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Geometry of charged rotating black hole

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Abstract. Some geometrical aspects of the Kerr-Newman black hole and its special cases have been studied. It is seen that the Gaussian curvature of the two or three dimensional induced metrics on some hypersurfaces outside of these black holes can be expressed in terms of the eigen values of the characteristic equation and depend upon the physical parameters which describe these black holes.

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§1. Introduction

In 1965, Newman et al. [1] have obtained a solution of Einstein-Maxwell equations. The corresponding metric in spherical coordinates (r, θ, ϕ, t) is given by [2]

$$
\begin{aligned}
(1.1)\qquad ds^2 = &\left(\frac{r^2 + a^2\cos^2\theta}{r^2 + a^2 + e^2 - 2mr}\right) dr^2 + (r^2 + a^2\cos^2\theta) d\theta^2 \\
&+ \sin^2\theta \left[r^2 + a^2 + \frac{a^2\sin^2\theta(2mr - e^2)}{r^2 + a^2\cos^2\theta}\right] d\phi^2 \\
&- \left(\frac{2a\sin^2\theta(2mr - e^2)}{r^2 + a^2\cos^2\theta}\right) d\phi dt - \left(1 - \frac{2mr - e^2}{r^2 + a^2\cos^2\theta}\right) dt^2.
\end{aligned}
$$

The solution (1.1) is now commonly known as Kerr-Newman solution and represents the exterior gravitational field of a charged rotating mass; it contains three real parameters; m (mass), e (charge) and a (angular momentum per unit mass). The Kerr-Newman metric is a generalization of other exact solutions of Einstein-Maxwell equations in general relativity and reduces to

(i) Kerr metric - if the charge e is zero,

(ii) Reissner-Nordström metric - if the angular momentum a is zero,

(iii) Schwarzschild metric - if both the charge e and angular momentum a are zero,

(iv) Minkowski metric - if the gravitational constant G is zero.

The Kerr-Newman solution with cosmological constant equal to zero, is also a special case of more general exact solution of the Einstein-Maxwell equations [1]. The Kerr-Newman solution (1.1) is of Petrov type D with non-null electromagnetic field. This solution defines a black hole with an event horizon only when the following relation is satisfied

$$a^2 + e^2 \leq m^2. \tag{1.2}$$

It is known that rotating black holes are formed due to the gravitational collapse of a massive spinning star or from the collapse of a collection of stars or gas with a total non-zero angular momentum. Since most of the stars rotate, it is expected that most of the black holes in nature are rotating and thus Kerr-Newman solution represents the gravitational field outside a charged rotating black hole.

Motivated by the all important role of rotating black holes, in this paper, we have studied some geometric aspects of Kerr-Newman black hole and discussed the special cases of this black hole.

§2. Kerr-Newman black hole

The non-zero components of the potential for the gravitation or the metric tensor for Kerr-Newman spacetime (1.1) in spherical coordinates (r, θ, ϕ, t) are given by

$$\begin{aligned} g_{11} &= \frac{C}{E}, \quad g_{22} = C, \quad g_{33} = \sin^2\theta[r^2 + a^2 + \frac{D}{C}a^2\sin^2\theta], \\ g_{44} &= -\left(1 - \frac{D}{C}\right), \quad g_{34} = g_{43} = -\frac{D}{C}(2a\sin^2\theta), \end{aligned} \tag{2.1}$$

where

$$C = r^2 + a^2\cos^2\theta, \quad D = 2mr - e^2, \quad E = r^2 + a^2 + e^2 - 2mr = r^2 + a^2 - D. \tag{2.2}$$

The non-zero components of the Christoffel symbols, using equations (2.1), can be calculated from the formula [5]

$$\Gamma^i_{jk} = g^{il}\Gamma_{ljk} = \frac{1}{2}g^{il}\left[\frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} + \frac{\partial g_{kl}}{\partial x^j}\right], \tag{2.3}$$

and are given by equations $(A1 - A17)$. While the non-zero components of the Riemann curvature tensor for the Kerr-Newman solution (1.1) can be calculated from the formula (cf., [5])

$$R_{ijkl} = \frac{1}{2}\left(\frac{\partial^2 g_{il}}{\partial x^j \partial x^k} + \frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} - \frac{\partial^2 g_{jl}}{\partial x^i \partial x^k}\right) + g_{mn}(\Gamma^m_{jk}\Gamma^n_{il} - \Gamma^m_{jl}\Gamma^n_{ik}) \tag{2.4}$$

by using equations (2.1) and $(A1 - A17)$, and are given by the equations $(B1 - B12)$. Equations $(A1 - A17)$ and $(B1 - B12)$ are listed in Appendix.

