{1}{2}\|y-x\|\int_0^1\left[f_a'(\|(1-t)x+ty\|)-f_a'(0)\right]dt. \tag{3.1}$$

*The constant $\frac{1}{2}$ is best possible in* (3.1).

*Proof.* For any $x,y\in\mathcal{B}$ and $n\ge 2$ we have from (2.1) that

$$\begin{aligned}\left\|y^n-\left(\frac{x+y}{2}\right)^n\right\| &\le n\left\|y-\frac{x+y}{2}\right\|\int_0^1\left\|(1-t)\frac{x+y}{2}+ty\right\|^{n-1}dt \\ &= \frac{1}{2}n\|y-x\|\int_0^1\left\|(1-t)\frac{x+y}{2}+ty\right\|^{n-1}dt\end{aligned} \tag{3.2}$$

and

$$\left\|x^n-\left(\frac{x+y}{2}\right)^n\right\| \le \frac{1}{2}n\|y-x\|\int_0^1\left\|(1-t)\frac{x+y}{2}+tx\right\|^{n-1}dt. \tag{3.3}$$

We add (3.2) with (3.3), use the triangle inequality and divide by 2 to get

$$\begin{aligned}&\left\|\frac{x^n+y^n}{2}-\left(\frac{x+y}{2}\right)^n\right\| \\ &\le \frac{1}{2}n\|y-x\| \\ &\times\frac{1}{2}\int_0^1\left[\left\|(1-t)\frac{x+y}{2}+ty\right\|^{n-1}+\left\|(1-t)\frac{x+y}{2}+tx\right\|^{n-1}\right]dt \\ &= \frac{1}{2}n\|y-x\| \\ &\times\frac{1}{2}\int_0^1\left[\left\|s\frac{x+y}{2}+(1-s)y\right\|^{n-1}+\left\|s\frac{x+y}{2}+(1-s)x\right\|^{n-1}\right]ds,\end{aligned} \tag{3.4}$$

where we used for the last equality the change of variable $s=1-t$.

Now, using the change of variable $s=2\tau$ we have

$$\frac{1}{2}\int_0^1\left\|s\frac{x+y}{2}+(1-s)x\right\|^{n-1}ds=\int_0^{1/2}\|(1-\tau)x+\tau y\|^{n-1}d\tau$$

and by the change of variable $s=1-v$ we have

$$\frac{1}{2}\int_0^1\left\|s\frac{x+y}{2}+(1-s)y\right\|^{n-1}ds=\frac{1}{2}\int_0^1\left\|(1-v)\frac{x+y}{2}+vy\right\|^{n-1}dv.$$

Moreover, if we make the change of variable $v = 2\tau - 1$ we also have

$$\frac{1}{2}\int_0^1 \left\| (1-v)\frac{x+y}{2} + vy \right\|^{n-1} dv = \int_{1/2}^1 \|(1-\tau)x + \tau y\|^{n-1} d\tau.$$

Therefore

$$\begin{aligned}&\frac{1}{2}\int_0^1 \left[ \left\| s\frac{x+y}{2} + (1-s)y \right\|^{n-1} + \left\| s\frac{x+y}{2} + (1-s)x \right\|^{n-1} \right] ds \\ &= \int_0^{1/2} \|(1-\tau)x + \tau y\|^{n-1} d\tau + \int_{1/2}^1 \|(1-\tau)x + \tau y\|^{n-1} d\tau \\ &= \int_0^1 \|(1-\tau)x + \tau y\|^{n-1} d\tau.\end{aligned}$$

Utilising (3.4) we get

$$\left\| \frac{x^n + y^n}{2} - \left(\frac{x+y}{2}\right)^n \right\| \le \begin{cases} 0 \ \ (n = 0, 1), \\ \frac{1}{2}n\|y-x\| \int_0^1 \|(1-t)x + ty\|^{n-1} dt \ \ (n \ge 2). \end{cases}$$

Now, by making use of an argument similar to the one from the proof of Corollary 2 we deduce that

$$\begin{aligned}\left\| \frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right) \right\| &\le \frac{1}{2}\|y-x\| \sum_{n=2}^{\infty} n|\alpha_n| \int_0^1 \|(1-t)x + ty\|^{n-1} dt \\ &= \frac{1}{2}\|y-x\| \int_0^1 \left[ f_a'(\|(1-t)x + ty\|) - f_a'(0) \right] dt\end{aligned}$$

and the inequality (3.1) is proved.

Now, assume that the inequality (3.1) holds with a constant $C > 0$, i.e.

$$\begin{aligned}&\left\| \frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right) \right\| \\ &\le C\|y-x\| \int_0^1 \left[ f_a'(\|(1-t)x + ty\|) - f_a'(0) \right] dt,\end{aligned} \tag{3.5}$$

for any $x, y \in \mathcal{B}$ with $\|x\|, \|y\| < R$.

If we take in (3.5) $f(z) = z^2$ and $y = -x$, $x \in \mathcal{B}$, $x \ne 0$, then we get from (3.5)

$$\|x^2\| \le 4C\|x\|^2 \int_0^1 |1 - 2t| dt = 2C\|x\|^2$$

since $\int_0^1 \left|t - \frac{1}{2}\right| dt = \frac{1}{4}$.

Consequently $\|x^2\| \le 2C\|x\|^2$ for any $x \in \mathcal{B}$, $x \ne 0$ which implies that $C \ge \frac{1}{2}$. □

We observe that $f_a'$ is monotonic nondecreasing and convex on the interval $[0,R)$ and since the function $\psi(t) := \|(1-t)x+ty\|$ is convex on $[0,1]$ then we have the sequence of inequalities:

$$
\begin{aligned}
&\left\|\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right\| \\
&\le \frac{1}{2}\|y-x\|\int_0^1 \left[f_a'(\|(1-t)x+ty\|)-f_a'(0)\right]dt \\
&\le \frac{1}{4}\|y-x\|\left[f_a'\left(\left\|\frac{x+y}{2}\right\|\right)+\frac{f_a'(\|x\|)+f_a'(\|y\|)}{2}-2f_a'(0)\right] \\
&\le \frac{1}{4}\|y-x\|\left[f_a'(\|x\|)+f_a'(\|y\|)-2f_a'(0)\right] \\
&\le \frac{1}{2}\|y-x\|\left[\max\left\{f_a'(\|x\|),f_a'(\|y\|)\right\}-f_a'(0)\right].
\end{aligned}
\tag{3.6}
$$

We also have

$$
\begin{aligned}
&\left\|\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right\| \\
&\le \frac{1}{2}\|y-x\|\int_0^1 \left[f_a'(\|(1-t)x+ty\|)-f_a'(0)\right]dt \\
&\le \frac{1}{2}\|y-x\|\int_0^1 \left[f_a'((1-t)\|x\|+t\|y\|)-f_a'(0)\right]dt \\
&\le \frac{1}{4}\|y-x\|\left[f_a'\left(\frac{\|x\|+\|y\|}{2}\right)+\frac{f_a'(\|x\|)+f_a'(\|y\|)}{2}-2f_a'(0)\right] \\
&\le \frac{1}{4}\|y-x\|\left[f_a'(\|x\|)+f_a'(\|y\|)-2f_a'(0)\right] \\
&\le \frac{1}{2}\|y-x\|\left[\max\left\{f_a'(\|x\|),f_a'(\|y\|)\right\}-f_a'(0)\right].
\end{aligned}
\tag{3.7}
$$

The inequalities (3.6) and (3.7) are sharp.

From (3.7) we get the following divided difference inequality as well:

$$
\begin{aligned}
&\left\|\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right\| \\
&\le \frac{1}{2}\|y-x\|\int_0^1 \left[f_a'(\|(1-t)x+ty\|)-f_a'(0)\right]dt \\
&\le \frac{1}{2}\|y-x\|\times\begin{cases}\left(\frac{f_a(\|y\|)-f_a(\|x\|)}{\|y\|-\|x\|}-f_a'(0)\right) & \text{if } \|y\|\ne\|x\|, \\ \\ \left(f_a'(\|x\|)-f_a'(0)\right) & \text{if } \|y\|=\|x\|.\end{cases}
\end{aligned}
\tag{3.8}
$$

**Remark 4.** If $\|x\|, \|y\| \leq M < R$, then from the inequalities (3.7) we have the simpler inequality

$$\left\| \frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right) \right\| \leq \frac{1}{2}\left(f_a'(M) - f_a'(0)\right)\|y-x\|. \tag{3.9}$$

The constant $\frac{1}{2}$ is best possible in (3.9).