Now for the non-singular case, we can use a 6-dimensional formalism in the pseudo-Euclidean space $\mathbb{R}^6$ and can move to the 6-dimensional formalism by making the identification

(2.5)

ij:	23	31	12	14	24	34
A:	1	2	3	4	5	6

We also introduce the metric tensor as

$$g_{ik}g_{jl} - g_{il}g_{jk} = g_{ijkl} \to \widehat{g}_{AB}, \tag{2.6}$$

where $A, B = 1, 2, 3, 4, 5, 6$ and g_{ij} are the components of the metric tensor at an arbitrary point of the charged rotating black hole. The tensor $\widehat{g}_{AB}$ $(A, B = 1, 2, 3, 4, 5, 6)$ is symmetric and non-singular.

The non-zero components of the metric tensor $\widehat{g}_{AB}$ are given by

$$\begin{aligned} \widehat{g}_{11}(x^\alpha) &= \sin^2\theta[(D+E)C + a^2 D\sin^2\theta], \\ \widehat{g}_{22}(x^\alpha) &= \frac{\sin^2\theta}{E}[(D+E)C + a^2 D\sin^2\theta], \\ \widehat{g}_{33}(x^\alpha) &= \frac{C^2}{E}, \quad \widehat{g}_{44}(x^\alpha) = \frac{D-C}{E}, \quad \widehat{g}_{55}(x^\alpha) = D - C, \\ \widehat{g}_{66}(x^\alpha) &= \frac{\sin^2\theta}{C^2}[\{(D+E)C + a^2 D\sin^2\theta\}(D-C) - (a^2 D\sin^2\theta)^2], \\ \widehat{g}_{24}(x^\alpha) &= \frac{-aD\sin^2\theta}{C}, \quad \widehat{g}_{15}(x^\alpha) = -aD\sin^2\theta. \end{aligned} \tag{2.7}$$

In a similar way, we can transform the components of the Riemann curvature tensor as $R_{ijkl} \to \widehat{R}_{AB}$. Thus, for example R_{1234} can be written as $\widehat{R}_{36}$

(using identification (2.5)) and is same as equation ($B2$). The non-zero components of the tensor $\widehat{R}_{AB}$ are $\widehat{R}_{11}, \widehat{R}_{22}, \widehat{R}_{33}, \widehat{R}_{44}, \widehat{R}_{55}, \widehat{R}_{66}, \widehat{R}_{36}, \widehat{R}_{24}, \widehat{R}_{21}, \widehat{R}_{25}, \widehat{R}_{41}, \widehat{R}_{45}, \widehat{R}_{15}$ and are given by ($B1-B12$) under the identification (2.5). This consideration enables us to find a canonical form of the λ-tensor $\widehat{R}_{AB} - \lambda\widehat{g}_{AB}$. The solutions of the characteristic equation $|\widehat{R}_{AB} - \lambda\widehat{g}_{AB}| = 0$ lead to the eigen values for the Kerr-Newman spacetime. As such it is difficult to find the eigen values as the calculations are very long but the procedure for finding the eigen values has been illustrated for the very similar case of Reissner-Nordström solution in the next section.

Consider now the case when $\theta = 0$ or $\theta = \pi$ then equation (1.1) reduces to (as θ is constant, $d\theta$=0)

$$ds^2 = Pdr^2 - P^{-1}dt^2, \tag{2.8}$$

where

$$P = \frac{r^2 + a^2}{r^2 + a^2 + e^2 - 2mr}. \tag{2.9}$$

The metric tensor, in coordinates (r, t) is given by

$$\widetilde{g}_{ij} = \begin{pmatrix} P & 0 \\ 0 & -\dfrac{1}{P} \end{pmatrix}, \tag{2.10}$$

where $i, j = 1, 4$. Thus the induced metric on the hypersurface $\widetilde{H}_\theta$ (when $\theta = 0$ or $\theta = \pi$) of the charged rotating black hole degenerate to a 2-dimensional metric.