If we consider the exponential function $\exp(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!}\lambda^n$, then for any $x, y \in \mathcal{B}$ we have:

$$\begin{aligned} &\left\| \frac{\exp(x)+\exp(y)}{2} - \exp\left(\frac{x+y}{2}\right) \right\| \\ &\leq \frac{1}{2}\|y-x\| \int_0^1 \left(\exp\left(\|(1-t)x+ty\|\right)-1\right) dt \\ &\leq \frac{1}{2}\|y-x\| \times \begin{cases} \left(\frac{1}{2}\left[\exp\left(\left\|\frac{x+y}{2}\right\|\right) + \frac{\exp(\|x\|)+\exp(\|y\|)}{2}\right] - 1\right), \\ \left(\frac{\exp(\|y\|)-\exp(\|x\|)}{\|y\|-\|x\|} - 1\right) \text{ if } \|y\| \neq \|x\|, \\ \left(\exp(\|x\|)-1\right) \text{ if } \|y\| = \|x\|, \end{cases} \\ &\leq \frac{1}{2}\|y-x\|\left[\exp(M)-1\right], \end{aligned} \tag{3.10}$$

where, for the last inequality we assume that $\|y\|, \|x\| \leq M$.

Now, if we consider the functions $(1-\lambda)^{-1}$ and $(1+\lambda)^{-1}$, then for any $x, y \in \mathcal{B}$ with $\|x\|, \|y\| < 1$ we have:

$$\begin{aligned} &\left\| \frac{(1\pm x)^{-1}+(1\pm y)^{-1}}{2} - \left(1 \pm \frac{x+y}{2}\right)^{-1} \right\| \\ &\leq \frac{1}{2}\|y-x\| \left[\int_0^1 \left(1-\|(1-t)x+ty\|\right)^{-2} - 1\right] dt \\ &\leq \frac{1}{2}\|y-x\| \\ &\times \begin{cases} \left(\frac{1}{2}\left[\left(1-\left\|\frac{x+y}{2}\right\|\right)^{-2} + \frac{(1-\|x\|)^{-2}+(1-\|y\|)^{-2}}{2}\right] - 1\right), \\ \left[(1-\|x\|)^{-1}(1-\|y\|)^{-1} \quad - 1\right] \text{ if } \|y\| \neq \|x\|, \\ \left[(1-\|x\|)^{-2} - 1\right] \text{ if } \|y\| = \|x\|, \end{cases} \\ &\leq \frac{1}{2}\|y-x\|\left[(1-M)^{-2}-1\right], \end{aligned} \tag{3.11}$$

where, for the last inequality we assume that $\|y\|, \|x\| \leq M < 1$.

## §4. Some Inequalities for Commuting Elements

For two commuting elements $x, y \in \mathcal{B}$ it is of interest to estimate the distance between $\frac{1}{2}\left[f\left(x^2\right)+f\left(y^2\right)\right]$ and $f(xy)$, namely the quantity

$$\left\|\frac{f\left(x^2\right)+f\left(y^2\right)}{2}-f(xy)\right\|,$$

where $f$ is a function defined on the Banach algebra $\mathcal{B}$.

We have the following result:

**Theorem 5.** *Let $f(\lambda)=\sum_{n=0}^{\infty}\alpha_n\lambda^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. For any $x, y \in \mathcal{B}$ with $xy=yx$ and $\left\|x^2\right\|, \left\|y^2\right\|, \|xy\|<R$ we have*

$$\begin{aligned}(4.1)\quad &\left\|\frac{f\left(x^2\right)+f\left(y^2\right)}{2}-f(xy)\right\| \\ &\le \frac{1}{2}\|y-x\|^2\left[\int_0^1 f_a'\left(\|(1-t)x+ty\|^2\right)dt\right. \\ &\left.+\int_0^1\|(1-t)x+ty\|^2 f_a''\left(\|(1-t)x+ty\|^2\right)dt\right].\end{aligned}$$

*Proof.* We have from (2.1), for $n\ge 1$, that

$$\begin{aligned}(4.2)\quad \|y^n-x^n\|^2 &\le n^2\|y-x\|^2\left(\int_0^1\|(1-t)x+ty\|^{n-1}dt\right)^2 \\ &\le n^2\|y-x\|^2\int_0^1\|(1-t)x+ty\|^{2(n-1)}dt\end{aligned}$$

for any $x, y \in \mathcal{B}$.