For a 2-dimensional case, the Riemann curvature tensor has only one independent non-zero component, which for the metric (2.8) is given by

$$\begin{aligned} R_{1414} = & \frac{2mr(3a^2 - r^2) + r^2e^2 - a^2e^2}{(r^2 + a^2)^3} \\ & - \frac{a(ma^2 + re^2 - mr^2)}{(r^2 + a^2)^3}\left[1 + \frac{ma^2 + re^2 - mr^2}{(r^2 + a^2 + e^2 - 2mr)}\right]. \end{aligned} \tag{2.11}$$

Moreover, the Gaussian curvature of the 2-dimensional metric is given by

$$\widetilde{K}(x^\alpha) = \frac{R_{1414}}{\det \widetilde{g}_{ij}} = -R_{1414}, \tag{2.12}$$

where det $(\widetilde{g}_{ij}) = -1$. It may be noted that this Gaussian curvature, expressed in terms of the only non-zero component of Riemann curvature tensor, depends on the parameters characterizing the charged rotating black hole.

§3. Special Cases

Since the Kerr-Newman solution (1.1) is the generalization of other known solutions of Einstein and Einstein-Maxwell equations, so we shall consider the following special cases. These considerations will help us to understand yet another geometrical aspect of black holes.

(i) Kerr black hole

When charge $e = 0$ then the metric (1.1) reduces to

$$ds^2 = \left(\frac{r^2 + a^2\cos^2\theta}{r^2 + a^2 - 2mr}\right) dr^2 + (r^2 + a^2\cos^2\theta)d\theta^2 + \sin^2\theta\left[r^2 + a^2 + \frac{a^2\sin^2\theta(2mr)}{r^2 + a^2\cos^2\theta}\right] d\phi^2 - \left(\frac{4amr\sin^2\theta}{r^2 + a^2\cos^2\theta}\right) d\phi dt - \left(1 - \frac{2mr}{r^2 + a^2\cos^2\theta}\right) dt^2. \tag{3.1}$$

Equation (3.1) is the well known Kerr metric.

The non-zero components of the Christoffel symbols and Riemann curvature tensor can easily be obtained for the solution (3.1) by taking $e = 0$ in equations $(A1 - A17)$ and $(B1 - B12)$, respectively. Using the identification (2.5), the components of metric tensor (2.6) in 6-dimensional formalism are given by

(3.2)

$$\widehat{g}_{11}(x^\alpha) = \sin^2\theta[(D^* + E^*)C + a^2D^*\sin^2\theta],$$

$$\widehat{g}_{22}(x^\alpha) = \frac{\sin^2\theta}{E^*}[(D^* + E^*)C + a^2D^*\sin^2\theta],$$

$$\widehat{g}_{33}(x^\alpha) = \frac{C^2}{E^*},\quad \widehat{g}_{44}(x^\alpha) = \frac{D^* - C}{E^*},\quad \widehat{g}_{55}(x^\alpha) = D^* - C,$$

$$\widehat{g}_{66}(x^\alpha) = \frac{\sin^2\theta}{C^2}[\{(D^* + E^*)C + a^2D^*\sin^2\theta\}(D^* - C) - (a^2D^*\sin^2\theta)^2],$$

$$\widehat{g}_{24}(x^\alpha) = \frac{-aD^*\sin^2\theta}{C},\quad \widehat{g}_{15}(x^\alpha) = -aD^*\sin^2\theta,$$

where

$$D^* = 2mr,\quad E^* = r^2 + a^2 - 2mr. \tag{3.3}$$

We can also transform the components of Riemann curvature tensor as $R_{ijkl} \to \widehat{R}_{AB}$. When $\theta = 0$ or $\theta = \pi$, then equation (3.1) leads to

$$ds^2 = Qdr^2 - Q^{-1}dt^2,$$

where

$$Q = \frac{r^2 + a^2}{r^2 + a^2 - 2mr}.$$

Thus the induced metric on the hypersurface of the Kerr black hole degenerates to a 2-dimensional metric; and the Gaussian curvature of this 2-dimensional metric is

$$K(x^\alpha) = -\frac{2mr(3a^2 - r^2)}{(r^2 + a^2)^3} + \frac{a(ma^2 - mr^2)}{(r^2 + a^2)^3}\left[1 + \frac{ma^2 - mr^2}{(r^2 + a^2 - 2mr)}\right].$$

This shows that the Gaussian curvature of the 2-dimensional metric depends upon the mass and angular momentum of the black hole.

(ii) Reissner-Nordström black hole

When $a = 0$, the Kerr-Newman metric, given by (1.1), reduces to

$$\begin{aligned} ds^2 = &\left(\frac{r^2}{r^2 + e^2 - 2mr}\right) dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \\ &- \left(\frac{r^2}{r^2 + e^2 - 2mr}\right)^{-1} dt^2. \end{aligned} \tag{3.4}$$

Equation (3.4) represents the metric of the charged black hole and is known as Reissner-Nordström black hole. We shall now discuss in detail the geometry of this black hole.

The gravitational potential for the metric (3.4) in spherical coordinates (r, θ, ϕ, t) is given by

$$g_{ij}(x^\alpha) = \begin{pmatrix} \frac{r^2}{r^2 + e^2 - 2mr} & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2 \sin^2\theta & 0 \\ 0 & 0 & 0 & -\frac{r^2 + e^2 - 2mr}{r^2} \end{pmatrix}. \tag{3.5}$$

The non-zero components of the Christoffel symbols and Riemann curvature tensor for the solution (3.14) are given by

$$
\begin{aligned}
\Gamma^1_{11} &= \frac{e^2 - mr^2}{r(r^2+e^2-2mr)},\\
\Gamma^1_{22} &= \frac{-(r^2+e^2-2mr)}{r},\\
\Gamma^1_{33} &= -\frac{(r^2+e^2-2mr)\sin^2\theta}{r},\\
\Gamma^1_{44} &= \frac{(r^2+e^2-2mr)(-mr+e^2)}{r^5},\\
\Gamma^2_{12} &= \Gamma^2_{21} = \frac{1}{r},\\
\Gamma^2_{33} &= -\sin\theta\cos\theta,\\
\Gamma^3_{13} &= \Gamma^3_{31} = \frac{1}{r},\\
\Gamma^3_{23} &= \Gamma^3_{32} = \cot\theta,\\
\Gamma^4_{14} &= \Gamma^4_{41} = \frac{(re^2 - mr^2)}{r(r^2+e^2-2mr)},
\end{aligned} \tag{3.6}
$$

and

$$
\begin{aligned}
R_{1212} &= \frac{e^2-mr}{r^2+e^2-2mr},\\
R_{1414} &= \frac{e^2-2mr}{r^4},\\
R_{2323} &= \sin^2\theta(2mr-e^2),\\
R_{2424} &= \frac{(mr-e^2)(r^2+e^2-2mr)}{r^4},\\
R_{3131} &= \sin^2\theta\left(\frac{e^2-mr}{r^2+e^2-2mr}\right),\\
R_{3434} &= \frac{\sin^2\theta(mr-e^2)(r^2+e^2-2mr)}{r^4}.
\end{aligned} \tag{3.7}
$$

Moreover, for the 6-dimensional formalism, using identification (2.5) and equation (2.6), the non-zero components of the metric tensor are

$$
\begin{aligned}
&\widehat{g}_{11}(x^\alpha) = r^4\sin^2\theta, \quad \widehat{g}_{22}(x^\alpha) = \frac{r^4\sin^2\theta}{r^2+e^2-2mr},\\
&\widehat{g}_{33}(x^\alpha) = \frac{r^4}{r^2+e^2-2mr}, \quad \widehat{g}_{44}(x^\alpha) = -1,\\
&\widehat{g}_{55}(x^\alpha) = 2mr - r^2 - e^2, \quad \widehat{g}_{66}(x^\alpha) = (2mr - r^2 - e^2)\sin^2\theta.
\end{aligned} \tag{3.8}
$$

While the non-zero components of $\widehat{R}_{AB}$ are given by

$$
\begin{aligned}
\widehat{R}_{11}(x^\alpha) &= \sin^2\theta(2mr - e^2),\\
\widehat{R}_{22}(x^\alpha) &= \sin^2\theta\left(\frac{e^2 - 2mr}{r^2 + e^2 - 2mr}\right),\\
\widehat{R}_{33}(x^\alpha) &= \frac{e^2 - mr}{r^2 + e^2 - 2mr},\\
\widehat{R}_{44}(x^\alpha) &= \frac{e^2 - 2mr}{r^4},\\
\widehat{R}_{55}(x^\alpha) &= \frac{(e^2 - 2mr)(r^2 + e^2 - 2mr)}{r^4},\\
\widehat{R}_{66}(x^\alpha) &= \frac{\sin^2\theta(mr - e^2)(r^2 + e^2 - 2mr)}{r^4}.
\end{aligned} \tag{3.9}
$$