The second inequality follows from the Cauchy-Bunyakovsky-Schwarz integral inequality

$$\left(\int_0^1 h(s)\,ds\right)^2\le\int_0^1 h^2(s)\,ds.$$

Since $x, y \in \mathcal{B}$ are commutative, then

$$(y^n-x^n)^2=y^{2n}-y^nx^n-x^ny^n+x^{2n}=2\left(\frac{y^{2n}+x^{2n}}{2}-(xy)^n\right),$$

which gives that

$$(4.3)\quad \left\|\frac{y^{2n}+x^{2n}}{2}-(xy)^n\right\|=\frac{1}{2}\left\|(y^n-x^n)^2\right\|\le\frac{1}{2}\|y^n-x^n\|^2$$

for $n \geq 1$.

Therefore, from (4.2) and (4.3) we have

$$\left\| \frac{y^{2n}+x^{2n}}{2} - (xy)^n \right\| \leq \frac{1}{2} n^2 \|y-x\|^2 \int_0^1 \|(1-t)\,x + ty\|^{2(n-1)}\, dt \tag{4.4}$$

for $n \geq 1$ and for any commuting elements $x, y \in \mathcal{B}$.

Using the generalized triangle inequality and the inequality (4.4) we have

$$\begin{aligned}
&\left\| \frac{1}{2}\left[\sum_{n=0}^{m} \alpha_n y^{2n} + \sum_{n=0}^{m} \alpha_n x^{2n}\right] - \sum_{n=0}^{m} \alpha_n (xy)^n \right\| \\
&= \left\| \sum_{n=1}^{m} \alpha_n \left[\frac{y^{2n}+x^{2n}}{2} - (xy)^n\right] \right\| \\
&\leq \sum_{n=1}^{m} |\alpha_n| \left\| \frac{y^{2n}+x^{2n}}{2} - (xy)^n \right\| \\
&\leq \frac{1}{2} \|y-x\|^2 \sum_{n=1}^{m} |\alpha_n|\, n^2 \int_0^1 \|(1-t)\,x+ty\|^{2(n-1)}\, dt \\
&= \frac{1}{2} \|y-x\|^2 \int_0^1 \sum_{n=1}^{m} n^2 |\alpha_n|\, \|(1-t)\,x+ty\|^{2(n-1)}\, dt.
\end{aligned} \tag{4.5}$$

Consider, for $u \neq 0$, the series

$$\sum_{n=0}^{\infty} n^2 \alpha_n u^{n-1} = \frac{1}{u} \sum_{n=0}^{\infty} n^2 \alpha_n u^n.$$

If we denote $g(u) := \sum_{n=0}^{\infty} \alpha_n u^n$, then

$$ug'(u) = \sum_{n=0}^{\infty} n \alpha_n u^n$$

and

$$u\left(ug'(u)\right)' = \sum_{n=0}^{\infty} n^2 \alpha_n u^n.$$

However

$$u\left(ug'(u)\right)' = ug'(u) + u^2 g''(u)$$

and then

$$\sum_{n=0}^{\infty} n^2 \alpha_n u^{n-1} = g'(u) + ug''(u)$$

for $u \neq 0$.

Utilising the above relations we can conclude that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^2 |\alpha_n| \|(1-t)x + ty\|^{2(n-1)} \\
&= \sum_{n=0}^{\infty} n^2 |\alpha_n| \|(1-t)x + ty\|^{2(n-1)} \\
&= f_a' \left( \|(1-t)x + ty\|^2 \right) + \|(1-t)x + ty\|^2 f_a'' \left( \|(1-t)x + ty\|^2 \right)
\end{aligned}$$

for almost any $t \in [0,1]$.

Since all the series whose partial sums are involved in (4.5) are convergent, then by letting $m \to \infty$ in (4.5) we get the desired inequality (4.1). □

**Remark 5.** If we use the notation

$$D_a^{(2)}(f)(u) := f_a'(u) + u f_a''(u), \; u \in D(0,R),$$

then the inequality (4.1) can be written in a simpler form as

$$\begin{aligned}
&\left\| \frac{f(x^2) + f(y^2)}{2} - f(xy) \right\| \\
&\leq \frac{1}{2} \|y - x\|^2 \int_0^1 D_a^{(2)}(f) \left( \|(1-t)x + ty\|^2 \right) dt,
\end{aligned} \tag{4.6}$$

where $x, y \in \mathcal{B}$ with $xy = yx$ and $\|x^2\|, \|y^2\|, \|xy\| < R$.