Now, it only remains to find the canonical form of the λ-tensor $\widehat{R}_{AB} - \lambda\widehat{g}_{AB}$. It can easily be shown that the solution of the characteristic equation

$$|\widehat{R}_{AB} - \lambda\widehat{g}_{AB}| = 0$$

is given by

$$\lambda_1(r) = \lambda_4(r) = \frac{2mr - e^2}{r^4} \tag{3.10}$$

or

$$\lambda_2(r) = \lambda_3(r) = \lambda_5(r) = \lambda_6(r) = \frac{3e^2 - mr}{r^4}. \tag{3.11}$$

It may be noted that $\lambda_i (i = 1, 2, 3, 4, 5, 6)$ can be treated as the eigen values (see [3]).

Consider now the case when $\theta = 0$ or $\theta = \pi$ (the hypersurface $\bar{H}_0$ or $\bar{H}_\pi$) so that the 4-dimensional space of the charged black hole (3.4) degenerates to 2-dimensional metric given by

$$ds^2 = \left(\frac{r^2}{r^2 + e^2 - 2mr}\right) dr^2 - \left(\frac{r^2}{r^2 + e^2 - 2mr}\right)^{-1} dt^2. \tag{3.12}$$

The only non-zero component of the Riemann curvature tensor for the 2-dimensional metric (3.12) is given by

$$\bar{R}_{1414}(x^\alpha) = \frac{e^2 - 2mr}{r^4}. \tag{3.13}$$

Thus, the Gaussian curvature $\bar{K}$ of the 2-dimensional induced metric on the hypersurface $\bar{H}_0$ or $\bar{H}_\pi$ is

$$\bar{K}(\bar{x}^\alpha) = \frac{\bar{R}_{1414}(x^\alpha)}{|\bar{g}_{ij}|} = \frac{2mr - e^2}{r^4}. \tag{3.14}$$

From equations (3.10) and (3.14) it may be noted that the curvature of the 2-dimensional metric on the hypersurface $\bar{H}_0$ or $\bar{H}_\pi$ induced from the Reissner-Nordström metric is related to the eigen values λ_1 or λ_4. That is the Gaussian curvature of the 2-dimensional metric is expressed in terms of the λ-tensor.

While for the case $2m < r < \infty$, $0 < \theta < \pi$ and $\phi = 0$, equation (3.4) reduces to

$$ds^2 = \left(\frac{r^2}{r^2 + e^2 - 2mr}\right) dr^2 + r^2 d\theta^2 - \left(\frac{r^2}{r^2 + e^2 - 2mr}\right)^{-1} dt^2. \tag{3.15}$$

Here

$$^*g_{ij} = \begin{pmatrix} \frac{r^2}{r^2 + e^2 - 2mr} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & -\frac{r^2 + e^2 - 2mr}{r^2} \end{pmatrix}.$$

The only non-zero components of the Riemann curvature tensor for the 3-dimensional metric (3.15) are given by

$$\begin{aligned} {}^*R_{1212}(x^\alpha) &= \frac{e^2 - mr}{r^2 + e^2 - 2mr}, \\ {}^*R_{1414}(x^\alpha) &= \frac{e^2 - 2mr}{r^4}, \\ {}^*R_{2424}(x^\alpha) &= \frac{(e^2 - mr)(r^2 + e^2 - 2mr)}{r^4}. \end{aligned} \tag{3.16}$$

So that the sectional curvatures for 3-dimensional metric at each point $x^\alpha \equiv (r, \theta, t)$ is given by the following three quantities

$${}^*K_1(x^\alpha) = \frac{{}^*R_{2424}(x^\alpha)}{|{}^*g_{24}|} = \frac{e^2 - mr}{r^4}, \tag{3.17a}$$

$${}^*K_2(x^\alpha) = \frac{{}^*R_{1414}(x^\alpha)}{|{}^*g_{14}|} = \frac{2mr - e^2}{r^4}, \tag{3.17b}$$

$${}^*K_4(x^\alpha) = \frac{{}^*R_{1212}(x^\alpha)}{|{}^*g_{12}|} = \frac{e^2 - mr}{r^4}, \tag{3.17c}$$

where $x^1 = r$, $x^2 = \theta$, $x^4 = t$ and $^*g_{24}$ denotes the sub-matrix of $^*g_{ij}$ corresponding to $x^1 = r$.

From equations (3.10), (3.11) and (3.17) it may be noted that the sectional curvature of the 3-dimensional spacetime of Reissner-Nordström black hole can be expressed in terms of a λ-tensor which happens to be the solutions (eigen-values) of the characteristic equation $|\widehat{R}_{AB} - \lambda\widehat{g}_{AB}| = 0$.

(iii) Schwarzschild Black hole

When we take $a = e = 0$ in equation (1.1), the Kerr-Newman black hole reduces to Schwarzschild black hole and in such case the gravitational field has been discussed by Borgiel [4].

§4. Conclusion

An attempt has been made to investigate some geometrical properties of charged rotating black holes. Different cases that arise from this black hole have been considered and the case of Reissner-Nordström solution (charged black hole) has been discussed in detail to illustrate the procedure for obtaining the solutions of the characteristic equation $|\widehat{R}_{AB} - \lambda\widehat{g}_{AB}| = 0$. It is seen that the sectional curvature of the two and three dimensional metrics on the above hypersurfaces induced from the Riessner-Nordström metric of these black holes can be expressed in terms of the solutions of the characteristic equation and depends upon the parameters which describe these black holes.

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Appendix

$$(A1) \quad \Gamma^1_{11} = \frac{C(m-r)+Er}{CE}.$$

$$(A2) \quad \Gamma^1_{12} = \frac{a^2AD\sin\theta\cos\theta}{(AB+a^2D^2\sin^2\theta)C}.$$

$$(A3) \quad \Gamma^1_{22} = \frac{-rE}{C}.$$

$$(A4) \quad \Gamma^1_{33} = \frac{E\sin^2\theta\{rC^2-(mC+rD)a^2\sin^2\theta\}}{C^3}.$$

$$(A5) \quad \Gamma^1_{34} = \frac{aE\sin^2\theta}{C^3}.$$

$$(A6) \quad \Gamma^1_{44} = \frac{E(mC-rD)}{C^3}.$$

$$(A7) \quad \Gamma^2_{11} = \frac{a^2\sin\theta\cos\theta}{CE}.$$

$$(A8) \quad \Gamma^2_{12} = \frac{r}{C}.$$

$$(A9) \quad \Gamma^2_{22} = \frac{-a^2\sin\theta\cos\theta}{C}.$$

$$(A10) \quad \Gamma^2_{33} = \frac{-\sin\theta\cos\theta\{(D+E)C^2+2a^2CD\sin^2\theta+a^4D\sin^4\theta}{C^3}.$$

$$(A11) \quad \Gamma^2_{34} = \frac{-aD(D+E)\sin\theta\cos\theta}{C^3}.$$

$$(A12) \quad \Gamma^2_{44} = \frac{a^2Dsin\theta\cos\theta}{C^3}.$$

$$(A13) \quad \Gamma^3_{13} = \frac{\{(a^2\sin^2\theta)(mC-rD)\}(B-D)}{(AB+a^2D^2\sin^2\theta)C} + \frac{rBC}{AB+a^2D^2\sin^2\theta}.$$

$$(A14) \quad \Gamma^3_{14} = \frac{a(mC-rD)(D-B)}{(AB+a^2D^2\sin^2\theta)C}.$$

$$(A15) \quad \Gamma^3_{23} = \frac{1}{2}\left\{\frac{-a^2D^2\sin 2\theta(C+a^2\sin^2\theta)}{(AB+a^2D^2\sin^2\theta)C\sin^2\theta}\right\} + \frac{1}{2}\left\{\frac{BC}{(AB+a^2D^2\sin^2\theta)\sin^2\theta}\right\} \left\{(D+E)\sin 2\theta+4a^2CD\sin^3\theta\cos\theta+\frac{a^4D\sin^2\theta\sin 2\theta}{C^2}\right\}.$$

$$(A16) \quad \Gamma^3_{24} = \frac{-a\sin\theta\cos\theta\{a^2D+(D+E)B\}}{(AB+a^2D^2\sin^2\theta)C}.$$

$$(A17) \quad \Gamma^4_{14} = \frac{(mC-rD)\{A+a^2D\sin^2\theta\}}{C\{AB+a^2D^2\sin^2\theta\}}.$$

$$(B1)\quad R_{1212} = \frac{a^2\cos 2\theta}{D} + \frac{C}{E^2}\left[\left\{\frac{a^2AD\sin\theta\cos\theta}{(AB+a^2D^2\sin^2\theta)C}\right\}^2 + \frac{rE(rE-C(r-m))}{C^2D}\right] + \left(\frac{r^2}{C} + \frac{a^2\sin^2\theta\cos\theta}{CE}\right) - 1.$$