**Remark 6.** Utilising the Hermite-Hadamard inequality for convex functions, we have

$$\begin{aligned}
&\int_0^1 \|(1-t)x + ty\|^{2(n-1)} dt \\
&\leq \frac{1}{2} \left[ \left\| \frac{x+y}{2} \right\|^{2(n-1)} + \frac{\|x\|^{2(n-1)} + \|y\|^{2(n-1)}}{2} \right] \\
&\leq \frac{\|x\|^{2(n-1)} + \|y\|^{2(n-1)}}{2} \leq \max \left\{ \|x\|^{2(n-1)}, \|y\|^{2(n-1)} \right\}
\end{aligned}$$

for any $n \geq 1$.

If we multiply this inequality with $n^2|\alpha_n|$ and sum, then we get

$$\begin{aligned}(4.7)\quad & \frac{1}{2}\|y-x\|^2\int_0^1 D_a^{(2)}(f)\left(\|(1-t)x+ty\|^2\right)dt \\ & \le \frac{1}{4}\|y-x\|^2 \\ & \times\left[D_a^{(2)}(f)\left(\left\|\frac{x+y}{2}\right\|^2\right)+\frac{D_a^{(2)}(f)\left(\|x\|^2\right)+D_a^{(2)}(f)\left(\|y\|^2\right)}{2}\right] \\ & \le \frac{1}{2}\|y-x\|^2\left[\frac{D_a^{(2)}(f)\left(\|x\|^2\right)+D_a^{(2)}(f)\left(\|y\|^2\right)}{2}\right] \\ & \le \frac{1}{2}\|y-x\|^2\max\left\{D_a^{(2)}(f)\left(\|x\|^2\right),D_a^{(2)}(f)\left(\|y\|^2\right)\right\},\end{aligned}$$

where $x,y\in\mathcal{B}$ with $xy=yx$ and $\|x^2\|,\|y^2\|,\|xy\|<R$, which provides some simpler upper bounds for the quantity

$$\left\|\frac{f(x^2)+f(y^2)}{2}-f(xy)\right\|.$$

Moreover, if we assume that $\|x\|,\|y\|\le M$ with $M^2<R$, then

$$\begin{aligned}D_a^{(2)}(f)\left(\|x\|^2\right),D_a^{(2)}(f)\left(\|y\|^2\right) &\le D_a^{(2)}(f)\left(M^2\right) \\ &= f_a'\left(M^2\right)+M^2 f_a''\left(M^2\right)\end{aligned}$$

and from (4.6) and (4.7) we get the simple inequality

$$(4.8)\quad \left\|\frac{f(x^2)+f(y^2)}{2}-f(xy)\right\|\le\frac{1}{2}\|y-x\|^2\left[f_a'\left(M^2\right)+M^2 f_a''\left(M^2\right)\right],$$

for any $x,y\in\mathcal{B}$ with $xy=yx$ and $\|x\|,\|y\|\le M$ with $M^2<R$.

If we consider the exponential function $\exp(\lambda)$, then for any $x,y\in\mathcal{B}$ with $xy=yx$ and $\|x\|,\|y\|\le M$ we have the inequality

$$(4.9)\quad \left\|\frac{\exp(x^2)+\exp(y^2)}{2}-\exp(xy)\right\|\le\frac{1}{2}\|y-x\|^2\left(1+M^2\right)\exp\left(M^2\right).$$

Now, if we consider the functions $(1-\lambda)^{-1}$ and $(1+\lambda)^{-1}$, then for any $x,y\in\mathcal{B}$ with $xy=yx$ and $\|x\|,\|y\|\le M<1$, we have the inequalities

$$(4.10)\quad \left\|\frac{\left(1\pm x^2\right)^{-1}+\left(1\pm y^2\right)^{-1}}{2}-(1\pm xy)^{-1}\right\|\le\frac{1}{2}\|y-x\|^2\frac{1+M^2}{(1-M^2)^3}.$$

**Acknowledgement.** The author would like to thank the anonymous referee for valuable comments that have been implemented in the final version of the paper.

## References


[1] H. Araki and S. Yamagami, *An inequality for Hilbert-Schmidt norm*, Commun. Math. Phys. **81** (1981), 89-96.

[2] R. Bhatia, *First and second order perturbation bounds for the operator absolute value*, Linear Algebra Appl. **208/209** (1994), 367-376.

[3] R. Bhatia, *Perturbation bounds for the operator absolute value.* Linear Algebra Appl. **226/228** (1995), 639–645.