$$(B2)\quad R_{1234} = \frac{-a(a^2m + 4e^2r - 6mr^2 + a^2m\cos 2\theta\sin 2\theta)}{F^2}.$$

$$(B3)\quad R_{1313} = -\sin^2\theta\left[1 + \frac{-a^2D\sin^2\theta + 4a^2(mC - rD)\sin\theta\cos\theta}{C^3}\right] + \frac{rE - C(r-m)}{BC^3}\sin^2\theta[rC^2 - a^2\sin^2\theta(mC + rD)] + \frac{a^2\sin^2\theta\cos^2\theta}{C^3D}[(D+E)C^2 + 2a^2CD\sin^2\theta + a^4D\sin^4\theta] + \sin^2\theta\left\{(D + E + \frac{a^2D\sin^2\theta}{C}\right\} \left\{\frac{(a^2\sin^2\theta(mC - rD))(B-D)}{(AB + a^2D^2\sin^2\theta)C} + \frac{rBC}{AB + a^2D^2\sin^2\theta}\right\}.$$

$$(B4)\quad R_{1314} = \frac{4a\cos 2\theta\sin^2\theta}{DF^3}[-a^6D + 2r^2\{-3e^4 + 4e^2(2m-r)r + mr^2(-4m+3r)\} + a^4\{e^4 + mr(-7+4mr-4r^2) + e^2(1-4mr+2r^2)\} + a^2\{e^4(1+r^2) + e^2r(-7r + r^3 - 8m - 4mr^2) + mr^2(12m + 4mr^2 - r - 2r^3)\} + a\{a^4D + e^4(1+r^4) + e^2r(r + r^3 - 8m - 4mr^2) + mr^2(12m + 4mr^2 - 7r - 2r^3)\} + a^2\{e^4 + mr(-7+4mr-4r^2) + e^2(1-4mr+2r^2)\}].$$

$$(B5)\quad R_{1323} = \frac{a^2\cos\theta\sin\theta}{2EC^3}\Big[2EF\{rE - a^2(m-r)\cos 2\theta\}\sin^2\theta + \left\{\frac{-2a^2D(D+E)(a^2m - 2rD + a^2m\cos 2\theta\sin^4\theta)}{F}\right\} - 4EC\{a^2r\cos^4\theta + \cos^2\theta(2r^3 + a^2m\sin^2\theta)\} + \frac{EC}{F^2}(3a^2 - ae^2 + 8mr + 8r^2 + 4E\cos 2\theta + a^2\cos^4\theta) \{a^4r\cos^4\theta + \cos^2\theta(2a^2r^3 + a^4m\sin^2\theta) + r(r^4 + a^2(e^2 - mr)\sin^2\theta)\} - 10a^2E\sin^2\theta\{a^4r\cos^4\theta + \cos^2\theta(2a^2r^3 + a^4m\sin^2\theta) + r^5 + a^2r(e^2 - mr)\sin^2\theta\} - 2\{r(mr - a^2 - e^2) - a(m-r)\cos^2\theta\}\{a^2r^4 + r^6 + a^2(D+E)\cos^4\theta + 2a^2r^2D\sin^2\theta + a^4D\sin^4\theta$$

$$+ 2a^2\cos^2\theta(a^2r^2 + r^4 - a^2e^2 sin^2\theta) + a^4 mr sin^2 2\theta\}\Big].$$

$$(B6) \quad R_{1414} = \frac{CD + 4r(mC - rD)}{C^3}$$
$$+ \left[\left\{\frac{a(mC - rD)(A + a^2D\sin^2\theta)}{C(AB + a^2D^2\sin^2\theta)}\right\}^2 \frac{D - C}{C}\right]$$
$$+ \sin^2\theta\left(E + D + \frac{a^2D\sin^2\theta}{C}\right)]\left[\frac{a(mC - rD)(D - B)}{C(AB + a^2D^2\sin^2\theta)}\right]^2$$
$$- \frac{a(mC - rD)(-rC + mC + rE)}{C^3} - \frac{a^4D\sin^2\theta\cos^2\theta}{C^3E}.$$