[4] R. Bhatia, D. Singh and K. B. Sinha, *Differentiation of operator functions and perturbation bounds.* Comm. Math. Phys. **191** (1998), no. 3, 603–611.

[5] R. Bhatia, *Matrix Analysis*, Springer Verlag, 1997.

[6] L. Ciurdariu, *A note concerning several Hermite-Hadamard inequalities for different types of convex functions.* Int. J. Math. Anal. **6** (2012), no. 33-36, 1623–1639.

[7] S. S. Dragomir, Y. J. Cho and S. S. Kim, *Inequalities of Hadamard's type for Lipschitzian mappings and their applications.* J. Math. Anal. Appl. **245** (2000), no. 2, 489–501.

[8] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, 2000.[Online http://rgmia.org/monographs/hermite_hadamard.html].

[9] R. Douglas, *Banach Algebra Techniques in Operator Theory*, Academic Press, 1972.

[10] Yu. B. Farforovskaya, *Estimates of the closeness of spectral decompositions of self-adjoint operators in the Kantorovich-Rubinshtein metric* (in Russian), Vesln. Leningrad. Gos. Univ. Ser. Mat. Mekh. Astronom. **4** (1967), 155-156.

[11] Yu. B. Farforovskaya, *An estimate of the norm* $\|f(B) - f(A)\|$ *for self-adjoint operators* $A$ *and* $B$ (in Russian) Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. **56** (1976), 143-162 .

[12] Yu. B. Farforovskaya and L. Nikolskaya, *Modulus of continuity of operator functions.* Algebra i Analiz **20** (2008), no. 3, 224–242; translation in St. Petersburg Math. J. **20** (2009), no. 3, 493–506.

[13] Y. Feng and W. Zhao, *Refinement of Hermite-Hadamard inequality.* Far East J. Math. Sci. **68** (2012), no. 2, 245–250.

[14] X. Gao, *A note on the Hermite-Hadamard inequality.* J. Math. Inequal. **4** (2010), no. 4, 587–591.

[15] S.-R. Hwang, K.-L. Tseng and K.-C. Hsu, *Hermite-Hadamard type and Fejér type inequalities for general weights (I).* J. Inequal. Appl. **2013**, 2013:170.

[16] T. Kato, *Continuity of the map $S \to |S|$ for linear operators*, Proc. Japan Acad. **49** (1973), 143-162.

[17] U. S. Kırmacı and R. Dikici, *On some Hermite-Hadamard type inequalities for twice differentiable mappings and applications.* Tamkang J. Math. **44** (2013), no. 1, 41–51.

[18] J. Mikusiński, *The Bochner Integral*, Birkhäuser Verlag, 1978.

[19] M. Muddassar, M. I. Bhatti and M. Iqbal, *Some new s-Hermite-Hadamard type inequalities for differentiable functions and their applications.* Proc. Pakistan Acad. Sci. **49** (2012), no. 1, 9–17.

[20] M. Matić and J. Pečarić, *Note on inequalities of Hadamard's type for Lipschitzian mappings.* Tamkang J. Math. **32** (2001), no. 2, 127–130.

[21] W. Rudin, *Functional Analysis*, McGraw Hill, 1973.

[22] M. Z. Sarikaya, *On new Hermite Hadamard Fejér type integral inequalities.* Stud. Univ. Babeş-Bolyai Math. **57** (2012), no. 3, 377–386.

[23] S. Wasowicz and A. Witkowski, *On some inequality of Hermite-Hadamard type.* Opuscula Math. **32** (2012), no. 3, 591–600.

[24] B.-Y. Xi and F. Qi, *Some integral inequalities of Hermite-Hadamard type for convex functions with applications to means.* J. Funct. Spaces Appl. **2012**, Art. ID 980438, 14 pp.

[25] G. Zabandan, A. Bodaghi and A. Kılıçman, *The Hermite-Hadamard inequality for r-convex functions.* J. Inequal. Appl. **2012**, 2012:215, 8 pp.

[26] C.-J. Zhao, W.-S. Cheung and X.-Y. Li, *On the Hermite-Hadamard type inequalities.* J. Inequal. Appl. **2013**, 2013:228.

[1]Mathematics, School of Engineering & Science
Victoria University, PO Box 14428
Melbourne City, MC 8001, Australia.
*E-mail*: `sever.dragomir@vu.edu.au`

[2]School of Computational & Applied Mathematics,
University of the Witwatersrand, Private Bag 3,
Johannesburg 2050, South Africa