$$(B7) \quad R_{1423} = \frac{-a\sin 2\theta}{2EF^3}[-9a^6m - 13a^4e^2m - 36a^4e^2r - 40a^2e^4r$$
$$+ 26a^4m^2r + 35a^4mr^2 + 120a^2e^2mr^2 - 60a^2e^2r^3 - 32e^4r^3$$
$$- 80a^2m^2r^3 + 76a^2mr^4 + 104e^2mr^4 - 24e^2r^5 - 80m^2r^5$$
$$+ 32mr^6 - 4a^2\{2a^4m + 3a^2(e^2(m - r) + mr(-2m + 3r))\}$$
$$+ r\{-2e^4 + e^2(10m - 3r)r + mr^2(-12m + 7r)\}\cos\theta$$
$$+ a^4mE\cos 4\theta].$$

$$(B8) \quad R_{2323} = -(D + E)\cos 2\theta - \frac{D}{C^4}[c^2\{a^2(C\cos 2\theta - 2a^2\sin^2\theta\cos^2\theta)$$
$$- a^4(3\sin^2\theta\cos^2\theta - \sin^4\theta)\}$$
$$- (a^2C\sin\theta\cos\theta + a^4\sin^3\theta\cos\theta)(-2a^2C\sin^2\theta\cos^2\theta)]$$
$$- \frac{rE\sin^2\theta}{C^3}(rC^2 - ma^2C\sin^2\theta - a^2rD\sin^2\theta)$$
$$- \frac{a^2\sin^2\theta\cos^2\theta}{C^3}[(D + E)C^2 + 2a^2CD\sin^2\theta + a^4D\sin^2\theta]$$
$$+ \frac{\sin^2\theta}{2}\left\{(D + E) + \frac{a^2D\sin^2\theta}{C}\right\}$$
$$\left[\left\{\frac{-a^2D^2(C\sin 2\theta + 2a^3\sin^3\theta\cos\theta)}{(AB + a^2D^2\sin^2\theta)C}\right\}\right.$$
$$+ \frac{BC}{(AB + a^2D^2\sin^2\theta)\sin^2\theta}\{2(D + E)\cos^2\theta$$
$$\left. + 4a^2CD\sin^3\theta\cos\theta + 2a^4D\sin^5\theta\cos\theta\}\right].$$

$$(B9)\quad R_{2324} = \frac{a}{2EC^3}\Bigg[2CDE(D+E)\cos^2\theta + \frac{1}{F^2}\Bigg\{-CDE(D+E)$$
$$\cos^2\theta(3a^4 - 4a^2e^2 + 8a^2mr + 8a^2r^2 + 8r^4 + 4a^2E\cos 2\theta$$
$$+ a^4\cos^4\theta - 2CDE(D+E)\sin^2\theta)$$
$$+ r(a^2 - D + a^2(m-r)\cos^2\theta)(E + a^2m\cos 2\theta)\sin^2\theta\Bigg\}$$
$$+ \Bigg\{\frac{4a^2D^2E\sin^4\theta\cos^2\theta}{F}\Bigg\} + \frac{5}{2a}a^2DE(D+E)\sin^2\theta\Bigg].$$

$$(B10)\quad R_{2424} = \frac{-a^2D(C\cos\theta + a^2 sin2\theta)}{C^3} + \frac{rE(mC - rD)}{C^3}$$
$$+ \frac{a^4D\sin^2\theta\cos^2\theta}{C^3}$$
$$+ \sin^2\theta\Bigg\{(E + D + \frac{a^2D\sin^2}{C}\Bigg\}\Bigg\{(aD^2\sin\theta\cos\theta)(a^2D - B)\Bigg\}^2.$$

$$(B11)\quad R_{3434} = \frac{1}{C^5}\Bigg[a^4E\sin^4\theta - (mC - rD)E\sin^2\theta\{rC^2 - a^2\sin^2\theta$$
$$(mC - rD)\} + \{aD(D+E)\sin\theta\cos\theta\}^2 - a^2D^2\sin^2\theta\cos^2\theta$$
$$\{C^2(D+E) + 2a^2CD\sin^2\theta + a^4D\sin^4\theta\}\Bigg].$$

$$(B12)\quad R_{1342} = -(R_{1234} + R_{1423}).$$

Here

$$A = (r^2 + a^2)(r^2 + a^2\cos^2\theta),$$
$$B = (r^2 + a^2\cos^2\theta) - (2mr - e^2),$$
$$C = (r^2 + a^2\cos^2\theta),$$
$$D = (2mr - e^2),$$
$$E = (r^2 + a^2 + e^2 - 2mr),$$
$$F = (a^2 + 2r^2 + a^2\cos 2\theta).$$

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